

# Leptonic invariants, neutrino mass-ordering and the octant of $\theta_{23}$

---

G.C. Branco,<sup>a,b</sup> M.N. Rebelo<sup>a,b</sup> and J.I. Silva-Marcos<sup>a</sup>

<sup>a</sup>*Departamento de Física and Centro de Física Teórica de Partículas (CFTP),  
Instituto Superior Técnico (IST), Universidade de Lisboa (UL),  
Av. Rovisco Pais, P-1049-001 Lisboa, Portugal*

<sup>b</sup>*Theory Department, CERN,  
CH 1211 Geneva 23, Switzerland*

*E-mail:* [gbranco@tecnico.ulisboa.pt](mailto:gbranco@tecnico.ulisboa.pt), [rebelo@tecnico.ulisboa.pt](mailto:rebelo@tecnico.ulisboa.pt),  
[juca@cftp.ist.utl.pt](mailto:juca@cftp.ist.utl.pt)

**ABSTRACT:** We point out that leptonic weak-basis invariants are an important tool for the study of the properties of lepton flavour models. In particular, we show that appropriately chosen invariants can give a clear indication of whether a particular lepton flavour model favours normal or inverted hierarchy for neutrino masses and what is the octant of  $\theta_{23}$ . These invariants can be evaluated in any conveniently chosen weak-basis and can also be expressed in terms of neutrino masses, charged lepton masses, mixing angles and CP violation phases.

**KEYWORDS:** Beyond Standard Model, Neutrino Physics

**ARXIV EPRINT:** [1705.07758](https://arxiv.org/abs/1705.07758)

---

## Contents

<b>1</b>	<b>Introduction</b>	<b>1</b>
<b>2</b>	<b>Invariants and the pattern of leptonic mixing and CP violation</b>	<b>2</b>
2.1	Introductory remarks	2
2.2	Leptonic weak-basis invariants	4
<b>3</b>	<b>Invariants sensitive to neutrino mass ordering and the <math>\theta_{23}</math> octant</b>	<b>6</b>
3.1	The neutrino mass ordering	6
3.2	The octant of $\theta_{23}$	7
<b>4</b>	<b>Application to specific Ansätze for Leptonic masses</b>	<b>8</b>
<b>5</b>	<b>Conclusions</b>	<b>10</b>

---

## 1 Introduction

Neutrino Physics is a very active field of research in Particle Physics, with a well established longterm program for future experiments. Although neutrino oscillations have provided solid evidence for leptonic mixing and for at least two non-vanishing neutrino masses, there are still some fundamental open questions. These include the establishment of the nature of neutrinos (Dirac or Majorana), determination of the pattern of neutrino masses (hierarchical or quasi-degenerate), settling of the ordering of neutrino masses (normal or inverted) and discovering leptonic CP violation.

We have reached a precision era for the measurement of leptonic mixing parameters and for the measurement of the squared mass differences of the three light neutrinos. Still, it is not yet clear whether it will be possible, in the near future, to determine the mass of the lightest neutrino and thus the scale of neutrino masses is not yet determined. However, it is by now established, both in laboratory experiments and via astrophysical bounds, that light neutrinos can at most have masses of the order of one eV. Several of these open questions have profound implications for Astrophysics and Cosmology.

Recently, the Daya Bay Reactor Neutrino Experiment [1] measured with certainty, for the first time, a nonzero value for the smallest leptonic mixing angle,  $\theta_{13}$ . At that time it was already known that the two other leptonic mixing angles were large. The fact that none of the three leptonic mixing angles vanishes opens up the possibility of observing leptonic CP violation of Dirac type in neutrino oscillation experiments. At present, there is a some likelihood indication for a Dirac phase of  $-\pi/2$ . Until recently all experimental results were in agreement with  $\theta_{23}$  corresponding to maximal mixing. However, there is a new measurement by NOvA [2] reporting that this value is excluded at  $2.6 \sigma$  CL.

On the theoretical side there have been many attempts at understanding the pattern of leptonic masses and mixing, through the introduction of family symmetries at the Lagrangian level or as symmetries of the leptonic mass matrices. In a bottom up approach, one may try to guess the family symmetries chosen by nature, from the input from experiment. One of the difficulties in pursuing this approach stems from the fact that the leptonic mass matrices change under weak-basis (WB) transformations. So even if there is a flavour symmetry chosen by nature, in what WB would the symmetry be evident?

In this paper we point out that leptonic WB invariants can be a very useful tool in the study of the pattern of leptonic masses and mixing, including leptonic CP violation. The paper is organised as follows. In the next section, we review leptonic CP-even and CP-odd WB invariants. In the CP-odd invariants, we include those which are sensitive to Dirac and Majorana-type CP violation. In the third section, we show how WB invariants provide a simple way of determining whether a given model favours normal or inverted neutrino mass ordering and also what it predicts for the  $\theta_{23}$  octant. In section 4, we illustrate the usefulness of the WB invariants, by applying them to specific Ansätze proposed in the literature. The summary and conclusions are presented in the last section.

## 2 Invariants and the pattern of leptonic mixing and CP violation

### 2.1 Introductory remarks

In the SM, the flavour structure of Yukawa couplings, in both the lepton and quark sectors, is not constrained by gauge symmetry. As a result, fermion masses and mixing are arbitrary. One may adopt a bottom up approach and attempt at extracting from experiment some hint of a flavour symmetry. One of the difficulties one encounters in this approach stems from the fact that one has the freedom to make weak basis (WB) transformations under which the flavour structure of Yukawa couplings change, but their physical content remains invariant. Let us consider the SM and assume that lepton number is violated by some physics beyond the SM, leading at low energies to an effective Majorana neutrino mass matrix. The leptonic mass terms are:

$$\mathcal{L}_{\text{mass}} = -\frac{1}{2}\nu_L^0 T C^{-1} m_\nu \nu_L^0 - \bar{\ell}_L^0 m_\ell \ell_R^0 + \text{h.c.}, \quad (2.1)$$

and the charged currents are:

$$\mathcal{L}_W = -\frac{g}{\sqrt{2}} W_\mu^+ \bar{\ell}_L^0 \gamma^\mu \nu_L^0 + \text{h.c.} \quad (2.2)$$

The WB transformations involving the leptonic fields are of the form:

$$\nu_L^0 \rightarrow V \nu_L^0, \quad \ell_L^0 \rightarrow V \ell_L^0, \quad \ell_R^0 \rightarrow W \ell_R^0 \quad (2.3)$$

with  $V$  and  $W$  unitary  $3 \times 3$  matrices. Under these transformations the leptonic mass terms transform as:

$$m_\nu \rightarrow V^T m_\nu V, \quad m_\ell \rightarrow V^\dagger m_\ell W \quad (2.4)$$

Parameter	Best fit	1 $\sigma$ range
$\Delta m_{21}^2$ [ $10^{-5}eV^2$ ]	7.60	7.42–7.79
$ \Delta m_{31}^2 $ [ $10^{-3}eV^2$ ](NH)	2.48	2.41–2.53
$ \Delta m_{31}^2 $ [ $10^{-3}eV^2$ ](IH)	2.38	2.32–2.43
$\sin^2 \theta_{12}$	0.323	0.307–0.339
$\sin^2 \theta_{23}$ (NH)	0.567	0.439 <sup>a</sup> –0.599
$\sin^2 \theta_{23}$ (IH)	0.573	0.530–0.598
$\sin^2 \theta_{13}$ (NH)	0.0234	0.0214–0.0254
$\sin^2 \theta_{13}$ (IH)	0.0240	0.0221–0.0259
$\delta$ (NH)	1.34 $\pi$	0.96–1.98 $\pi$
$\delta$ (IH)	1.48 $\pi$	1.16–1.82 $\pi$

**Table 1.** Neutrino oscillation parameter summary, taken from ref. [4]. For  $\Delta m_{31}^2$ ,  $\sin^2 \theta_{23}$ ,  $\sin^2 \theta_{13}$ , and  $\delta$  the upper (lower) row corresponds to normal (inverted) neutrino mass hierarchy. <sup>a</sup>There is a local minimum in the first octant,  $\sin^2 \theta_{23} = 0.467$  with  $\Delta\chi^2 = 0.28$  with respect to the global minimum.

Leptonic mixing and CP violation in the leptonic sector are parametrised by the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix,  $U_{\text{PMNS}}$ , which contains three mixing angles and three CP violating phases, two of the phases reflecting the Majorana character of neutrinos.

Following the standard parametrisation [3] the matrix  $U_{\text{PMNS}}$  can be denoted as:

$$U_{\text{PMNS}} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix} \cdot P \quad (2.5)$$

with P given by

$$P = \text{diag} (1, e^{i\alpha_{21}}, e^{i\alpha_{31}}) \quad (2.6)$$

where  $c_{ij} = \cos \theta_{ij}$ ,  $s_{ij} = \sin \theta_{ij}$  the angles  $\theta_{ij} = [0, \pi/2]$ ,  $\delta = [0, 2\pi]$  is a Dirac-type CP violating phase and  $\alpha_{21}$ ,  $\alpha_{31}$  denote phases associated to the Majorana character of neutrinos. Neutrino oscillation experiments are sensitive to the mixing parameters with the exception of the CP violating phase  $\alpha_{21}$ ,  $\alpha_{31}$ . There is no loss of generality in adopting the convention that  $\theta_{ij}$  are all in the first quadrant.

In table 1 we summarise the present knowledge concerning neutrino masses and leptonic mixing. In the literature, there are three global phenomenological fits on  $\theta_{12}$ ,  $\theta_{23}$ ,  $\theta_{13}$ , and  $\delta$  [4–6]. The specific bounds vary slightly from reference to reference. For definiteness we present those of ref. [4]. Neutrino oscillations give information about differences of squared masses:

$$\Delta m_{21}^2 \equiv \Delta_{21}^2 \equiv m_2^2 - m_1^2, \quad \Delta m_{31}^2 \equiv \Delta_{31}^2 \equiv m_3^2 - m_1^2 \quad (2.7)$$

The sign of  $\Delta m_{31}^2$  is not yet known. The best fit values of some of the parameters listed in table 1 depend on the sign of  $\Delta m_{31}^2$ . For a positive sign the ordering is called normal

(NH), for a negative sign the ordering is called inverted (IH). The association of the terms normal and inverted to each one of the signs reflects a prejudice, since from a theoretical point of view, no ordering can a priori be considered more natural, as discussed in ref. [7].

At this stage, it is worth recalling the main differences between rephasing invariant quantities in the cases of Majorana and Dirac neutrinos. Let us start by considering unitarity triangles, assuming that the  $U_{\text{PMNS}}$  is a 3x3 unitary matrix. It is well known that there are many frameworks, including for example the seesaw type one [8–12] where this is not exactly true, since there are small deviations from 3x3 unitarity. With a unitary  $U_{\text{PMNS}}$ , one has six leptonic unitarity triangles, three corresponding to orthogonality of rows and another three for orthogonality of columns. The triangles corresponding to orthogonality of rows are often called Dirac triangles and are very similar to the unitarity triangles in the quark sector. Under rephasing of the charged lepton fields, the leptonic Dirac triangles rotate and thus the direction of their sides have no physical meaning. Analytically, they correspond to quantities like  $\arg(U_{e1}U_{\mu1}^*)$  which are not rephasing invariant. The phases which are physically meaningful in these Dirac triangles are the internal angles of the triangles which analytically correspond to the arguments of invariant leptonic quartets like  $(U_{e2}U_{\mu3}U_{e3}^*U_{\mu2}^*)$ . In the Majorana triangles, one encounters a very different situation [13]. In these triangles the directions of the sides are physically meaningful and do not change under the rephasing of the charged lepton fields. Recall that one cannot rephase Majorana neutrinos. Analytically these directions correspond to rephasing-invariant bilinears like  $(U_{e1}U_{e2}^*)$ . Therefore, the most rigorous definition of Majorana phases is that they correspond to arguments of the rephasing invariant bilinears  $(U_{\ell j}U_{\ell k}^*)$ . It can be seen that, independently of unitarity, there are only six independent Majorana phases in a  $3 \times 3$   $U_{\text{PMNS}}$ . Assuming unitarity, it has been shown that from the knowledge of six independent Majorana phases one can construct the full PMNS matrix, including moduli and phases [14].

## 2.2 Leptonic weak-basis invariants

In this subsection, we describe the WB invariants which can fix the lepton mixing and CP violation in the leptonic sector. We consider WB invariants written in terms of the charged lepton mass matrix and the effective neutrino mass matrix and not WB invariants written in terms of the neutrino mass matrices [15–19] appearing in the framework of the seesaw mechanism.

It can be shown that the following four weak basis (WB) invariants completely define four independent moduli of  $U_{\text{PMNS}}$  [20]:

$$I_1 = \text{Tr}[H_\ell H_\nu], \quad I_2 = \text{Tr}[H_\ell^2 H_\nu], \quad (2.8)$$

$$I_3 = \text{Tr}[H_\ell H_\nu^2], \quad I_4 = \text{Tr}[H_\ell^2 H_\nu^2] \quad (2.9)$$

where  $H_\nu = m_\nu^* m_\nu^T$  and  $H_\ell = m_\ell m_\ell^\dagger$ . These four WB invariants are physical quantities and can be expressed in terms of charged lepton and neutrino masses and moduli of  $U_{\text{PMNS}}$ . From the knowledge of the four invariants and the charged lepton and neutrino masses, one can derive all the moduli of  $U_{\text{PMNS}}$ , using 3x3 unitarity. From the knowledge of the moduli

one can then readily evaluate the common area of all unitarity triangles which in turn gives the strength of leptonic CP violation of the Dirac type. This is entirely analogous to the situation in the quark sector [20]. Although the four invariants of eqs. (2.8) and (2.9), together with  $3 \times 3$  unitarity completely fix the moduli of  $U_{\text{PMNS}}$  and the strength of leptonic CP violation of Dirac-type, there is still a two-fold ambiguity, since the sign of CP violation is not fixed. This ambiguity can be lifted by calculating [21]:

$$I^{CP} \equiv \text{Tr} [m_\nu^* \cdot m_\nu^T, h_\ell]^3 \quad (2.10)$$

At this stage it should be pointed out that there is an important difference between the lepton and quark sectors. While in the quark sector one can overdetermine the CKM matrix from experiment, in the case of the lepton sector with Majorana neutrinos, one cannot completely determine  $U_{\text{PMNS}}$  from feasible experiments. This is related to the appalling fact, emphasised by Glashow et al, [22] that it is not possible to fully reconstruct the neutrino mass matrix from feasible experiments. It is instructive to write explicitly the strength of Dirac type CP violation in terms of four independent moduli of  $U_{\text{PMNS}}$ . Choosing as independent moduli  $U_{e2}, U_{e3}, U_{\mu3}, U_{\mu2}$ , one obtains [20]:

$$\text{Im } Q \equiv \text{Im} (U_{e2} U_{\mu3} U_{e3}^* U_{\mu2}^*) = \sqrt{|U_{e2}|^2 |U_{\mu3}|^2 |U_{e3}|^2 |U_{\mu2}|^2 - R^2}, \quad (2.11)$$

$$R = (1 - |U_{e2}|^2 - |U_{\mu3}|^2 - |U_{e3}|^2 - |U_{\mu2}|^2 + |U_{e2}|^2 |U_{\mu3}|^2 + |U_{e3}|^2 |U_{\mu2}|^2)/2 \quad (2.12)$$

Experimentally one can extract information on the real and imaginary parts of such quartets from neutrino oscillation experiments. This is to be compared to the quark sector where one can choose as input moduli  $|V_{us}|, |V_{ub}|, |V_{cb}|$  and  $|V_{td}|$ . The first three moduli can be measured from strange particles, and B-meson decays, while  $|V_{td}|$  can be measured from  $B_d - \bar{B}_d$  mixing. Of course, the measurement of  $|V_{td}|$  through this meson mixing can be affected by New Physics contributions to this mixing. In spite of the scarcity of leptonic measurements, a nice aspect of the leptonic sector is the absence of ‘‘hadronic uncertainties’’. For example, it is remarkable that the present experimental measurement  $|U_{e3}|$  has a smaller percent error than the measurement of  $|V_{ub}|$ , in spite of the enormous effort from both theorists and experimentalists to measure  $|V_{ub}|$ .

So far, we have only considered WB invariants which fix the strength of CP violation of Dirac-type. In the case of Majorana neutrinos one has two extra phases which, as emphasised before, have to do with the fact that for Majorana neutrinos there is only freedom to rephase the charged leptons and therefore the phase of the bilinear  $(U_{\ell j} U_{\ell k}^*)$  has physical meaning.

It is possible to derive WB invariant CP-odd conditions sensitive to the three CP violating phases present in eq. (2.5). This was done in ref. [23] where it was shown that the following set of conditions is necessary and sufficient for CP invariance in the case of three generations, for nonzero and nondegenerate masses:

$$\text{Im } \text{Tr} [h_\ell \cdot m_\nu^* \cdot m_\nu \cdot m_\nu^* \cdot h_\ell^* \cdot m_\nu] = 0 \quad (2.13)$$

$$\text{Im } \text{Tr} [h_\ell \cdot (m_\nu^* \cdot m_\nu)^2 \cdot (m_\nu^* \cdot h_\ell^* \cdot m_\nu)] = 0 \quad (2.14)$$

$$\text{Im } \text{Tr} [h_\ell \cdot (m_\nu^* \cdot m_\nu)^2 \cdot (m_\nu^* \cdot h_\ell^* \cdot m_\nu)(m_\nu^* \cdot m_\nu)] = 0 \quad (2.15)$$

$$\text{Im } \text{Tr} [(m_\nu \cdot h_\ell \cdot m_\nu^*) + (h_\ell^* \cdot m_\nu \cdot m_\nu^*)] = 0 \quad (2.16)$$

Selecting a minimal set of necessary and sufficient conditions for CP invariance is not trivial and was provided later on, in ref. [24]:

$$\text{Tr} [m_\nu^* \cdot m_\nu^T, h_\ell]^3 = 0 \tag{2.17}$$

$$\text{Tr} [m_\nu \cdot h_\ell \cdot m_\nu^*, h_\ell^*]^3 = 0 \tag{2.18}$$

$$\text{ImTr} (h_\ell \cdot m_\nu^* \cdot m_\nu \cdot m_\nu^* \cdot h_\ell^* \cdot m_\nu) = 0 \tag{2.19}$$

The first of these three equations is similar to the condition derived for the quark sector. It is sensitive to the Dirac-type phase and insensitive to the Majorana-type phases. The second and third equations are sensitive to both Dirac and Majorana type phases. The second equation was first derived in ref. [25] in the context of three degenerate neutrinos which, as was shown, still allows for leptonic mixing and Majorana type CP violation. The third equation coincides with eq. (2.13) which was derived for the first time in ref. [23] where it was shown that it is the necessary and sufficient condition for CP conservation in the case of two generations. Recall that for two generations only, Majorana-type CP violation can occur.

### 3 Invariants sensitive to neutrino mass ordering and the $\theta_{23}$ octant

In the previous section we summarised important information on weak basis invariants that was already known and that have proved to be extremely useful. In this section we discuss a set of new invariants sensitive to the neutrino mass ordering and to the octant in which the angle  $\theta_{23}$  lies. One important feature of the new invariants is the fact that their building blocks are analogous to the invariants found in ref. [20] for the quark sector.

#### 3.1 The neutrino mass ordering

One of the outstanding questions is the neutrino mass ordering. It is not yet known whether or not the mass of  $m_3$  is higher than the mass of  $m_1$  (and of  $m_2$ ). This refers to the neutrino mass ordering associated to the  $U_{\text{PMNS}}$  angles as given in table 1. The scale of neutrino masses is also not yet fixed. The highest hierarchy is obtained when the lightest neutrino mass is close to zero. However, it may happen that the three neutrino masses are almost degenerate. Almost degeneracy requires the mass scale to be higher than the square root of  $|\Delta m_{31}^2|$ .

Depending on the neutrino mass ordering, it is useful to consider different parametrisations for the neutrino masses. For normal ordering the following parametrisation is useful:

$$\begin{aligned} m_1^2 &= \Delta^2 x & \Delta^2 &= |\Delta_{31}^2| \\ m_2^2 &= \Delta^2 (x + \epsilon) ; & \epsilon &= \frac{\Delta_{21}^2}{|\Delta_{31}^2|} ; & 0 \leq x \\ m_3^2 &= \Delta^2 (x + 1) \end{aligned} \tag{3.1}$$

For inverted ordering, we use the following parametrisation for the neutrino masses:

$$\begin{aligned} m_1^2 &= \Delta^2 (x' + 1) \\ m_2^2 &= \Delta^2 (x' + 1 + \epsilon) ; & 0 \leq x' \\ m_3^2 &= \Delta^2 x' \end{aligned} \tag{3.2}$$

It can be easily shown that the sign of the following WB invariant:

$$\tilde{I}_1 \equiv \text{Tr}[H_\ell H_\nu] - \frac{1}{3}\text{Tr}[H_\ell]\text{Tr}[H_\nu] \quad (3.3)$$

indicates the ordering of the neutrino masses. Since  $\tilde{I}_1$  is a WB invariant it can be computed in any particular WB. It is instructive to compute  $\tilde{I}_1$  in the basis where the charged lepton mass matrix is diagonal. In this basis we have:

$$H_\ell = \text{diag} (m_e^2, m_\mu^2, m_\tau^2), \quad H_\nu = U_{\text{PMNS}} d_\nu^2 U_{\text{PMNS}}^\dagger \quad (3.4)$$

with  $d_\nu^2 = \text{diag} (m_1^2, m_2^2, m_3^2)$ . This allows us to express  $\tilde{I}_1$  in terms of physical observables:

$$\text{Tr}[H_\nu] = m_1^2 + m_2^2 + m_3^2, \quad \text{Tr}[H_\ell] = m_e^2 + m_\mu^2 + m_\tau^2 \quad (3.5)$$

$$\text{Tr}[H_\ell H_\nu] = m_e^2 m_k^2 |U_{1k}|^2 + m_\mu^2 m_k^2 |U_{2k}|^2 + m_\tau^2 m_k^2 |U_{3k}|^2 \quad (3.6)$$

where summation in  $k$  is implied and the  $U_{ij}$  stand for the entries of  $U_{\text{PMNS}}$ . Then, it is straightforward to calculate  $\tilde{I}_1$ . We find:

$$\begin{aligned} \tilde{I}_1 = & m_\tau^2 \left[ \Delta_{31}^2 \left( |U_{33}|^2 - \frac{1}{3} \right) + \Delta_{21}^2 \left( |U_{32}|^2 - \frac{1}{3} \right) \right] + \\ & + m_\mu^2 \left[ \Delta_{31}^2 \left( |U_{23}|^2 - \frac{1}{3} \right) + \Delta_{21}^2 \left( |U_{22}|^2 - \frac{1}{3} \right) \right] + \\ & + m_e^2 \left[ \Delta_{31}^2 \left( |U_{13}|^2 - \frac{1}{3} \right) + \Delta_{21}^2 \left( |U_{12}|^2 - \frac{1}{3} \right) \right] \end{aligned} \quad (3.7)$$

Taking into account the value of charged lepton masses:

$$m_e^2 = 2.6 \times 10^{-7} \text{GeV}^2, \quad m_\mu^2 = 1.1 \times 10^{-2} \text{GeV}^2, \quad m_\tau^2 = 3 \text{GeV}^2, \quad (3.8)$$

it is clear that we can safely neglect the terms in  $m_e^2$  and  $m_\mu^2$  in the determination of the sign of  $\tilde{I}_1$ . Furthermore, experimentally we know that  $|U_{33}|^2 > 1/3$ . Therefore, it follows that the sign of  $\tilde{I}_1$  gives the sign of  $\Delta_{31}^2$ . It is interesting to note that for exact tribimaximal mixing [26] which leads to leptonic mixings close to the experimental values:

$$U_{\text{TBM}} = \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \quad (3.9)$$

one has the following expression for  $\tilde{I}_1$ :

$$\tilde{I}_1 \equiv \text{Tr}[H_\ell H_\nu] - \frac{1}{3}\text{Tr}[H_\ell]\text{Tr}[H_\nu] \simeq \frac{1}{6} m_\tau^2 \Delta_{31}^2 \quad (3.10)$$

### 3.2 The octant of $\theta_{23}$

Despite great experimental progress in the determination of the neutrino oscillation parameters, two of these still remain poorly known — the atmospheric mixing angle  $\theta_{23}$  and



	$\tilde{I}_2 > 0$	$\tilde{I}_2 < 0$
$\tilde{I}_1 > 0$	NO, $\theta_{23} < \pi/4$	NO, $\theta_{23} > \pi/4$
$\tilde{I}_1 < 0$	IO, $\theta_{23} > \pi/4$	IO, $\theta_{23} < \pi/4$

**Table 2.** Combination of the two invariants. NO stands for normal ordering, IO for inverted ordering.

the CP violating Dirac type phase  $\delta$ . The forthcoming neutrino oscillation experiments are expected to significantly improve their measurements. Concerning the angle  $\theta_{23}$  there are two degenerate solutions known as the octant problem [27]. One, the lower octant solution corresponds to  $\theta_{23} < \pi/4$ , the other, the higher octant solution corresponds to  $\theta_{23} > \pi/4$ . The recent measurement of the angle  $\theta_{13}$  [1] and the fact that it is not too small gives grounds for optimism concerning the possibility of resolving the octant issue in forthcoming neutrino experiments [28, 29].

It is remarkable that there is a WB invariant which is sensitive to the  $\theta_{23}$  octant, namely:

$$\tilde{I}_2 \equiv \text{Tr}[H_\ell] \text{Tr}[H_\ell^2 H_\nu] - \text{Tr}[H_\ell^2] \text{Tr}[H_\ell H_\nu] \quad (3.11)$$

We find

$$\begin{aligned} \tilde{I}_2 = & \Delta_{31}^2 \left[ m_\tau^2 m_\mu^2 (m_\tau^2 - m_\mu^2) \left( |U_{33}|^2 - |U_{23}|^2 \right) + m_\tau^2 m_e^2 (m_\tau^2 - m_e^2) \left( |U_{33}|^2 - |U_{13}|^2 \right) + \right. \\ & \left. + m_\mu^2 m_e^2 (m_\mu^2 - m_e^2) \left( |U_{23}|^2 - |U_{13}|^2 \right) \right] + \\ & + \Delta_{21}^2 \left[ m_\tau^2 m_\mu^2 (m_\tau^2 - m_\mu^2) \left( |U_{32}|^2 - |U_{22}|^2 \right) + m_\tau^2 m_e^2 (m_\tau^2 - m_e^2) \left( |U_{32}|^2 - |U_{12}|^2 \right) \right. \\ & \left. + m_\mu^2 m_e^2 (m_\mu^2 - m_e^2) \left( |U_{22}|^2 - |U_{12}|^2 \right) \right] \quad (3.12) \end{aligned}$$

It is clear that the sign of  $\tilde{I}_2$  gives the sign of  $\left( |U_{33}|^2 - |U_{23}|^2 \right)$ , once the sign of  $\Delta_{31}^2$  is known from  $\tilde{I}_1$ . In table 2, we illustrate how the knowledge of the sign of  $\tilde{I}_1$  and  $\tilde{I}_2$  determines the neutrino mass ordering as well as the  $\theta_{23}$  octant.

#### 4 Application to specific Ansätze for Leptonic masses

Recently, various analysis of the prediction of neutrino mass textures have been presented in the literature. Typically a random scan [30, 31] is performed, with the input of the parameters of the Ansätze, leading to the determination of the various predictions of the Ansätze for a selected number of physical parameters. The invariants  $\tilde{I}_1$  and  $\tilde{I}_2$  are a complementary tool for these analysis, providing a simple determination of the favoured neutrino mass ordering and the octant of  $\theta_{23}$ .

For illustrative purposes, we use the two invariants  $\tilde{I}_i$  in the case of two specific Ansätze, studied in ref. [22], which predict a definite neutrino mass ordering.

In ref. [22] the authors considered neutrino mass matrices with the maximal allowed number of zero textures in the WB where the charged lepton mass matrix is already diagonal. They concluded that no more than two independent zero textures were viable. Furthermore, out of the fifteen different choices only seven could accommodate the known experimental constraints. Texture zeros in  $m_\nu$  lead to predictions. In both examples the neutrino mass ordering is fixed by the chosen texture and therefore, as we are going to show the sign of  $\tilde{I}_1$  is fixed.

In the case of texture  $A_1$  defined as [22]:

$$m_\nu = \begin{pmatrix} 0 & 0 & a \\ 0 & b & c \\ a & c & d \end{pmatrix} \quad (4.1)$$

the computation of the invariants  $\tilde{I}_1$  and  $\tilde{I}_2$  leads in leading order to:

$$\tilde{I}_1 \simeq \frac{1}{3} m_\tau^2 (|a|^2 + |c|^2 + 2|d|^2 - |b|^2) \simeq \frac{1}{6} m_\tau^2 \Delta_{31}^2 \quad (4.2)$$

$$\tilde{I}_2 \simeq m_\tau^2 m_\mu^2 (m_\tau^2 - m_\mu^2) (|a|^2 + |d|^2 - |b|^2) \simeq \Delta_{31}^2 m_\tau^2 m_\mu^2 (m_\tau^2 - m_\mu^2) (|U_{33}|^2 - |U_{23}|^2) \quad (4.3)$$

From eq. (4.3) we get to a good approximation that:

$$|b|^2 = |a|^2 + |d|^2 - (|U_{33}|^2 - |U_{23}|^2) \Delta_{31}^2 \quad (4.4)$$

Replacing  $|b|^2$  into eq. (4.2) we get:

$$|c|^2 + |d|^2 \simeq \left[ \frac{1}{2} + (|U_{33}|^2 - |U_{23}|^2) \right] \Delta_{31}^2 \quad (4.5)$$

The lefthand side is positive definite and we know experimentally (see table 1) that the term in brackets on the righthand side cannot be negative, so we conclude that in this case  $\Delta_{31}^2$  must be positive.

Another interesting example is case C, which corresponds to the following texture [22]:

$$m_\nu = \begin{pmatrix} a & c_1 & c_2 \\ c_1 & 0 & c_3 \\ c_2 & c_3 & 0 \end{pmatrix} \quad (4.6)$$

Computing the invariants  $\tilde{I}_1$  and  $\tilde{I}_2$  we obtain for the leading order terms:

$$\tilde{I}_1 \simeq \frac{1}{3} m_\tau^2 (-|a|^2 - 2|c_1|^2 + |c_2|^2 + |c_3|^2) \simeq \frac{1}{6} m_\tau^2 \Delta_{31}^2 \quad (4.7)$$

$$\tilde{I}_2 \simeq m_\tau^2 m_\mu^2 (m_\tau^2 - m_\mu^2) (|c_2|^2 - |c_1|^2) \simeq \Delta_{31}^2 m_\tau^2 m_\mu^2 (m_\tau^2 - m_\mu^2) (|U_{33}|^2 - |U_{23}|^2) \quad (4.8)$$

In this case it is not straightforward to apply the previous procedure since  $\tilde{I}_1$  and  $\tilde{I}_2$  cannot be simply related. However close to tribimaximal mixing it can be shown that:

$$|a|^2 \simeq |c_3|^2 \quad |c_1|^2 \simeq |c_2|^2 \quad (4.9)$$

so that:

$$m_1^2 \simeq |c_3|^2 + 2|c_1|^2 \quad m_2^2 \simeq |c_3|^2 + 2|c_1|^2 \quad m_3^2 \simeq |c_3|^2 \quad (4.10)$$

and

$$\tilde{I}_1 \simeq \frac{1}{3}m_\tau^2 \left(-|c_1|^2\right) < 0 \quad (4.11)$$

which indicates that this texture favours inverted order.

## 5 Conclusions

We have emphasised that WB invariants can play an important role in the study of lepton masses and mixing, including CP violation. The great advantage of these invariants stems from the fact that they can be directly evaluated in any conveniently chosen weak basis, without involving the diagonalisation of complex mass matrices. The invariants are physical quantities and can be expressed in terms neutrino masses, charged lepton masses, mixing angles and CP violating phases. We first review the four WB invariants which, together with the assumption of  $3 \times 3$  unitarity of  $U_{\text{PMNS}}$  matrix, can completely fix all the moduli of  $U_{\text{PMNS}}$ . From these moduli, one can evaluate the common area of all leptonic unitarity triangles. This area gives the strength of leptonic CP violation of Dirac type, but it does not fix the sign of CP violation. This sign can be fixed by a CP-odd leptonic WB invariant. We have also described the WB invariants which can probe CP violation of Majorana type, emphasising that this CP violation has to do with the orientation of the sides of Majorana-type unitarity triangles. For Majorana neutrinos, this orientation is physically meaningful and is associated to the arguments of bilinears of  $U_{\text{PMNS}}$  matrix elements. Finally, we have shown that one can construct additional WB invariants which can determine whether the neutrino mass ordering is normal or inverted and also determine the octant of  $\theta_{23}$ . These invariants are then used to study specific texture-zero Ansätze for the neutrino mass matrices. The invariants proposed in this paper are a powerful tool for model builders, having the potential for simplifying, or even replacing, the numerical study of the parameter study of a given model and the analysis of their phenomenological consequences. This is illustrated for two particular leptonic textures proposed in the literature, but the method can be applied to any model, since it is based on invariants which do not depend on the particular weak basis in which the model is formulated.

## Acknowledgments

This work was partially supported by Fundação para a Ciência e a Tecnologia (FCT, Portugal) through the projects CERN/FIS-NUC/0010/2015, CFTP-FCT Unit 777 (UID/FIS/00777/2013) which are partially funded through POCTI (FEDER), COMPETE, QREN and EU.

**Open Access.** This article is distributed under the terms of the Creative Commons Attribution License ([CC-BY 4.0](https://creativecommons.org/licenses/by/4.0/)), which permits any use, distribution and reproduction in any medium, provided the original author(s) and source are credited.

## References

- [1] DAYA BAY collaboration, F.P. An et al., *Observation of electron-antineutrino disappearance at Daya Bay*, *Phys. Rev. Lett.* **108** (2012) 171803 [[arXiv:1203.1669](#)] [[INSPIRE](#)].
- [2] NOVA collaboration, P. Adamson et al., *Measurement of the neutrino mixing angle  $\theta_{23}$  in NOvA*, *Phys. Rev. Lett.* **118** (2017) 151802 [[arXiv:1701.05891](#)] [[INSPIRE](#)].
- [3] PARTICLE DATA GROUP collaboration, C. Patrignani et al., *Review of Particle Physics*, *Chin. Phys. C* **40** (2016) 100001 [[INSPIRE](#)].
- [4] D.V. Forero, M. Tortola and J.W.F. Valle, *Neutrino oscillations refitted*, *Phys. Rev. D* **90** (2014) 093006 [[arXiv:1405.7540](#)] [[INSPIRE](#)].
- [5] F. Capozzi, G.L. Fogli, E. Lisi, A. Marrone, D. Montanino and A. Palazzo, *Status of three-neutrino oscillation parameters, circa 2013*, *Phys. Rev. D* **89** (2014) 093018 [[arXiv:1312.2878](#)] [[INSPIRE](#)].
- [6] M.C. Gonzalez-Garcia, M. Maltoni and T. Schwetz, *Global Analyses of Neutrino Oscillation Experiments*, *Nucl. Phys. B* **908** (2016) 199 [[arXiv:1512.06856](#)] [[INSPIRE](#)].
- [7] G.C. Branco and J.I. Silva-Marcos, *Invariants, Alignment and the Pattern of Fermion Masses and Mixing*, *Phys. Lett. B* **715** (2012) 315 [[arXiv:1112.1631](#)] [[INSPIRE](#)].
- [8] P. Minkowski,  *$\mu \rightarrow e\gamma$  at a Rate of One Out of  $10^9$  Muon Decays?*, *Phys. Lett. B* **67** (1977) 421 [[INSPIRE](#)].
- [9] T. Yanagida, *Horizontal symmetry and masses of neutrinos*, *Conf. Proc. C* **7902131** (1979) 95 [[INSPIRE](#)].
- [10] S.L. Glashow, *The Future of Elementary Particle Physics*, in *Quarks And Leptons. Proceedings, Summer Institute, Cargese, France, 9–29 July 1979*, M. Levy, J.L. Basdevant, D. Speiser, J. Weyers, R. Gastmans and M. Jacob eds., pp. 687–713 [*NATO Sci. Ser. B* **61** (1980) 1].
- [11] M. Gell-Mann, P. Ramond and R. Slansky, *Complex Spinors and Unified Theories*, *Conf. Proc. C* **790927** (1979) 315 [[arXiv:1306.4669](#)] [[INSPIRE](#)].
- [12] R.N. Mohapatra and G. Senjanović, *Neutrino Mass and Spontaneous Parity Violation*, *Phys. Rev. Lett.* **44** (1980) 912 [[INSPIRE](#)].
- [13] J.A. Aguilar-Saavedra and G.C. Branco, *Unitarity triangles and geometrical description of CP-violation with Majorana neutrinos*, *Phys. Rev. D* **62** (2000) 096009 [[hep-ph/0007025](#)] [[INSPIRE](#)].
- [14] G.C. Branco and M.N. Rebelo, *Building the full PMNS Matrix from six independent Majorana-type phases*, *Phys. Rev. D* **79** (2009) 013001 [[arXiv:0809.2799](#)] [[INSPIRE](#)].
- [15] A. Pilaftsis, *CP violation and baryogenesis due to heavy Majorana neutrinos*, *Phys. Rev. D* **56** (1997) 5431 [[hep-ph/9707235](#)] [[INSPIRE](#)].
- [16] G.C. Branco, T. Morozumi, B.M. Nobre and M.N. Rebelo, *A Bridge between CP-violation at low-energies and leptogenesis*, *Nucl. Phys. B* **617** (2001) 475 [[hep-ph/0107164](#)] [[INSPIRE](#)].
- [17] S. Davidson and R. Kitano, *Leptogenesis and a Jarlskog invariant*, *JHEP* **03** (2004) 020 [[hep-ph/0312007](#)] [[INSPIRE](#)].
- [18] G.C. Branco, M.N. Rebelo and J.I. Silva-Marcos, *Leptogenesis, Yukawa textures and weak basis invariants*, *Phys. Lett. B* **633** (2006) 345 [[hep-ph/0510412](#)] [[INSPIRE](#)].

- [19] Y. Wang and Z.-z. Xing, *Commutators of lepton mass matrices associated with the seesaw and leptogenesis mechanisms*, *Phys. Rev. D* **89** (2014) 097301 [[arXiv:1404.0109](#)] [[INSPIRE](#)].
- [20] G.C. Branco and L. Lavoura, *Rephasing Invariant Parametrization of the Quark Mixing Matrix*, *Phys. Lett. B* **208** (1988) 123 [[INSPIRE](#)].
- [21] J. Bernabeu, G.C. Branco and M. Gronau, *CP restrictions on quark mass matrices*, *Phys. Lett. B* **169** (1986) 243 [[INSPIRE](#)].
- [22] P.H. Frampton, S.L. Glashow and D. Marfatia, *Zeros of the neutrino mass matrix*, *Phys. Lett. B* **536** (2002) 79 [[hep-ph/0201008](#)] [[INSPIRE](#)].
- [23] G.C. Branco, L. Lavoura and M.N. Rebelo, *Majorana Neutrinos and CP Violation in the Leptonic Sector*, *Phys. Lett. B* **180** (1986) 264 [[INSPIRE](#)].
- [24] H.K. Dreiner, J.S. Kim, O. Lebedev and M. Thormeier, *Supersymmetric Jarlskog invariants: The Neutrino sector*, *Phys. Rev. D* **76** (2007) 015006 [[hep-ph/0703074](#)] [[INSPIRE](#)].
- [25] G.C. Branco, M.N. Rebelo and J.I. Silva-Marcos, *Degenerate and quasidegenerate Majorana neutrinos*, *Phys. Rev. Lett.* **82** (1999) 683 [[hep-ph/9810328](#)] [[INSPIRE](#)].
- [26] P.F. Harrison, D.H. Perkins and W.G. Scott, *Tri-bimaximal mixing and the neutrino oscillation data*, *Phys. Lett. B* **530** (2002) 167 [[hep-ph/0202074](#)] [[INSPIRE](#)].
- [27] G.L. Fogli and E. Lisi, *Tests of three flavor mixing in long baseline neutrino oscillation experiments*, *Phys. Rev. D* **54** (1996) 3667 [[hep-ph/9604415](#)] [[INSPIRE](#)].
- [28] H. Minakata, H. Sugiyama, O. Yasuda, K. Inoue and F. Suekane, *Reactor measurement of  $\theta_{13}$  and its complementarity to long baseline experiments*, *Phys. Rev. D* **68** (2003) 033017 [*Erratum ibid.* **D 70** (2004) 059901] [[hep-ph/0211111](#)] [[INSPIRE](#)].
- [29] S.S. Chatterjee, P. Pasquini and J.W.F. Valle, *Resolving the atmospheric octant by an improved measurement of the reactor angle*, *Phys. Rev. D* **96** (2017) 011303 [[arXiv:1703.03435](#)] [[INSPIRE](#)].
- [30] L.M. Cebola, D. Emmanuel-Costa and R.G. Felipe, *Confronting predictive texture zeros in lepton mass matrices with current data*, *Phys. Rev. D* **92** (2015) 025005 [[arXiv:1504.06594](#)] [[INSPIRE](#)].
- [31] L.M. Cebola, D. Emmanuel-Costa and R.G. Felipe, *Neutrino observables from predictive flavour patterns*, *Eur. Phys. J. C* **76** (2016) 156 [[arXiv:1601.06150](#)] [[INSPIRE](#)].