JHEP

Published for SISSA by Springer

Received: October 14, 2015
Accepted: October 22, 2015
Published: November 18, 2015

Charm in deep-inelastic scattering

Richard D. Ball,[a,b] **Marco Bonvini**[c] **and Luca Rottoli**[c]

[a] *The Higgs Centre for Theoretical Physics, University of Edinburgh, JCMB, KB, Mayfield Rd, Edinburgh EH9 3JZ, Scotland*

[b] *PH Department, TH Unit, CERN, CH-1211 Geneva 23, Switzerland*

[c] *Rudolf Peierls Centre for Theoretical Physics, University of Oxford, 1 Keble Road, OX1 3NP Oxford, U.K.*

E-mail: rdb@ph.ed.ac.uk, marco.bonvini@physics.ox.ac.uk, luca.rottoli@physics.ox.ac.uk

Abstract: We show how to extend systematically the FONLL scheme for inclusion of heavy quark mass effects in DIS to account for the possible effects of an intrinsic charm component in the nucleon. We show that when there is no intrinsic charm, FONLL is equivalent to S-ACOT to any order in perturbation theory, while when an intrinsic charm component is included FONLL is identical to ACOT, again to all orders in perturbation theory. We discuss in detail the inclusion of top and bottom quarks to construct a variable flavour number scheme, and give explicit expressions for the construction of the structure functions F_2^c, F_L^c and F_3^c to NNLO.

Keywords: Deep Inelastic Scattering, QCD, Parton Model

ArXiv ePrint: 1510.02491

doi:10.1007/JHEP11(2015)122

JHEP11(2015)122

Contents

1 Introduction

An accurate treatment of heavy quark mass effects is an essential ingredient of modern PDF fits [1–5]. Global PDF fits require the computation of physical cross sections over a range of perturbative scales Q^2 in order to incorporate a wide range of data from fixed target experiments up to LHC. As these scales pass through (or close to) the thresholds for charm, bottom and top, precision results require the incorporation of heavy quark mass effects close to threshold, $Q^2 \sim m^2$, and the resummation of collinear logarithms at scales far above the threshold, $Q^2 \gg m^2$, m being the mass of the heavy quark. This is achieved through

JHEP11(2015)122

the use of a so-called variable flavour number scheme (VFNS): calculations involving heavy quarks in DIS in different schemes with different numbers of active flavours participating to DGLAP evolution are combined to derive an expression for the coefficient functions which is valid both close to threshold, and far above it. A number of such VFNSs have been proposed for DIS structure functions, including ACOT [6, 7], S-ACOT [8, 9], TR and TR′ [10, 11], and FONLL [12–14].

A common feature of these various VFNSs is that they assume that the heavy quark PDF is generated entirely perturbatively above threshold. This assumption is reasonable enough for top and bottom, since both sit well within the accepted region of validity of perturbative QCD, and an entirely perturbative treatment is appropriate. By contrast, the distribution of gluons, and up, down and strange quarks in the proton is clearly nonperturbative, and can only be determined empirically through PDF fits.

The charm quark plays a special role, since the charm threshold sits at the borderline between perturbative and nonperturbative behaviour. While at high scales most charm is generated perturbatively through photo-gluon fusion (so that at HERA for example charm contributes up to 25% of the measured structure functions), closer to threshold it is difficult to rule out a priori a small nonperturbative component. Ideally one would like to admit the possibility of an initial charm PDF at threshold, which then evolves perturbatively to higher scales. The initial charm PDF could then be determined by fitting to data, just like the gluon and light quark PDFs. While over the years a variety of nonperturbative models of this 'intrinsic charm' have been proposed [15–17], and various attempts have been made at an empirical determination [18–21], so far no conclusive evidence has been found.

In this paper we will construct a VFNS which can incorporate intrinsic heavy quark PDFs, specifically intrinsic charm. We will take as given the existence to all orders in perturbation theory of the usual massless $\overline{\rm MS}$ factorization, and the complementary massive factorization proven in [22]. We then compare the ACOT and FONLL constructions, all the time taking into account the possibility of an intrinsic component of the charm PDF. We find in this way that for a specified renormalization and factorization scheme (namely $\overline{\rm MS}$), the FONLL [12, 13] and ACOT [6, 7] constructions give formally identical results, to all orders in perturbation theory. Moreover, in the limit of vanishing intrinsic charm, the original FONLL procedure [14] is precisely equivalent to the S-ACOT prescription [9, 23], again to all orders in perturbation theory: the only difference between them is in formally subleading terms implemented through a damping factor (FONLL) or a phenomenological χ-rescaling (S-ACOT-χ, [24]), which parametrize subleading ambiguities in the implementation of the condition of zero intrinsic charm. The TR prescription in its original formulation [10] was only specified at NLO, while at NNLO [11] it essentially reduces to S-ACOT [25]: as far as we are aware there is no formal extension to all orders, so this prescription will not be considered further here.

The basic formalism of the schemes used for fixed order and resummed results and their matching is developed in section 2. Then in section 3 we present the FONLL formalism, we show the formal equivalence of FONLL and ACOT, the simplifications evident in the limit of no intrinsic charm, and in particular show that when all charm is generated perturbatively, FONLL is equivalent to S-ACOT. The inclusion of top and bottom quarks

JHEP11(2015)122

is discussed in section 4. Conclusions are drawn in section 5. In the appendices we derive some technical results on matrix inversion, and write down explicit results for the structure functions F_2^c, F_L^c and F_3^c to NNLO.

2 Heavy quarks and factorization

The definition of light and heavy quarks is somewhat arbitrary: being 'light' or 'heavy' is a relative concept. In the context of initial state factorization, a convenient definition of 'light' quark is a quark whose mass $m \lesssim \Lambda_{\rm QCD}$, such that a perturbative treatment is not applicable. According to this definition, the up, down and strange quarks are light. Light quarks can be taken to be massless, because the factorization theorem is accurate up to $\mathcal{O}(\Lambda^2_{\rm QCD}/Q^2)$ corrections, and light quark mass corrections are higher twist effects $\mathcal{O}(m^2/Q^2)$. Consistently, it is natural to define as 'heavy' a quark whose mass $m \gg \Lambda_{\rm QCD}$, such that $\alpha_s(m^2)$ is in the perturbative regime. With this definition, the bottom and top quarks are heavy, and their description can be carried out using perturbation theory.

The charm mass m_c sits somewhere around the boundary of the region of validity of perturbative QCD: if we denote the initial scale of perturbative parton evolution by Q_0, such that for $Q > Q_0$ evolution is perturbative, while for $Q < Q_0$ nonperturbative behaviour sets in, then $Q_0 \sim m_c$. For this reason, the charm is special, since it is not heavy enough to fully trust perturbation theory, but not light enough that its mass can be ignored. Therefore, we cannot safely assume that at Q_0 the charm distributions $c(x, Q_0)$ and $\bar{c}(x, Q_0)$ are strictly zero, even if Q_0 is below the threshold for perturbative charm production, since nonzero distributions (commonly called 'intrinsic charm') may be generated by nonperturbative effects. To take this into account we need to treat charm in the same way that we treat the light partons $q = u, d, s$, $\bar{q} = \bar{u}, \bar{d}, \bar{s}$ and g, with an initial (fitted) PDF at Q_0, evolved up to scales $Q > Q_0$ using perturbation theory. However, unlike the other light partons, charm mass effects cannot be neglected for scales Q which are not much larger than m_c, as typically encountered in DIS experiments.

The computation of coefficient functions can be performed in different factorization and renormalization schemes, all leading to results for physical cross sections which must be equivalent to all orders in perturbation theory, and must therefore differ at finite order only by higher order corrections. For renormalization, the quark mass does not play an important role, since a UV divergent massless-quark loop would be still divergent even if the quark were massive. Renormalization can be performed in $\overline{\rm MS}$ for all quark families; however, all flavours would then participate to α_s evolution at any scale, resulting in unphysical heavy quark effects at scales much smaller than the heavy quark mass. It is therefore more appropriate to use $\overline{\rm MS}$ only for quarks with masses lighter than the renormalization scale $\mu_{\rm R}$, and use the CWZ scheme [26] for quarks heavier than $\mu_{\rm R}$, since this scheme, being based on a subtraction at zero external momenta, ensures decoupling of the heavy quark with mass m for scales $\mu_{\rm R} \ll m$. In this way, resummation of large logarithms of $\mu_{\rm R}/m$ with $\mu_{\rm R} \gg m$ is achieved, while the analogous logarithms when $m \gg \mu_{\rm R}$ are power suppressed as $\mu_{\rm R}^2/m^2$. Since the choice of which renormalization scheme to use with each quark depends on the relative size of the quark mass and the renormalization

JHEP11(2015)122

scale, which varies dynamically, a variable flavour number (renormalization) scheme is generated.

On the other hand, the quark mass acts as an IR regulator. This means that radiative corrections involving massive quarks are finite. Thus while for massless quarks factorization is mandatory, for massive quarks one may choose whether to factorize massive collinear logarithms or not. In principle, there is nothing wrong with using standard massless factorization for the light quarks, while keeping massive collinear logarithms in the coefficient functions: this is the so called 3 flavour scheme (3FS), discussed in section 2.1. However, the collinear logarithms (appearing as single logarithms, so there are k logarithms of Q^2/m^2 at order $\alpha_s^k(Q^2)$) can become large at high scales, spoiling the perturbative convergence of the 3FS result. In this case, it is more appropriate to factorize and resum the collinear logarithms associated in the first place to the charm, then the bottom and at very high scales the top as well, leading to 4, 5 and 6 flavour schemes respectively. The 4FS is discussed in section 2.2, with particular emphasis on the charm quark (since the bottom and the top are treated identically in the 3FS and 4FS). This will give us the opportunity to discuss the issues related to a possible intrinsic component of the charm PDF. The extension of this discussion to the bottom and top quark is straightforward, so the details are postponed to section 4.

2.1 The 3 flavour scheme

As already discussed, since the heavy quark mass regulates the IR behaviour, there is no need to factorize the (finite) collinear logarithms due to splittings involving the charm, bottom and top quarks. One can therefore use standard ($\overline{\rm MS}$) massless factorization for the gluon and light quarks only, and leave explicit collinear logarithms due to massive charm, bottom and top quarks unfactorized in the coefficient function. Together with the adoption of the decoupling scheme for UV renormalization of charm, bottom and top loops, this gives the so-called 3 flavour scheme (3FS) [8, 26]. In the context of heavy quark factorization, this is also often called the 'massive scheme', since the quark mass dependence of each massive quark, and in particular the charm, is exact. The coefficient functions will then contain unresummed (and at high scales potentially large) mass collinear logarithms.

In this scheme, only the 3 flavours of light quarks (plus the gluon) evolve with standard DGLAP equations, as a consequence of the massless $\overline{\rm MS}$ factorization acting only on those flavours. The contribution of the charm, bottom and top quarks (both in loops and trees) is evaluated at fixed order in perturbation theory, without subtraction and resummation of the related collinear logarithms. The 3FS is thus useful in the threshold region of the charm, where in particular the effects of the charm quark mass are treated explicitly, but breaks down at higher scales due to large unresummed logarithms of Q^2/m_c^2. Explicitly we then have, for the generic structure function,[1]

$$F^{(3)}(Q^2, m_c^2) = \sum_{i=g,q,\bar{q},c,\bar{c}} C_i^{(3)}\left(\tfrac{m_c^2}{Q^2}, \alpha_s^{(3)}(Q^2)\right) \otimes f_i^{(3)}(Q^2) \tag{2.1}$$

[1]Throughout this paper we will use consistently a superscript (n) to denote a coefficient function in a scheme with n active flavours.

JHEP11(2015)122

where $\otimes$ is the usual x-space convolution, and we have suppressed all explicit dependence on x. The coefficient functions $C_i^{(3)}$ include the effects of the charm mass order by order in perturbation theory, both in tree diagrams (for example a charm quark emerging from the proton and being struck by a virtual photon) and in loops (for example a photon-gluon fusion creating a charm-anticharm pair, or a virtual charm loop in a gluon propagator). This dependence includes thresholds: for example the contribution to the coefficient function from photon-gluon fusion includes a factor of $\theta(W^2-4m_c^2)$, to ensure that it vanishes below threshold. In writing eq. (2.1) we include from the start an explicit contribution from a charm PDF (i.e., the sum runs also over $i = c, \bar{c}$): if all charm were generated perturbatively we would set this contribution to zero, and the structure function would then depend only on the light PDFs $f_i^{(3)}$ with $i = g, q, \bar{q}$. We write explicitly only the dependence on the charm mass m_c, since this is our main focus here, but we note in passing that the structure function and the coefficient functions can also depend on the bottom and top quark masses as well through virtual loops and, when kinematically allowed, pair production.

The label (3) means that there are only 3 'active' quarks, by which we mean that they evolve as

$$f_i^{(3)}(Q^2) = \sum_{j=g,q,\bar{q}} \Gamma_{ij}^{(3)}\left(Q^2, Q_0^2\right) \otimes f_j^{(3)}(Q_0^2), \qquad i = g, q, \bar{q}, \tag{2.2}$$

where $\Gamma_{ij}^{(3)}\left(Q^2, Q_0^2\right)$ is the solution of the DGLAP equation with three active flavours. The charm is still present but not active, and in particular the Q^2 dependence of the charm contribution to the structure function is all in the coefficient function, so $f_{c,\bar{c}}^{(3)}$ are independent of Q^2 for all Q^2. Since in eq. (2.1) we have four flavours, even though only three are active, it is convenient to write

$$f_i^{(3)}(Q^2) = \sum_{j=g,q,\bar{q},c,\bar{c}} \bar{\Gamma}_{ij}^{(3)}\left(Q^2, Q_0^2\right) \otimes f_j^{(3)}(Q_0^2), \qquad i = g, q, \bar{q}, c, \bar{c}, \tag{2.3}$$

where

$$\bar{\Gamma}_{ij}^{(3)}(Q^2, Q_0^2) = \begin{cases} \Gamma_{ij}^{(3)}(Q^2, Q_0^2) & i,j = g, q, \bar{q} \\ \delta_{ij}, & i,j = c, \bar{c} \\ 0, & \text{otherwise.} \end{cases} \tag{2.4}$$

The massive coefficient functions $C_i^{(3)}(m_c^2/Q^2, \alpha_s^{(3)}(Q^2))$ are computed to a fixed order in perturbation theory, retaining the full mass dependence of the diagrams, including in particular the kinematic thresholds arising from the charm quarks in the final state. They are fully known to $\mathcal{O}(\alpha_s^2)$ [27, 28] for incoming light partons, but only to $\mathcal{O}(\alpha_s)$ for incoming heavy partons [29, 30]. If there is no initial state (intrinsic) charm, the structure function eq. (2.1) does not include contributions from $C_{c,\bar{c}}^{(3)}(m_c^2/Q^2, \alpha_s^{(3)}(Q^2))$. In this case, below the threshold for charm pair production the only charm mass effect is through virtual loops; the adoption of the CWZ renormalization scheme ensures that the charm quark decouples completely below threshold $W^2 < m_c^2$ and thus at low scales $Q^2 \ll m_c^2$, so that in this limit

$$F^{(3)}(Q^2, m_c^2) = \sum_{i=g,q,\bar{q}} C_i^{(3)}\left(\alpha_s^{(3)}(Q^2)\right) \otimes f_i^{(3)}(Q^2) + \mathcal{O}\left(\tfrac{Q^2}{m_c^2}\right), \tag{2.5}$$

JHEP11(2015)122

where the charm mass dependence has completely disappeared from the coefficient function. While the heavy limit is clearly not perturbative in the case of charm, it applies equivalently to bottom and top.

2.2 The 4 flavour scheme

In the 3FS the finite mass logarithms arising from the splittings of the charm quark appear at fixed order in the coefficient functions $C_i^{(3)}$. Explicitly, the massive coefficients have a decomposition

$$C_i^{(3)}\left(\tfrac{m_c^2}{Q^2}, \alpha_s^{(3)}(Q^2)\right) = \sum_{k=0}^{\infty}\left[\alpha_s^{(3)}(Q^2)\right]^k \sum_{j=0}^{k} A_{i,k,j}\left(\tfrac{m_c^2}{Q^2}\right)\log^j \tfrac{m_c^2}{Q^2}, \tag{2.6}$$

where the dependence on the logarithms has been made fully explicit, and the coefficients $A_{i,k,j}(m_c^2/Q^2)$ admit a power expansion on their argument. At large scales $Q^2 \gg m_c^2$, these logarithms become large, eventually spoiling the convergence of the fixed-order result eq. (2.1). In this regime resummation of the collinear logarithms is necessary for reliable predictions.

Therefore, at scales higher than the charm mass, it is advisable (and eventually necessary) to use a different factorization scheme where these logarithms are factorized into the definition of the PDFs, and resummed through PDF evolution, just as for the light partons. This would be mandatory if one considered the charm quark as a massless flavour, as appropriate in the high energy limit $Q^2 \gg m_c^2$: in this limit, all collinear divergences including those from charm quarks have to be subtracted. Using standard massless $\overline{\text{MS}}$ subtractions, the resulting evolution equation reads

$$f_i^{(4)}(Q^2) = \sum_{j=g,q,\bar{q},c,\bar{c}} \Gamma_{ij}^{(4)}\left(Q^2, Q_0^2\right) \otimes f_j^{(4)}(Q_0^2) \tag{2.7}$$

where $\Gamma_{ij}^{(4)}$ is the DGLAP evolution factor to a given order in perturbation theory for four active (massless) flavours, resumming all collinear logarithms of Q^2/Q_0^2, including those generated by charm splittings. Collinear logarithms due to bottom and top quarks, however, are not resummed in this scheme, and will therefore continue to appear at fixed order in the coefficient functions.

We now focus on the computation of the coefficient functions. In the high energy limit where all four active flavours are considered massless, we can obtain the structure functions using standard massless $\overline{\text{MS}}$ collinear counterterms also for the charm quark, up to corrections suppressed by powers of m_c^2/Q^2. We thus get

$$\begin{aligned} F^{(4)}(Q^2, m_c^2) &= F^{(4)}(Q^2, 0) + \mathcal{O}\left(\tfrac{m_c^2}{Q^2}\right) \\ F^{(4)}(Q^2, 0) &= \sum_{i=g,q,\bar{q},c,\bar{c}} C_i^{(4)}\left(0, \alpha_s^{(4)}(Q^2)\right) \otimes f_i^{(4)}(Q^2), \end{aligned} \tag{2.8}$$

where $C_i^{(4)}(0, \alpha_s^{(4)}(Q^2))$ are the usual massless scheme coefficient functions, analogous to the $C_i^{(3)}$ of eq. (2.1) but with an additional massless quark, evaluated to the given order in

JHEP11(2015)122

perturbation theory in the four flavour running coupling $\alpha_s^{(4)}(Q^2)$. Here, the first argument has been set to zero to remind us that the charm mass has been neglected (while the bottom and top masses are finite). These massless coefficient functions have been computed to $\mathcal{O}(\alpha_s^3)$ [31]. Note that, while $C_i^{(3)}(m_c^2/Q^2, \alpha_s^{(3)})$ is logarithmically divergent when $m_c^2 \to 0$, $C_i^{(4)}(0, \alpha_s^{(4)})$ is finite, due to the subtraction of the collinear divergences.

While eq. (2.8) is acceptable at high scales where the corrections of $\mathcal{O}(m_c^2/Q^2)$ are negligible, it is not legitimate for lower scales closer to the charm mass. In order to make it valid at all scales, the neglected power corrections must be reinstated, at least at fixed order (and this is sufficient, because at high scales they vanish faster than the growth of the logarithms). This is the approach adopted in the FONLL prescription, and also in the recently proposed derivation of ref. [32]. Once this is done, and the missing power corrections are expressed in terms of the same 4 flavour PDFs evolving as in eq. (2.7), we must have a factorized result of the form

$$F^{(4)}(Q^2, m_c^2) = \sum_{i=g,q,\bar{q},c,\bar{c}} C_i^{(4)}\left(\tfrac{m_c^2}{Q^2}, \alpha_s^{(4)}(Q^2)\right) \otimes f_i^{(4)}(Q^2), \tag{2.9}$$

where $C_i^{(4)}(m_c^2/Q^2, \alpha_s^{(4)})$ are coefficient functions which include the effects of the charm mass. As we shall see later in section 3, the exact form of the mass-dependent part of these coefficient functions is not uniquely fixed in the case of perturbatively generated charm: this has led to the construction of several different (though equally valid) formulations in the literature, such as ACOT [6, 7], S-ACOT [8, 9], TR and TR′ [10, 11], FONLL [12–14], and the recent formulation of ref. [32].

A particular form of the coefficient functions, which does not depend on any assumptions about intrinsic charm, is the one obtained in the ACOT scheme [7], which uses a special factorization scheme the existence of which has been proved to all orders in perturbation theory by Collins [8]. These coefficient functions are obtained by using massless collinear counterterms for the light partons, and massive collinear counterterms (using the quark mass as an infrared regulator) for the charm quark, and then applying the usual subtraction procedure while keeping charm mass dependence everywhere. The resulting anomalous dimensions correspond to the DGLAP anomalous dimensions, and lead therefore to the same evolution eq. (2.7). Hence, the Collins (ACOT) result can be interpreted as a massive extension of the massless $\overline{\text{MS}}$ factorization scheme. We will regard at eq. (2.9) as the result obtained in this scheme.

Note that, since in both eq. (2.8) and eq. (2.9) all collinear singularities are factorised into the PDFs eq. (2.7), then if $Q^2 \gg m_c^2$,

$$C_i^{(4)}\left(\tfrac{m_c^2}{Q^2}, \alpha_s^{(4)}(Q^2)\right) = C_i^{(4)}\left(0, \alpha_s^{(4)}(Q^2)\right) + \mathcal{O}\left(\tfrac{m_c^2}{Q^2}\right). \tag{2.10}$$

It can be observed that, since the coefficient functions $C_i^{(4)}(m_c^2/Q^2, \alpha_s^{(4)}(Q^2))$ contain the correct mass dependence and do not contain mass logarithms, the result eq. (2.9) performs the collinear resummation of charm massive logarithms as the massless result eq. (2.8), but it additionally includes the exact charm mass dependence as the massive result eq. (2.1). In this respect, this result already provides a satisfactory treatment of heavy quarks in the

JHEP11(2015)122

initial state. Practically, however, it has never been used beyond NLO, due to complications in the computation of the massive coefficient functions $C_i^{(4)}(m_c^2/Q^2, \alpha_s^{(4)}(Q^2))$.

2.3 Matching

The two results eqs. (2.1) and (2.9) are alternative expressions for the same structure function, written in terms of different ingredients, specifically α_s and the PDFs. It is the purpose of this section to relate these ingredients in the two schemes.

This proceeds in two stages: first we must match the two renormalization and factorization schemes at some matching scale $\mu_c^2 \sim m_c^2$, and then we evolve to Q^2. For the running coupling this gives the relation

$$\alpha_s^{(3)}(\mu_c^2) = \alpha_s^{(4)}(\mu_c^2) + \sum_{p=2}^{\infty} a_p \left(\alpha_s^{(4)}(\mu_c^2)\right)^p, \tag{2.11}$$

with coefficients a_p that are readily computed order by order, and are known up to four loops [33]. The relation between $\alpha_s^{(3)}$ and $\alpha_s^{(4)}$ at the generic scale Q^2 can be obtained using renormalization group evolution from μ_c^2 to Q^2. Given these coefficients, we can choose to expand any perturbative quantity either in powers of $\alpha_s^{(3)}$ or in terms of $\alpha_s^{(4)}$, by using the relation eq. (2.11) or its inverse. Clearly when comparing coefficients in perturbative expansions, it is necessary to expand all quantities consistently. In what follows we will leave all α_s dependence implicit, reinstating it only when we perform explicit perturbative expansions in appendix B.

For the factorization the matching condition is likewise

$$f_i^{(4)}(\mu_c^2) = \sum_{j=g,q,\bar{q},c,\bar{c}} K_{ij}\left(\frac{m_c^2}{\mu_c^2}\right) \otimes f_j^{(3)}(\mu_c^2), \tag{2.12}$$

$$K_{ij}\left(\frac{m_c^2}{\mu_c^2}\right) = \delta_{ij} + \sum_{p=1}^{\infty} \left(\alpha_s^{(4)}(\mu_c^2)\right)^p K_{ij}^p\left(\frac{m_c^2}{\mu_c^2}\right), \tag{2.13}$$

where the coefficients $K_{ij}^p(m_c^2/\mu_c^2)$ are determined perturbatively, by requiring that the 3FS result eq. (2.1) and the 4FS result eq. (2.9) are equal order by order in (the same) α_s. The computation can be simplified by taking the massless limit: inserting eq. (2.12) in eq. (2.8), we recover eq. (2.1) up to power suppressed contributions provided

$$\sum_{i=g,q,\bar{q},c,\bar{c}} C_i^{(4)}(0) \otimes K_{ij}\left(\frac{m_c^2}{Q^2}\right) = C_j^{(3,0)}\left(\frac{m_c^2}{Q^2}\right) \tag{2.14}$$

where in the right hand side $C_j^{(3,0)}$ is just $C_j^{(3)}$, but with all power suppressed contributions be set to zero, keeping only mass independent terms and the mass logarithms. Using the explicit form eq. (2.6), we can write exactly

$$\sum_{i=g,q,\bar{q},c,\bar{c}} C_i^{(4)}(0) \otimes K_{ij}\left(\frac{m_c^2}{Q^2}\right) = \sum_{k=0}^{\infty} \left(\alpha_s^{(4)}(Q^2)\right)^k \sum_{l=0}^{k} A_{j,k,l}(0) \log^l \frac{m_c^2}{Q^2} \tag{2.15}$$

JHEP11(2015)122

where the power suppressed contributions have been removed by computing the coefficients $A_{j,k,l}$ for $m_c = 0$. Thus, the matching coefficents $K_{ij}(m_c^2/\mu_c^2)$ depend on its argument through the logarithms $\log(m_c^2/\mu_c^2)$ which are present and unresummed in the 3FS coefficients $C_i^{(3)}(m_c^2/Q^2)$. Inverting eq. (2.14) we can write

$$\lim_{m_c^2 \to 0} \sum_{j=g,q,\bar{q},c,\bar{c}} C_j^{(3)}\left(\tfrac{m_c^2}{Q^2}\right) \otimes K_{ji}^{-1}\left(\tfrac{m_c^2}{Q^2}\right) = C_i^{(4)}(0). \tag{2.16}$$

which shows that $K_{ij}^{-1}(m_c^2/Q^2)$ factor out the potentially large logarithms from the massive 3FS coefficient functions in order to ensure the correct massless limit, where all collinear logarithms have been cancelled.

In practice the matching coefficients $K_{ij}(m_c^2/Q^2)$ are computed by comparing calculations of deep inelastic coefficients functions in the 3FS and 4FS to a given order in perturbation theory, and using eq. (2.14) or equivalently eq. (2.16).[2] The components of K_{ij} with any i (light or heavy) and $j = g, q, \bar{q}$ are fully known to $\mathcal{O}(\alpha_s^2)$ [12, 34], and some of them also to $\mathcal{O}(\alpha_s^3)$ [35–38]. On the other hand, the components K_{ic} and $K_{i\bar{c}}$ for any value of i are only known to $\mathcal{O}(\alpha_s)$ [39]. The off-diagonal components with a gluon and a heavy quark, namely K_{cg}, $K_{\bar{c}g}$, K_{gc} and $K_{g\bar{c}}$, start contributing at $\mathcal{O}(\alpha_s)$, while all other off-diagonal components are nonzero only at $\mathcal{O}(\alpha_s^2)$. The diagonal quantities are all of the form $K_{ii} = 1 + \mathcal{O}(\alpha_s)$: while K_{gg} gets a contribution at $\mathcal{O}(\alpha_s)$ due to a heavy quark loop, and $K_{cc} = K_{\bar{c}\bar{c}}$ are nontrivial at $\mathcal{O}(\alpha_s)$, the light quark components get corrections only at $\mathcal{O}(\alpha_s^2)$.

It is important to realise that the matching condition eq. (2.12) only holds (to any fixed order) at the particular matching scale $\mu_c \sim m_c$, otherwise there would be unresummed large logarithms. The 4FS PDFs $f_j^{(4)}(Q^2)$ at the generic scale $Q^2 > \mu_c^2$ are then obtained by evolving up with DGLAP evolution, eq. (2.7),

$$f_i^{(4)}(Q^2) = \sum_{j,k=g,q,\bar{q},c,\bar{c}} \Gamma_{ij}(Q^2, \mu_c^2) \otimes K_{jk}\left(\tfrac{m_c^2}{\mu_c^2}\right) \otimes f_k^{(3)}(\mu_c^2), \tag{2.17}$$

as standard in common VFNS PDF sets. Note that, to any fixed order, the 4FS PDFs implicitly depend on the charm mass m_c and the charm threshold μ_c, though the last dependence is formally higher order. To transform $f_j^{(3)}(Q^2)$ to $f_j^{(4)}(Q^2)$ we must also evolve to Q^2 the 3FS PDFs using eq. (2.3). We thus find that

$$f_i^{(4)}(Q^2) = \sum_{j=g,q,\bar{q},c,\bar{c}} T_{ij}(Q^2, \mu_c^2, m_c^2) \otimes f_j^{(3)}(Q^2), \tag{2.18}$$

where we introduced the transformation matrix

$$T_{ij}(Q^2, \mu_c^2, m_c^2) = \sum_{k,l=g,q,\bar{q},c,\bar{c}} \Gamma_{ik}^{(4)}(Q^2, \mu_c^2) \otimes K_{kl}\left(\tfrac{m_c^2}{\mu_c^2}\right) \otimes \bar{\Gamma}_{lj}^{(3)}(\mu_c^2, Q^2), \tag{2.19}$$

and we have used the fact that the evolution matrices can be inverted by evolving backwards: $\Gamma_{ij}(\mu_c^2, Q^2)$ is the inverse of $\Gamma_{ij}(Q^2, \mu_c^2)$. Note that while $K_{ij}(m_c^2/\mu_c^2)$ contains no

[2] Alternatively, these coefficients can be computed as a matching between two effective theories of QCD, as described in ref. [32].

JHEP11(2015)122

large logarithms (since $\mu_c^2 \sim m_c^2$), the large logarithms of Q^2/μ_c^2 resummed in the evolution factors are mismatched, so $T_{ij}(Q^2, \mu_c^2, m_c^2)$ also resums large logarithms. If the evolution factors are expanded to any given fixed order in α_s, the μ_c dependence of T_{ij} disappears and $T_{ij}(Q^2, \mu_c^2, m_c^2) = K_{ij}(m_c^2/Q^2)$.

Having established the relation between the PDFs in the two schemes, eq. (2.18), it is now interesting to use it to write the massive 3FS result eq. (2.1) in terms of the 4FS PDFs $f_i^{(4)}(Q^2)$ (and also in terms of $\alpha_s^{(4)}(Q^2)$ through eq. (2.11), though we leave the dependence implicit). This will be needed for the FONLL construction described in the next section. Substituting the inverse of the transformation eq. (2.18) into eq. (2.1), we get

$$F^{(3)}(Q^2, m_c^2) = \sum_{i,j=g,q,\bar{q},c,\bar{c}} C_i^{(3)}\left(\tfrac{m_c^2}{Q^2}\right) \otimes T_{ij}^{-1}(Q^2, \mu_c^2, m_c^2) \otimes f_j^{(4)}(Q^2), \tag{2.20}$$

where

$$T_{ij}^{-1}(Q^2, \mu_c^2, m_c^2) = \sum_{k,l=g,q,\bar{q},c,\bar{c}} \bar{\Gamma}_{ik}^{(3)}(Q^2, \mu_c^2) \otimes K_{kl}^{-1}\left(\tfrac{m_c^2}{\mu_c^2}\right) \otimes \Gamma_{lj}^{(4)}(\mu_c^2, Q^2), \tag{2.21}$$

and K_{ij}^{-1} is obtained from eq. (2.13) by inverting term by term. The large logarithms of m_c^2/Q^2 in $C_i^{(3)}$ must then cancel term by term with corresponding large logarithms in T_{ij}^{-1}, resulting from the mismatch of the two evolution factors $\Gamma_{ij}^{(4)}$ and $\bar{\Gamma}_{ij}^{(3)}$: in other words T_{ij}^{-1} provides the correct subtraction terms for $C_i^{(3)}$. Moreover T_{ij}^{-1} also gives automatically the correct finite parts of the subtraction.

Comparing eq. (2.20) with eq. (2.9), and noting that to a given order in resummed perturbation theory the structure function (being physical) must be independent of the scheme, we see immediately that

$$C_i^{(4)}\left(\tfrac{m_c^2}{Q^2}\right) = \sum_{j=g,q,\bar{q},c,\bar{c}} C_j^{(3)}\left(\tfrac{m_c^2}{Q^2}\right) \otimes T_{ji}^{-1}(Q^2, \mu_c^2, m_c^2). \tag{2.22}$$

Note that once the coefficient function is expanded out to fixed order in α_s, there is nothing to prevent us from setting $\mu_c^2 = Q^2$ in eq. (2.22): this simplifies the expressions by setting both evolution factors to unity, so that

$$C_i^{(4)}\left(\tfrac{m_c^2}{Q^2}\right) = \sum_{j=g,q,\bar{q},c,\bar{c}} C_j^{(3)}\left(\tfrac{m_c^2}{Q^2}\right) \otimes K_{ji}^{-1}\left(\tfrac{m_c^2}{Q^2}\right), \tag{2.23}$$

the large logarithms of Q^2/m_c^2 now being those in K_{ij}^{-1}. When truncated to any given fixed order in perturbation theory, eq. (2.22) and eq. (2.23) will yield identical results for the coefficient functions $C_i^{(4)}$, independent of the matching scale μ_c.

It is interesting to observe that the massive 4FS result eq. (2.9), introduced originally as the result of a collinear factorization with massive quarks, has now been derived from the massive 3FS result eq. (2.1) after scheme change, eq. (2.23), which removes all its collinear logarithms. The matching condition eq. (2.16) then ensures that in the massless limit

$$\lim_{m_c \to 0} C_i^{(4)}\left(\tfrac{m_c^2}{Q^2}\right) = C_i^{(4)}(0), \tag{2.24}$$

JHEP11(2015)122

as expected from eq. (2.10). For this to work properly it is essential that the coefficient function $C_i^{(3)}$ and matching matrix K_{ij} are always evaluated to the same fixed order: if the coefficient function has terms of higher order than the matching matrix there will be uncancelled logarithms, while if the matching matrix has terms of higher order than the coefficient function it will be trying to cancel logarithms which are not there. This observation is of great importance if one wishes to combine the results obtained in different schemes, as done in the FONLL prescription.

We have thus shown that starting from the massive 3FS result eq. (2.1), and re-expressing it in terms of 4FS PDFs, we obtain a result equivalent to the massive 4FS result eq. (2.9): in other words we can use the resummation of the massless collinear logarithms performed in the massless 4FS by the evolution eq. (2.7) to resum the large logarithms in the massive 3FS. The result is at the heart of the ACOT scheme [6, 7]: formally order by order in perturbation theory

$$\begin{aligned} F_{\mathrm{ACOT}}(Q^2, m_c^2) &= \sum_{i=g,q,\bar{q},c,\bar{c}} C_i^{(4)}\left(\tfrac{m_c^2}{Q^2}\right) \otimes f_i^{(4)}(Q^2), \\ &= \sum_{i,j=g,q,\bar{q},c,\bar{c}} C_i^{(3)}\left(\tfrac{m_c^2}{Q^2}\right) \otimes K_{ij}^{-1}\left(\tfrac{m_c^2}{Q^2}\right) \otimes f_j^{(4)}(Q^2). \end{aligned} \tag{2.25}$$

The form of eq. (2.25) is interesting: it combines the PDFs $f_i^{(4)}$ evolved in the 4FS with the coefficient function $C_i^{(3)}$ computed in the massive 3FS, the matching conditions K_{ij} linking the two schemes subtracting the unresummed collinear logarithms from the massive coefficient functions so that the coefficients convoluted with the PDFs are free from collinear logarithms and thus have a well behaved perturbative expansion.

3 Combining fixed order and resummation

In the previous section we showed that a consistent scheme change relates a 3FS calculation, which does not factorizes the collinear logarithms due to the charm quark, to a 4FS calculation, in which the collinear logarithms are resummed and the massive effects are included into the 4FS coefficient function. An alternative way of combining massive coefficient functions in the 3FS with massless coefficient functions in the 4FS is the FONLL construction [12, 13].

In most applications so far FONLL has been used with the assumption that charm is generated entirely perturbatively (so there is no 'intrinsic' charm): the structure function in the 3FS can then be expressed entirely in terms of light partons. We thus require a precise all-order definition of what we mean by 'zero intrinsic charm' before we can obtain definite all-order results.

Here we will summarize the main features of the FONLL construction in section 3.1, and explain its relation to ACOT in section 3.2. We then discuss the definition of intrinsic charm, and the transition from 3FS PDFs to 4FS PDFs in section 3.3, and present FONLL results for structure functions without intrinsic charm in section 3.4 (corresponding to the NNLO results in [14], now formally generalized to all orders), and the corrections necessary when intrinsic charm is included in section 3.5. We then go on to show the relation of the

JHEP11(2015)122

FONLL scheme without intrinsic charm and the S-ACOT schemes in section 3.6. We finally discuss in section 3.7 a phenomenological damping factor included in the original FONLL formulation.

3.1 The FONLL construction

The general FONLL construction [12, 13] is based on the observation that to evaluate cross-sections consistently both in the threshold and the high energy region, it is sufficient to any given order in perturbation theory to add the massive 3FS result (which includes all charm mass effects to fixed order) to the massless 4FS result (which performs the resummation of all large logarithms at high energy), and then subtract any doubly counted contributions.

The FONLL construction only involves physical quantities computed in well-defined (massive 3F or massless 4F) factorization schemes, and thus side steps issues related to the existence of more novel factorization schemes. In particular this means that in FONLL it is straightforward to write down expressions at any order in perturbation theory: all one has to do is evaluate the relevant massive diagrams in the massive scheme, and combine them linearly with the corresponding massless calculations. The only nontrivial part is then to identify the double counting.

Structure functions calculated with four flavours in the FONLL method are thus given by

$$F_{\rm FONLL}\left(Q^2,m_c^2\right) = F^{(4)}\left(Q^2,0\right) + F^{(3)}\left(Q^2,m_c^2\right) - \text{d.c.}\,. \tag{3.1}$$

The double counting term can be obtained as the massless limit of the massive 3FS result, and corresponds to the fixed-order expansion of the massless 4FS result. The massless limit of the massive coefficient functions is however divergent, due to the presence of unsubtracted massive collinear logarithms. A proper definition of this term is given by

$$F^{(3,0)}(Q^2,m_c^2) = \sum_{i=g,q,\bar{q},c,\bar{c}} C_i^{(3,0)}\left(\tfrac{m_c^2}{Q^2}\right) \otimes f_i^{(3)}(Q^2), \tag{3.2}$$

where we used eq. (2.14) to define the (singular) massless limit $C_i^{(3,0)}(m_c^2/Q^2)$ of the massive coefficient functions $C_i^{(3)}(m_c^2/Q^2)$. In this limit all terms which vanish as $m_c^2 \to 0$ are removed, and all that remains are the finite terms and collinear logarithms, as explicitly shown in eq. (2.15).

The structure functions in the FONLL prescription are thus given by

$$F_{\rm FONLL}\left(Q^2,m_c^2\right) = F^{(4)}\left(Q^2,0\right) + \left[F^{(3)}\left(Q^2,m_c^2\right) - F^{(3,0)}\left(Q^2,m_c^2\right)\right]. \tag{3.3}$$

Clearly eq. (3.3) does what we want it to: in particular when $Q^2 \gg m_c^2$ the terms in square brackets vanish as a power of m_c^2/Q^2, and we recover the massless coefficient function in the 4FS. Likewise, when $Q^2 \sim m_c^2$, we can write

$$\begin{aligned} F_{\rm FONLL}\left(Q^2,m_c^2\right) &= F^{(3)}\left(Q^2,m_c^2\right) + \left[F^{(4)}\left(Q^2,0\right) - F^{(3,0)}\left(Q^2,m_c^2\right)\right] \\ &\equiv F^{(3)}\left(Q^2,m_c^2\right) + F^{(d)}\left(Q^2,m_c^2\right). \end{aligned} \tag{3.4}$$

JHEP11(2015)122

where in ref. [14] the term in square brackets is referred to as the 'difference term' $F^{(d)}$. While this term is nonzero for $Q^2 \sim m_c^2$, it is subleading in $\alpha_s(Q^2)$, since when $Q^2 \sim m_c^2$ there are no large logarithms. Thus eq. (3.3) gives a structure function which is correct at high energy, up to power suppressed corrections, and correct in the threshold region up to subleading corrections.

3.2 Comparison to ACOT

One way of using the FONLL construction would be simply to compute the three ingredients $F^{(3)}$, $F^{(3,0)}$ and $F^{(4)}$, using the factorized expression in the 3-flavour and 4-flavour schemes, eqs. (2.1), (3.2), (2.8), and combine them linearly according to eq. (3.3). In practice this is awkward, because it means one has to work simultaneously with PDFs in two different schemes. Thus instead it is more convenient to use the matching of the two schemes, eq. (2.12), to write eq. (3.3) in terms of PDFs in the 4FS and thus in the form eq. (2.9). In this way we can identify explicit expressions for the mass dependent coefficient functions $C_i^{(4)}(m_c^2/Q^2)$.

We showed in section 2.3 that using the scheme change we can write $F^{(3)}$ in the form

$$F^{(3)}(Q^2, m_c^2) = \sum_{i,j=g,q,\bar{q},c,\bar{c}} C_i^{(3)}\left(\tfrac{m_c^2}{Q^2}\right) \otimes K_{ij}^{-1}\left(\tfrac{m_c^2}{Q^2}\right) \otimes f_j^{(4)}(Q^2). \tag{3.5}$$

Consistently, in the massless limit, we have

$$F^{(3,0)}(Q^2, m_c^2) = \sum_{i,j=g,q,\bar{q},c,\bar{c}} C_i^{(3,0)}\left(\tfrac{m_c^2}{Q^2}\right) \otimes K_{ij}^{-1}\left(\tfrac{m_c^2}{Q^2}\right) \otimes f_j^{(4)}(Q^2). \tag{3.6}$$

Substituting into eq. (3.3), we have

$$\begin{aligned} F_{\text{FONLL}}\left(Q^2, m_c^2\right) = & \sum_{i,j=g,q,\bar{q},c,\bar{c}} \left[C_i^{(3)}\left(\tfrac{m_c^2}{Q^2}\right) - C_i^{(3,0)}\left(\tfrac{m_c^2}{Q^2}\right)\right] \otimes K_{ij}^{-1}\left(\tfrac{m_c^2}{Q^2}\right) \otimes f_j^{(4)}(Q^2) \\ & + \sum_{i=g,q,\bar{q},c,\bar{c}} C_i^{(4)}(0) \otimes f_i^{(4)}(Q^2). \end{aligned} \tag{3.7}$$

Now however we find an interesting simplification: the difference term vanishes identically, since due to the matching condition eq. (2.12) the terms with coefficient functions in the 4FS precisely cancel those from the massless limit of the 3FS, eq. (2.14). Eq. (3.7) can thus be written simply as

$$\begin{aligned} F_{\text{FONLL}}(Q^2, m_c^2) &= \sum_{i,j=g,q,\bar{q},c,\bar{c}} C_i^{(3)}\left(\tfrac{m_c^2}{Q^2}\right) \otimes K_{ij}^{-1}\left(\tfrac{m_c^2}{Q^2}\right) \otimes f_j^{(4)}(Q^2) \\ &= \sum_{i=g,q,\bar{q},c,\bar{c}} C_i^{(4)}\left(\tfrac{m_c^2}{Q^2}\right) \otimes f_i^{(4)}(Q^2) \\ &= F_{\text{ACOT}}\left(Q^2, m_c^2\right), \end{aligned} \tag{3.8}$$

with no massless coefficient functions at all. This shows that when we make no theoretical assumption about the 4FS PDFs at the initial scale, the FONLL construction of the structure function is equivalent to ACOT order by order in perturbation theory. Indeed,

JHEP11(2015)122

the above manipulations can be viewed as an alternative all-order derivation of the ACOT result using the FONLL construction. The only essential ingredients are the existence of the 3-flavour and 4-flavour factorization schemes, and the matching relations which allow them to be related together.

In retrospect this result should not have been too surprising. When $Q^2 \gg m_c^2$, the massless 4FS PDFs $f_i^{(4)}(Q^2)$ $i, j = g, q, \bar{q}, c, \bar{c}$ resum all large logarithms through solution of the evolution equations eq. (2.7). It follows, as explained after eq. (2.23), that the coefficient functions convoluted with these PDFs, which contain all dependence on m_c^2, must be free of large logarithms, and thus have a well behaved perturbative expansion in $\alpha_s^{(4)}(Q^2)$. Since the structure function is a physical quantity, the coefficients in the perturbative expansion of the coefficient function must (for a given factorization and renomalization scheme) be then unique to each order, which is indeed what we find.

It is important to note however that in deriving this result we made no assumption about the origin of charm, and in particular we did not assume that charm is generated purely perturbatively. To make contact with other work on FONLL, in particular ref. [14], where this assumption is an integral part of the construction, we first need to define carefully what we mean when we assume that all charm is generated perturbatively, i.e. when there is no 'intrinsic' charm. This will tell us how to modify the expressions given in ref. [14] to incorporate intrinsic charm, and in turn help us to understand better the role of intrinsic charm in the formulation of ACOT.

3.3 Intrinsic charm

In the previous sections we made no attempt to distinguish between extrinsic (perturbative) and intrinsic charm: we have been agnostic about the nature of the initial charm distributions at Q_0, in either 3- or 4-flavour schemes, which can be fitted at that scale and then perturbatively evolved to the scale Q. However in the more conventional formalisms there are no fitted charm PDFs: zero 'intrinsic' charm is an implicit part of the construction. The definition of intrinsic charm is in truth rather ambiguous, and any condition of zero intrinsic charm must reflect this ambiguity: conditions can be made in different schemes and at different scales, and will in general all differ by subleading terms. Thus the only formalism devoid of such ambiguities is the complete 4-flavour formalism adopted above, where the charm PDF takes part to DGLAP evolution and it is fitted together with the light flavour PDFs. Nevertheless it is interesting to consider the case of zero intrinsic charm, appropriately defined, in order to make contact with previous work, in particular ref. [14].

We first note that due to factorization all the information about the nature of the target hadron, and in particular whether or not it contains intrinsic charm, is contained in its PDFs. Since in a particular renormalization and factorization scheme the PDF evolution is also target independent, the only place where intrinsic charm can enter is in the boundary conditions for the perturbative evolution of the PDFs.

This necessarily implies that once the massive coefficient functions $C_i^{(4)}(m_c^2/Q^2)$ have been correctly computed to a given order for calculations with intrinsic charm, the very same coefficient functions must also hold to the same order when there is no intrinsic charm.

JHEP11(2015)122

The same is true of the matching matrix $K_{ij}(m_c^2/Q^2)$, since this can be defined entirely in terms of coefficient functions. This is a straightforward consequence of factorization: coefficient functions are by construction hard cross sections, and are thus independent of the target hadron. Unfortunately however the converse is not true: coefficient functions computed to a given order for the special case of no intrinsic charm might need correcting in the more general case when intrinsic charm is included. We shall see below that this is indeed the case.

A naive definition of zero intrinsic charm would be that it vanish in the 4-flavour scheme at the initial scale: $f_c^{(4)}(x,Q_0^2) = f_{\bar{c}}^{(4)}(x,Q_0^2) = 0$. Unfortunately this definition is rather ambiguous: instead of the rather arbitrary starting scale Q_0, one might instead prefer the scale of the charm mass m_c, or indeed the threshold scale $W = 2m_c$. However once the scale is chosen, this would mean that in the 4FS all charm is 'extrinsic', i.e. it is generated dynamically by perturbative evolution, so $f_c^{(4)}(x,Q^2)$ and $f_{\bar{c}}^{(4)}(x,Q^2)$ can be expressed entirely in terms of light quark PDFs.

A much better characterization of intrinsic charm is to define it as the charm PDF in the 3FS, where the charm PDF does not evolve, and thus, one could argue, there is no extrinsic charm. Thus, zero intrinsic charm means that

$$f_c^{(3)} = f_{\bar{c}}^{(3)} = 0. \tag{3.9}$$

This is a very natural assumption to make, since in the massive 3FS scheme the charm PDFs are scale independent, so there is no ambiguity about scale choice. Unfortunately it means that there is in general no scale at which the 4FS charm PDFs vanish, as can be seen from the matching condition eq. (2.12): if charm in one scheme is zero, in the other it will be generally nonzero already at $\mathcal{O}(\alpha_s)$ if $\mu \neq m_c$, due to the off-diagonal term K_{cg} in the matching condition, and at $\mathcal{O}(\alpha_s^2)$ even for $\mu = m_c$ due to the presence of non-logarithmic terms at this order.

The condition eq. (3.9) can however always be turned into a nonzero boundary condition for perturbative massless evolution. To see how this works, we first write out the matching conditions eq. (2.12) separating out the light partons from the charm partons:

$$f_i^{(4)}(\mu_c^2) = \sum_{j=g,q,\bar{q}} K_{ij}\left(\tfrac{m_c^2}{\mu_c^2}\right) \otimes f_j^{(3)}(\mu_c^2) + \sum_{j=c,\bar{c}} K_{ij}\left(\tfrac{m_c^2}{\mu_c^2}\right) \otimes f_j^{(3)}, \qquad i = g,q,\bar{q}, \tag{3.10}$$

$$f_i^{(4)}(\mu_c^2) = \sum_{j=g,q,\bar{q}} K_{ij}\left(\tfrac{m_c^2}{\mu_c^2}\right) \otimes f_j^{(3)}(\mu_c^2) + \sum_{j=c,\bar{c}} K_{ij}\left(\tfrac{m_c^2}{\mu_c^2}\right) \otimes f_j^{(3)}, \qquad i = c,\bar{c}. \tag{3.11}$$

When there is no intrinsic charm, eq. (3.9), the second term in each of these equations vanishes, and in particular eq. (3.11) becomes simply

$$f_c^{(4)}(\mu_c^2) = \sum_{j=g,q,\bar{q}} K_{cj}\left(\tfrac{m_c^2}{\mu_c^2}\right) \otimes f_j^{(3)}(\mu_c^2), \qquad \left(f_{c,\bar{c}}^{(3)} = 0\right). \tag{3.12}$$

Thus when there is no intrinsic charm case, it is possible (and often convenient) to invert eq. (3.10) to express the light 3FS PDFs in terms of *only light* 4FS PDFs, as

$$f_i^{(3)}(\mu_c^2) = \sum_{j=g,q,\bar{q}} \tilde{K}_{ij}^{-1}\left(\tfrac{m_c^2}{\mu_c^2}\right) \otimes f_j^{(4)}(\mu_c^2), \qquad i = q,\bar{q},g, \qquad \left(f_{c,\bar{c}}^{(3)} = 0\right), \tag{3.13}$$

JHEP11(2015)122

where $\tilde{K}_{ij}$ is the matching matrix restricted to the subspace of light partons, $i,j = g,q,\bar{q}$, so that the inverse is taken in this subspace. Substituting into eq. (3.12) we find, for $Q_0 \sim m_c$

$$f_c^{(4)}(Q_0^2) = \sum_{j,k=g,q,\bar{q}} K_{cj}\left(\frac{m_c^2}{Q_0^2}\right) \otimes \tilde{K}^{-1}_{jk}\left(\frac{m_c^2}{Q_0^2}\right) \otimes f_k^{(4)}(Q_0^2), \qquad \left(f_{c,\bar{c}}^{(3)} = 0\right), \tag{3.14}$$

which is the required boundary condition expressing the charm PDF in terms of the light PDFs at the starting scale. Note that if we choose $Q_0 = m_c$, $f_c^{(4)}(Q_0^2)$ will be $O(\alpha_s^2)$ and thus presumably very small: still, it is always nonzero in general. However all charm, at any scale, is still determined perturbatively from the light parton PDFs, and is thus extrinsic. The condition eq. (3.14) may be consistently applied at any scale $Q_0 \sim m_c$: changes in Q_0 only introduce subleading corrections. However it will not hold for $Q_0 \gg m_c$, since then these formally subleading corrections will be accompanied by large logarithms.

3.4 FONLL with zero intrinsic charm

Now that we have a definition of intrinsic charm, we can apply the FONLL construction under the assumption that all charm is generated perturbatively [14]. The treatment given here will be entirely explicit, to any order in perturbation theory.

When there is no intrinsic charm, it is possible to write the structure function in the massive scheme, $F^{(3)}$, entirely in terms of the light partons in the 4FS: combining eq. (2.1) and eq. (3.13),

$$\begin{aligned} F^{(3)}(Q^2, m_c^2)\Big|_{\text{zic}} &= \sum_{i=g,q,\bar{q}} C_i^{(3)}\left(\frac{m_c^2}{Q^2}\right) \otimes f_i^{(3)}(Q^2) \\ &= \sum_{i,j=g,q,\bar{q}} C_i^{(3)}\left(\frac{m_c^2}{Q^2}\right) \otimes \tilde{K}^{-1}_{ij}\left(\frac{m_c^2}{Q^2}\right) \otimes f_j^{(4)}(Q^2), \end{aligned} \tag{3.15}$$

where the subscript 'zic' stands for 'zero intrinsic charm'. Thus when there is no intrinsic charm the coefficient functions $C_{c,\bar{c}}^{(3)}$ are not needed, and likewise the matching terms K_{ic}, $K_{i\bar{c}}$. Of course these terms are still nonzero, but when eq. (3.9) holds they are no longer needed for the evaluation of $F^{(3)}$, and can thus be ignored. Like eq. (2.1), this expression only holds for $Q^2 \sim m_c^2$: although the light PDFs are in the 4FS, and thus resum collinear logarithms in the light sector, eq. (3.13) is fixed order, so the large logarithms of m_c^2/Q^2 in the heavy quark sector are not resummed.

In ref. [14] eq. (3.15) is written as

$$F^{(3)}(Q^2, m_c^2)\Big|_{\text{zic}} = \sum_{i=g,q,\bar{q}} B_i^{(4)}\left(\frac{m_c^2}{Q^2}\right) \otimes f_i^{(4)}(Q^2), \tag{3.16}$$

having defined

$$B_i^{(4)}\left(\frac{m_c^2}{Q^2}\right) = \sum_{j=g,q,\bar{q}} C_j^{(3)}\left(\frac{m_c^2}{Q^2}\right) \otimes \tilde{K}^{-1}_{ji}\left(\frac{m_c^2}{Q^2}\right), \tag{3.17}$$

with inverse

$$C_i^{(3)}\left(\frac{m_c^2}{Q^2}\right) = \sum_{j=g,q,\bar{q}} B_j^{(4)}\left(\frac{m_c^2}{Q^2}\right) \otimes K_{ji}\left(\frac{m_c^2}{Q^2}\right), \qquad i = g,q,\bar{q}. \tag{3.18}$$

JHEP11(2015)122

The similarity with the more general ACOT expression eq. (2.25) is obvious: the only difference in fact is that the sum extends only over the light partons, and the inverse of the matching matrix, $\tilde{K}^{-1}$, is likewise taken in the light parton subspace.

Substituting eq. (3.16) in the general expression eq. (3.3), and eq. (2.8) for the massless term, we immediately obtain the FONLL expression for the structure function when there is no intrinsic charm:

$$F_{\text{FONLL}}(Q^2, m_c^2)\Big|_{\text{zic}} = \sum_{i=g,q,\bar{q}} \left[B_i^{(4)}\left(\tfrac{m_c^2}{Q^2}\right) - B_i^{(4,0)}\left(\tfrac{m_c^2}{Q^2}\right) + C_i^{(4)}(0) \right] \otimes f_i^{(4)}(Q^2) + \sum_{i=c,\bar{c}} C_i^{(4)}(0) \otimes f_i^{(4)}(Q^2). \tag{3.19}$$

Here, in analogy to eq. (3.17),

$$\begin{aligned} B_i^{(4,0)}\left(\tfrac{m_c^2}{Q^2}\right) &= \sum_{j=g,q,\bar{q}} C_j^{(3,0)}\left(\tfrac{m_c^2}{Q^2}\right) \otimes \tilde{K}_{ji}^{-1}\left(\tfrac{m_c^2}{Q^2}\right) \\ &= \sum_{j=g,q,\bar{q}} \sum_{k=g,q,\bar{q},c,\bar{c}} C_k^{(4)}(0) \otimes K_{kj}\left(\tfrac{m_c^2}{Q^2}\right) \otimes \tilde{K}_{ji}^{-1}\left(\tfrac{m_c^2}{Q^2}\right) \\ &= C_i^{(4)}(0) + \sum_{k=c,\bar{c}} \sum_{j=g,q,\bar{q}} C_k^{(4)}(0) \otimes K_{kj}\left(\tfrac{m_c^2}{Q^2}\right) \otimes \tilde{K}_{ji}^{-1}\left(\tfrac{m_c^2}{Q^2}\right), \end{aligned} \tag{3.20}$$

where we have used eq. (2.14) in the second step, and eq. (A.11) (see appendix A) in the last. For $Q^2 \gg m_c^2$, the first two terms in eq. (3.19) are manifestly of order m_c^2/Q^2, and thus the massless limit eq. (2.8) is recovered. The diagrams contributing to the FONLL result in the zero intrinsic charm case are shown schematically in figure 1 (upper row).

Substituting the last line into eq. (3.19) gives the alternative expression

$$\begin{aligned} F_{\text{FONLL}}(Q^2, m_c^2)\Big|_{\text{zic}} = &\sum_{i=g,q,\bar{q}} B_i^{(4)}\left(\tfrac{m_c^2}{Q^2}\right) \otimes f_i^{(4)}(Q^2) + \sum_{i=c,\bar{c}} C_i^{(4)}(0) \otimes f_i^{(4)}(Q^2) \\ &- \sum_{k=c,\bar{c}} \sum_{i,j=g,q,\bar{q}} C_k^{(4)}(0) \otimes K_{kj}\left(\tfrac{m_c^2}{Q^2}\right) \otimes \tilde{K}_{ji}^{-1}\left(\tfrac{m_c^2}{Q^2}\right) \otimes f_i^{(4)}(Q^2). \end{aligned} \tag{3.21}$$

The last two terms are the difference term eq. (3.4). At the initial scale Q_0 the difference term is thus precisely zero, due to the boundary condition eq. (3.14) used when there is no intrinsic charm. For $Q > Q_0$, since the 4FS PDFs are constrained to evolve using 4FS evolution eq. (2.7), generating charm perturbatively, they will contain higher order logarithms which do not cancel: with the matching matrix truncated to order α_s^p,

$$f_i^{(4)}(Q^2) - \sum_{j,k=g,q,\bar{q}} K_{ij}\left(\tfrac{m_c^2}{Q^2}\right) \otimes \tilde{K}_{jk}^{-1}\left(\tfrac{m_c^2}{Q^2}\right) \otimes f_k^{(4)}(Q^2) \sim \mathcal{O}\left(\alpha_s^{p+1} \log^{p+1} \tfrac{m_c^2}{Q^2}\right). \tag{3.22}$$

It follows that whenever $Q \sim m_c$, so that $\log(m_c^2/Q^2)$ is not too large, the difference term is always subleading, as required.

JHEP11(2015)122

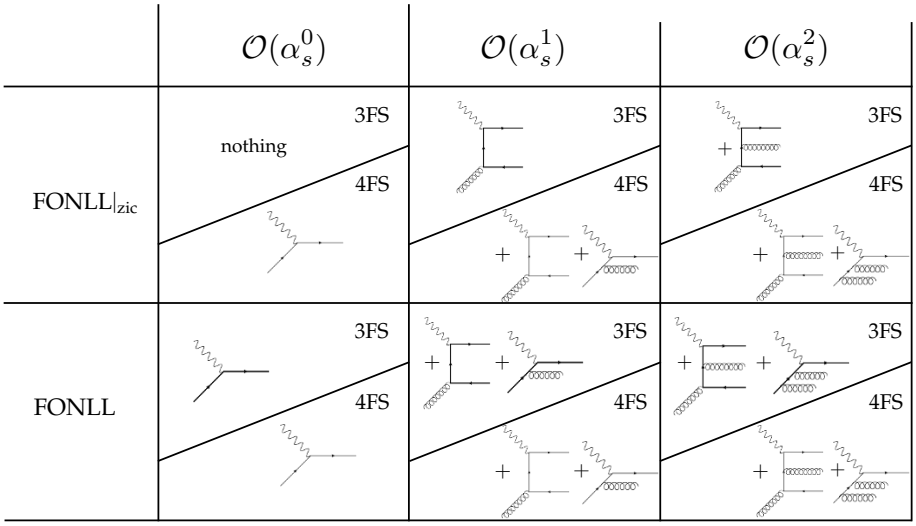

Figure 1. Representative diagrams included at each order in α_s in the massive 3FS and massless 4FS components entering the FONLL formula eq. (3.3), in the simpler case of no intrinsic charm (upper line, corresponding to eq. (3.19)) and in the full case (lower line, corresponding to eq. (3.32)). All quark lines represent the charm quark: bold lines means that mass effects are retained in the coefficient function, while light lines means that the charm mass has been set to zero. Diagrams with corrections to the gluon propagator through a virtual heavy quark loop are not shown.

3.5 FONLL including intrinsic charm

When we drop the assumption that the intrinsic charm is zero, we can go through the same argument as in the previous section expressing $F^{(3)}$ in terms of $f_i^{(4)}$, but keeping the nonzero $f_{c,\bar{c}}^{(3)}$ terms: we then find

$$\begin{aligned} F^{(3)}(Q^2,m_c^2) &= \sum_{i=g,q,\bar{q}} C_i^{(3)}\left(\tfrac{m_c^2}{Q^2}\right)\otimes f_i^{(3)}(Q^2) + \sum_{i=c,\bar{c}} C_i^{(3)}\left(\tfrac{m_c^2}{Q^2}\right)\otimes f_i^{(3)} \\ &= \sum_{i,k=g,q,\bar{q}} B_k^{(4)}\left(\tfrac{m_c^2}{Q^2}\right)\otimes K_{ki}\left(\tfrac{m_c^2}{Q^2}\right)\otimes f_i^{(3)}(Q^2) + \sum_{i=c,\bar{c}} C_i^{(3)}\left(\tfrac{m_c^2}{Q^2}\right)\otimes f_i^{(3)} \\ &= \sum_{k=g,q,\bar{q}} B_k^{(4)}\left(\tfrac{m_c^2}{Q^2}\right)\otimes f_k^{(4)}(Q^2) \\ &\quad + \sum_{i=c,\bar{c}}\left[C_i^{(3)}\left(\tfrac{m_c^2}{Q^2}\right) - \sum_{k=g,q,\bar{q}} B_k^{(4)}\left(\tfrac{m_c^2}{Q^2}\right)\otimes K_{ki}\left(\tfrac{m_c^2}{Q^2}\right)\right]\otimes f_i^{(3)}, \end{aligned} \tag{3.23}$$

where in the second line we used eq. (3.18), and in the third eq. (3.10). We recognize in the first term the zero intrinsic charm result, eq. (3.16), so defining

$$F^{(3)}(Q^2,m_c^2) = F^{(3)}(Q^2,m_c^2)\Big|_{\text{zic}} + \Delta F^{(3)}(Q^2,m_c^2), \tag{3.24}$$

the intrinsic charm contribution to the structure function in the massive 3FS scheme is

$$\Delta F^{(3)}(Q^2,m_c^2) = \sum_{i=c,\bar{c}}\left[C_i^{(3)}\left(\tfrac{m_c^2}{Q^2}\right) - \sum_{j=g,q,\bar{q}} B_j^{(4)}\left(\tfrac{m_c^2}{Q^2}\right)\otimes K_{ji}\left(\tfrac{m_c^2}{Q^2}\right)\right]\otimes f_i^{(3)}. \tag{3.25}$$

JHEP11(2015)122

Note that this precise form derives from having used the coefficient functions $B^{(4)}$ for writing the zero intrinsic charm contribution.

We can no longer use eq. (3.13) to express $f^{(3)}_{c,\bar{c}}$ in terms of 4FS PDFs: instead we have to use the inverse of eq. (2.12): for $i = c, \bar{c}$

$$\begin{aligned} f_i^{(3)} &= \sum_{j=g,q,\bar{q},c,\bar{c}} K^{-1}_{ij}\left(\tfrac{m_c^2}{Q^2}\right) \otimes f_j^{(4)}(Q^2) \\ &= \sum_{j=c,\bar{c}} K^{-1}_{ij} \otimes \left[f_j^{(4)}(Q^2) - \sum_{k,l=g,q,\bar{q}} K_{jk}\left(\tfrac{m_c^2}{Q^2}\right) \otimes \tilde{K}^{-1}_{kl}\left(\tfrac{m_c^2}{Q^2}\right) \otimes f_l^{(4)}(Q^2) \right], \end{aligned} \tag{3.26}$$

where in the second line we used eq. (A.9) in appendix A, or equivalently solved eqs. (3.10), (3.11) for $f_i^{(3)}$. We thus find that

$$\begin{aligned} \Delta F^{(3)}(Q^2, m_c^2) = \sum_{i=c,\bar{c}} \left[C_i^{(3)}\left(\tfrac{m_c^2}{Q^2}\right) - \sum_{k=g,q,\bar{q}} B_k^{(4)}\left(\tfrac{m_c^2}{Q^2}\right) \otimes K_{ki}\left(\tfrac{m_c^2}{Q^2}\right) \right] \\ \otimes \sum_{j=g,q,\bar{q},c,\bar{c}} K^{-1}_{ij}\left(\tfrac{m_c^2}{Q^2}\right) \otimes f_j^{(4)}(Q^2). \end{aligned} \tag{3.27}$$

Using eq. (3.27) in the generic FONLL formula eq. (3.3) then gives

$$F_{\rm FONLL}(Q^2, m_c^2) = F_{\rm FONLL}(Q^2, m_c^2)\Big|_{\rm zic} + \Delta F_{\rm FONLL}(Q^2, m_c^2), \tag{3.28}$$

where the intrinsic charm contribution

$$\begin{aligned} \Delta F_{\rm FONLL}\left(Q^2, m_c^2\right) = \sum_{i,j=c,\bar{c}} \Bigg[\left(C_i^{(3)}\left(\tfrac{m_c^2}{Q^2}\right) - C_i^{(3,0)}\left(\tfrac{m_c^2}{Q^2}\right) \right) \\ - \sum_{m=g,q,\bar{q}} \left(B_m^{(4)}\left(\tfrac{m_c^2}{Q^2}\right) - B_m^{(4,0)}\left(\tfrac{m_c^2}{Q^2}\right) \right) \otimes K_{mi}\left(\tfrac{m_c^2}{Q^2}\right) \Bigg] \\ \otimes K^{-1}_{ij}\left(\tfrac{m_c^2}{Q^2}\right) \otimes \left[f_j^{(4)}(Q^2) - \sum_{k,l=g,q,\bar{q}} K_{jk}\left(\tfrac{m_c^2}{Q^2}\right) \otimes \tilde{K}^{-1}_{kl}\left(\tfrac{m_c^2}{Q^2}\right) \otimes f_l^{(4)}(Q^2) \right]. \end{aligned} \tag{3.29}$$

At large Q^2 this term is manifestly $\mathcal{O}(m_c^2/Q^2)$, so has no effect on the high energy limit. Near threshold it is in general $\mathcal{O}(1)$, unless the intrinsic charm vanishes: then using eq. (3.22) it is easy to see that $\Delta F_{\rm FONLL}$ is subleading (down by one power of $\alpha_s(Q^2)$).

We stress that eq. (3.24), with the two terms given by eqs. (3.19), (3.29), is actually identical to the simple expression eq. (3.8) order by order in perturbation theory. In a sense this is obvious because both equations have been derived with the FONLL construction, eq. (3.3), systematically rewriting expressions involving 3FS PDFs in terms of 4FS PDFs using the matching conditions eq. (2.12). This said, it is perhaps useful, if only as a cross check, to verify by explicit computation that the two expression are indeed identical.

To do this, it is clearly sufficient to show that eq. (3.24), with the two terms given by eq. (3.16) and eq. (3.27), is equivalent to eq. (3.5). We first rewrite it, by collecting the coefficients of $f_i^{(4)}$ (and suppressing the arguments of the functions to lighten the notation),

JHEP11(2015)122

as

$$
\begin{aligned}
F^{(3)} &= \sum_{k=g,q,\bar{q}} \left[B_k^{(4)} + \sum_{i=c,\bar{c}} \left(C_i^{(3)} - \sum_{j=g,q,\bar{q}} B_j^{(4)} \otimes K_{ji} \right) \otimes K_{ik}^{-1} \right] \otimes f_k^{(4)} \\
&\quad + \sum_{i,j=c,\bar{c}} \left(C_i^{(3)} - \sum_{k=g,q,\bar{q}} B_k^{(4)} \otimes K_{ki} \right) \otimes K_{ij}^{-1} \otimes f_j^{(4)} \\
&= \sum_{k=g,q,\bar{q}} \left[\sum_{i=g,q,\bar{q}} C_i^{(3)} \otimes \left(\tilde{K}_{ik}^{-1} - \sum_{j=g,q,\bar{q}} \sum_{l=c,\bar{c}} \tilde{K}_{ij}^{-1} \otimes K_{jl} \otimes K_{lk}^{-1} \right) + \sum_{i=c,\bar{c}} C_i^{(3)} \otimes K_{ik}^{-1} \right] \otimes f_k^{(4)} \\
&\quad + \sum_{k=c,\bar{c}} \left[\sum_{i=c,\bar{c}} C_i^{(3)} \otimes K_{ik}^{-1} - \sum_{i,j=g,q,\bar{q}} \sum_{l=c,\bar{c}} C_i^{(3)} \otimes \tilde{K}_{ij}^{-1} \otimes K_{jl} \otimes K_{lk}^{-1} \right] \otimes f_k^{(4)}, \quad (3.30)
\end{aligned}
$$

using eq. (3.17). Now using the expressions in appendix A, specifically eqs. (A.9), (A.6) in the first line and eq. (A.8) in the second, we find

$$
\begin{aligned}
F^{(3)} &= \sum_{k=g,q,\bar{q}} \left[\sum_{i=g,q,\bar{q}} C_i^{(3)} \otimes K_{ik}^{-1} + \sum_{i=c,\bar{c}} C_i^{(3)} \otimes K_{ik}^{-1} \right] \otimes f_k^{(4)} \\
&\quad + \sum_{k=c,\bar{c}} \left[\sum_{i=g,q,\bar{q}} C_i^{(3)} \otimes K_{ik}^{-1} + \sum_{i=c,\bar{c}} C_i^{(3)} \otimes K_{ik}^{-1} \right] \otimes f_k^{(4)} \\
&= \sum_{i,k=g,q,\bar{q},c,\bar{c}} C_i^{(3)} \otimes K_{ik}^{-1} \otimes f_k^{(4)}, \quad (3.31)
\end{aligned}
$$

which is the desired result, eq. (3.5).

It follows that the formulation of FONLL in ref. [14], eq. (3.19), in which it is assumed that all charm is generated perturbatively, plus the extra 'intrinsic charm' contribution eq. (3.29), is identical to the full ACOT result through eq. (3.8), irrespective of any condition on the charm at the initial scale.

Note that this means that we can write the full FONLL (or equivalently ACOT) expression eq. (3.28) as simply

$$
\begin{aligned}
F_{\text{FONLL}}\left(Q^2, m_c^2\right) &= \sum_{i=g,q,\bar{q}} B_i^{(4)} \otimes f_i^{(4)}(Q^2) \\
&\quad + \sum_{i=c,\bar{c}} \left[C_i^{(3)} - \sum_{k=g,q,\bar{q}} B_k^{(4)} \otimes K_{ki} \right] \otimes \sum_{j=g,q,\bar{q},c,\bar{c}} K_{ij}^{-1} \otimes f_j^{(4)}, \quad (3.32)
\end{aligned}
$$

which follows directly from eq. (3.23) and the observation made in section 3.2 that all the massless contributions (and thus the difference term in eq. (3.4)) cancel when no assumption is made about instrinsic charm. The second term is now essential to obtain the correct high Q^2 behaviour, even in the limit of zero intrinsic charm. The diagrams contributing to the full FONLL result eq. (3.32) are shown schematically in figure 1 (lower row).

Comparison of the ACOT representation eq. (2.25) with the original zero intrinsic charm representation of FONLL eq. (3.19) gives us a new way to understand the origin of intrinsic charm contribution eq. (3.29): taking the difference, and using the definitions

JHEP11(2015)122

eq. (3.17) and eq. (3.20)

$$\begin{aligned}\Delta F_{\text{FONLL}} &= \sum_{i,j=g,q,\bar{q},c,\bar{c}} \left[C_i^{(3)} - C_i^{(3,0)}\right] \otimes K_{ij}^{-1} \otimes f_j^{(4)} - \sum_{i,j=g,q,\bar{q}} \left[C_i^{(3)} - C_i^{(3,0)}\right] \otimes \tilde{K}_{ij}^{-1} \otimes f_j^{(4)} \\ &= \sum_{i,j=g,q,\bar{q},c,\bar{c}} \left[C_i^{(3)} - C_i^{(3,0)}\right] \otimes \left[K_{ij}^{-1} - \widehat{K_{ij}^{-1}}\right] \otimes f_j^{(4)}, \end{aligned} \tag{3.33}$$

where the matrix $\widehat{K_{ij}^{-1}}$ acts as a projector onto the space of light partons

$$\widehat{K_{ij}^{-1}} = \begin{cases} \tilde{K}_{ij}^{-1} & i,j = g,q,\bar{q} \\ 0, & \text{otherwise.} \end{cases} \tag{3.34}$$

The expression eq. (3.33) is particularly transparent: when intrinsic charm is included the massless coefficient functions in the charm sector must be mass corrected, with additional collinear subtractions for the incoming charm quarks lines, these subtractions being factorized multiplicatively.

From eq. (3.33) it is also possible to derive another useful form of ΔF_{FONLL} in terms of 4FS coefficient functions. Substituting the inverse of eq. (2.23),

$$C_i^{(3)}\left(\tfrac{m_c^2}{Q^2}\right) = \sum_{j=g,q,\bar{q},c,\bar{c}} C_j^{(4)}\left(\tfrac{m_c^2}{Q^2}\right) \otimes K_{ji}\left(\tfrac{m_c^2}{Q^2}\right), \tag{3.35}$$

and eq. (2.14) into eq. (3.33) we find immediately

$$\Delta F_{\text{FONLL}}\left(Q^2, m_c^2\right) = \sum_{i=g,q,\bar{q},c,\bar{c}} \left[C_i^{(4)}\left(\tfrac{m_c^2}{Q^2}\right) - C_i^{(4)}(0)\right] \otimes \left[f_i^{(4)} - \sum_{k,l=g,q,\bar{q}} K_{ik} \otimes \tilde{K}_{kl}^{-1} \otimes f_l^{(4)}\right]. \tag{3.36}$$

When $i = g, q, \bar{q}$ in the sum, the difference in the second square brackets vanishes, because of eq. (A.11). Therefore eq. (3.36) simplifies to

$$\Delta F_{\text{FONLL}}\left(Q^2, m_c^2\right) = \sum_{i=c,\bar{c}} \left[C_i^{(4)}\left(\tfrac{m_c^2}{Q^2}\right) - C_i^{(4)}(0)\right] \otimes \left[f_i^{(4)} - \sum_{k,l=g,q,\bar{q}} K_{ik} \otimes \tilde{K}_{kl}^{-1} \otimes f_l^{(4)}\right]. \tag{3.37}$$

This is a very compact expression, and manifestly shows that the missing mass corrections in eq. (3.19) due to intrinsic charm are entirely determined by the mass dependence of the charm initiated contribution in the 4FS.

3.6 Comparison to S-ACOT

Finally, we consider the connection to S-ACOT [8, 9, 23], a simplified variant of ACOT whose validity is based on the assumption that the charm is generated perturbatively. Under this assumption, the authors of ref. [9] claim that in the construction of the ACOT (massive 4FS) coefficient functions the mass dependence in all diagrams with an incoming charm quark can be systematically ignored, i.e. $C_{c,\bar{c}}^{(4)}(m_c^2/Q^2)$ can be replaced with $C_{c,\bar{c}}^{(4)}(0)$ in all steps of the construction.

More precisely, the structure functions in S-ACOT are written as in eq. (2.9),

$$F_{\text{S-ACOT}}(Q^2, m_c^2) = \sum_{i=g,q,\bar{q},c,\bar{c}} \bar{C}_i^{(4)}\left(\tfrac{m_c^2}{Q^2}\right) \otimes f_i^{(4)}(Q^2) \tag{3.38}$$

JHEP11(2015)122

but with new coefficient functions $\bar{C}_i^{(4)}$. Those must be determined by consistency with the unresummed result, eq. (2.1); using eq. (2.12) we find

$$C_i^{(3)}\left(\frac{m_c^2}{Q^2}\right) = \sum_{i=g,q,\bar{q},c,\bar{c}} \bar{C}_j^{(4)}\left(\frac{m_c^2}{Q^2}\right) \otimes K_{ji}\left(\frac{m_c^2}{Q^2}\right). \tag{3.39}$$

In the general case which can account for intrinsic charm, this has a unique solution, $\bar{C}_i^{(4)} = C_i^{(4)}$ for $i = g, q, \bar{q}, c, \bar{c}$, giving back ACOT. However, in the absence of intrinsic charm, $f_{c,\bar{c}}^{(3)} = 0$, and thus eq. (3.39) can only be derived for $i = g, q, \bar{q}$. This means that the system of eq. (3.39) is under-constrained, in the sense that solving it for $\bar{C}_j^{(4)}(m_c^2/Q^2)$ is ambiguous: there are only 7 equations for 9 unknowns. This is another manifestation of the ambiguity in inverting eq. (3.12) discussed in section 3.3. When this is the case, we are free to choose two of the coefficient functions $\bar{C}_j^{(4)}(m_c^2/Q^2)$ as we please, subject only to the constraint that we recover the massless coefficient functions when $Q^2 \gg m_c^2$. The most natural choice is then the S-ACOT simplification [9]

$$\bar{C}_i^{(4)}\left(\frac{m_c^2}{Q^2}\right) = C_i^{(4)}(0), \qquad i = c, \bar{c}. \tag{3.40}$$

Given this, we can then solve eq. (3.39) for the remaining components, giving immediately

$$\bar{C}_i^{(4)}\left(\frac{m_c^2}{Q^2}\right) = \sum_{j=g,q,\bar{q}} \left[C_i^{(3)}\left(\frac{m_c^2}{Q^2}\right) - \sum_{k=c,\bar{c}} C_k^{(4)}(0) \otimes K_{kj}\right] \otimes \tilde{K}_{ji}^{-1}, \qquad i = g, q, \bar{q}. \tag{3.41}$$

These are the S-ACOT coefficient functions, which can be determined order by order in perturbation theory starting from the massive 3FS coefficients for light channels and the massless 4FS coefficients for the heavy channel. The physical interpretation is very transparent: the collinear logarithms due to the charm quark are completely subtracted off $C_i^{(3)}$, while power suppressed contributions are all left untouched.

We now want to compare this result to FONLL. We saw in section 3.4 that when all charm is generated perturbatively, eq. (3.19) does not depend on $C_{c,\bar{c}}^{(4)}(m_c^2/Q^2)$: in the full FONLL expression eq. (3.28) (which we just showed is equivalent to ACOT) all the mass dependence of the incoming charm quark lines is contained in the 4FS coefficient functions $C_{c,\bar{c}}^{(4)}(m_c^2/Q^2)$ in the ΔF_{FONLL} term eq. (3.37). When we set $C_{c,\bar{c}}^{(4)}(m_c^2/Q^2) \to C_{c,\bar{c}}^{(4)}(0)$ as in S-ACOT, ΔF_{FONLL} vanishes identically, whether or not we have intrinsic charm (i.e. whether or not the PDF term in square brackets in eq. (3.37) vanishes). It follows that S-ACOT is equivalent order by order in perturbation theory to FONLL as formulated in ref. [14] with all charm generated perturbatively, eq. (3.19). This of course accounts for the numerical equivalence of FONLL and S-ACOT at NLO discovered in ref. [40].[3]

The equivalence between FONLL with zero intrinsic charm eq. (3.19) and S-ACOT eq. (3.38) implies the relation

$$\bar{C}_i^{(4)}\left(\frac{m_c^2}{Q^2}\right) = B_i^{(4)}\left(\frac{m_c^2}{Q^2}\right) - B_i^{(4,0)}\left(\frac{m_c^2}{Q^2}\right) + C_i^{(4)}(0), \tag{3.42}$$

[3]There are other sources of differences at finite order between FONLL and S-ACOT, due to a damping factor adopted in ref. [14] and the χ rescaling sometimes used in practical applications of S-ACOT. See discussion in section 3.7.

JHEP11(2015)122

for $i = g, q, \bar{q}$. It is straightforward to check that eq. (3.42) is identical to eq. (3.41), as expected. It is interesting to observe that while in S-ACOT the subtraction of charm-induced collinear logarithms is achieved identifying the logarithms in a factorized form, in the FONLL formulation the collinear logarithms are subtracted using the 'massless limit' $B_i^{(4,0)}$ which includes also constant (non-log) terms, which are then restored through $C_i^{(4)}(0)$.

In summary, we have shown that

$$F_{\text{ACOT}} \equiv F_{\text{FONLL}} \tag{3.43}$$

$$F_{\text{S-ACOT}} \equiv F_{\text{FONLL}}\Big|_{\text{zic}} \tag{3.44}$$

to all orders in perturbation theory. The first of these equivalences is a direct consequence of the fact that both ACOT and FONLL express their final result in terms of PDFs factorised in the 4FS: since the PDFs are then formally identical, the coefficient functions must also be identical, order by order in perturbation theory. The second equivalence is more subtle: it states that suppressing the mass dependence in the coefficient functions with incoming charm is actually equivalent to the simplified result obtained when there is no intrinsic charm PDF, where suppressed contributions proportional to eq. (3.22) are neglected. It is particularly useful, since it means that it is unnecessary to compute massive coefficient functions with incoming charm if all charm is generated perturbatively: in this situation S-ACOT is exact. However reliable calculations with a fitted charm distribution do require knowledge of these coefficient functions: in this circumstance S-ACOT can only be an approximation (and not necessarily a very reliable one [41]).

3.7 Damping factor

Our discussion so far was mostly formal, focussing on all-order expressions. When the various contributions are computed at finite order, higher order interference terms may spoil the accuracy of the results, as discussed in section 2.3. To avoid this problem, the computations in ref. [14] also include a phenomenological damping factor: in place of eq. (3.4) one writes

$$\begin{aligned} F_{\text{FONLL}}(Q^2, m_c^2) &= F^{(3)}(Q^2, m_c^2) + D\left(\tfrac{m_c^2}{Q^2}\right)\left[F^{(4)}(Q^2, 0) - F^{(3,0)}(Q^2, m_c^2)\right] \\ &\equiv F^{(3)}(Q^2, m_c^2) + D\left(\tfrac{m_c^2}{Q^2}\right) F^{(d)}(Q^2, m_c^2)\,, \end{aligned} \tag{3.45}$$

where

$$D\left(\tfrac{m_c^2}{Q^2}\right) = \Theta(Q^2 - m_c^2)\left(1 - \tfrac{m_c^2}{Q^2}\right)^2 \tag{3.46}$$

suppresses the difference term $F^{(d)}$ close to threshold, i.e. when $Q^2 \sim m_c^2$. This is the region where the resummation of collinear logarithms, added to the fixed-order result $F^{(3)}(Q^2, m_c^2)$ through the difference term $F^{(d)}(Q^2, m_c^2)$, is not needed and can therefore be artificially suppressed. This suppression turns out to be important when working at $\mathcal{O}(\alpha_s)$, where the $\mathcal{O}(\alpha_s^2)$ interference terms are sizeable, but becomes almost negligible already at $\mathcal{O}(\alpha_s^2)$, where the $\mathcal{O}(\alpha_s^2)$ interference terms are small.

JHEP11(2015)122

With this damping eq. (3.19) becomes

$$F_{\rm FONLL}(Q^2,m_c^2)\Big|_{\rm zic} = \sum_{i=g,q,\bar{q}} B_i^{(4)}\left(\tfrac{m_c^2}{Q^2}\right)\otimes f_i^{(4)}(Q^2) + D\left(\tfrac{m_c^2}{Q^2}\right)\Bigg[\sum_{i=g,q,\bar{q}}\left(C_i^{(4)}(0) - B_i^{(4,0)}\left(\tfrac{m_c^2}{Q^2}\right)\right)\otimes f_i^{(4)}(Q^2) + \sum_{i=c,\bar{c}} C_i^{(4)}(0)\otimes f_i^{(4)}(Q^2)\Bigg] \tag{3.47}$$

and, likewise, eq. (3.29) becomes

$$\Delta F_{\rm FONLL}(Q^2,m_c^2) = \sum_{i=c,\bar{c}}\Bigg[\left(C_i^{(3)}\left(\tfrac{m_c^2}{Q^2}\right) - D\left(\tfrac{m_c^2}{Q^2}\right)C_i^{(3,0)}\left(\tfrac{m_c^2}{Q^2}\right)\right) - \sum_{m=g,q,\bar{q}}\left(B_m^{(4)}\left(\tfrac{m_c^2}{Q^2}\right) - D\left(\tfrac{m_c^2}{Q^2}\right)B_m^{(4,0)}\left(\tfrac{m_c^2}{Q^2}\right)\right)\otimes K_{mi}\left(\tfrac{m_c^2}{Q^2}\right)\Bigg] \otimes \sum_{j=g,q,\bar{q},c,\bar{c}} K_{ij}^{-1}\left(\tfrac{m_c^2}{Q^2}\right)\otimes f_j^{(4)}(Q^2)\,. \tag{3.48}$$

However since when when add $\Delta F_{\rm FONLL}$ to $F_{\rm FONLL}\big|_{\rm zic}$ the difference term vanishes identically (and thus for example in eq. (3.32) there are no massless terms to damp), it is clear that when there is intrinsic charm the damping has no effect whatsoever.

It may seem paradoxical that while the limit of zero intrinsic charm should be unique, the zero intrinsic charm result eq. (3.47) clearly depends on the arbitrary function D. The reason of course is that in taking the limit the ΔF term eq. (3.48) is suppressed, since it becomes formally subleading: although small, it is still not entirely negligible, and indeed must be similar in size to the subleading variation achieved through changing the damping factor D. The FONLL damping factor is thus another manifestation of the ambiguity in the treatment of the zero intrinsic charm limit, discussed in the previous section.

The χ-rescaling prescription plays a similar role in S-ACOT-χ. In this case, however, rather than damping the (massless) resummation contribution, the massive kinematics is restored in those contributions which are computed in the massless limit. In this way, the S-ACOT-χ result becomes closer to ACOT, since (the dominant) part of the neglected power corrections are reinstated. This means that, even in the presence of intrinsic charm, S-ACOT-χ can be a reasonable approximation. Given the restriction of the exact massive results for the charm initiated contributions to $\mathcal{O}(\alpha_s)$ (the $\mathcal{O}(\alpha_s^2)$ diagrams with incoming massive quark lines have yet to be calculated), the usage of S-ACOT-χ might be a useful tool for improving the accuracy of calculations in the presence of intrinsic charm to $\mathcal{O}(\alpha_s^2)$ and beyond (see e.g. ref. [42]).

4 From charm to bottom and top

So far in this paper we have not considered the third generation quarks, ignoring in particular top and bottom mass dependence in the coefficient functions. This means in effect that we assumed that top and bottom were infinitely heavy, so that we were always well

JHEP11(2015)122

below threshold for their production. Virtual effects are suppressed by powers of the quark mass, provided a decoupling renormalization scheme is used for these quarks.

In practice, the bottom quark is not much heavier than the charm quark, so, while the top quark can be safely ignored at small scales, considering the bottom quark to be infinitely heavy is not a very good approximation even at the charm threshold. Virtual bottom quark loops cost a power of α_s in a gluon propagator, so bottom mass effects appear first at NNLO. Their effects are thus small, but not completely negligible.

It is the purpose of this section to extend the discussion in section 2 and section 3 to include the bottom and top quarks, in a complete and coherent framework. We first concentrate on the bottom quark, and then generalise to the top quark. We will also discuss the possibility of intrinsic beauty.

4.1 The bottom quark

In both the schemes (3FS and 4FS) discussed in section 2, bottom quark effects can appear in the perturbative coefficient functions through additional diagrams, either as virtual loops or, when kinematically allowed, through pair production. In both schemes the UV divergences due to bottom loops are renormalized in the decoupling (CWZ) scheme, so the bottom mass effects are formally suppressed as Q^2/m_b^2 when $Q^2 \ll m_b^2$. However in practice this condition is never really satisfied, and while pair production vanishes below threshold, the effect of virtual bottom quark loops should be included at NNLO and beyond.

Hence, all coefficient functions discussed so far implicitly include a dependence on the bottom mass. We thus replace

$$C_i^{(3)}\left(\tfrac{m_c^2}{Q^2}\right) \to C_i^{(3)}\left(\tfrac{m_c^2}{Q^2}, \tfrac{m_b^2}{Q^2}\right) \tag{4.1}$$

$$C_i^{(4)}\left(\tfrac{m_c^2}{Q^2}\right) \to C_i^{(4)}\left(\tfrac{m_c^2}{Q^2}, \tfrac{m_b^2}{Q^2}\right) \tag{4.2}$$

and equivalently the massless limit $C_i^{(4)}(0) \to C_i^{(4)}(0, m_b^2/Q^2)$; the same extension applies also for derived quantities such as $B_i^{(4)}$ or $C_i^{(3,0)}$. The m_b dependence is computed at fixed order: below threshold it can appear at $\mathcal{O}(\alpha_s^2)$ through a 1-loop correction to a gluon propagator, while above threshold it will appear already as an $\mathcal{O}(\alpha_s)$ contribution to the structure function (e.g. F_2) through the production of bottom quarks in the final state (see figure 1).

Since one naturally performs the resummation of charm collinear logarithms first, the 4FS plays for beauty the same role that the 3FS plays for charm. Entirely analogous to eq. (2.1) we thus have

$$F^{(4)}(Q^2, m_c^2, m_b^2) = \sum_{i=g,q',\bar{q}',b,\bar{b}} C_i^{(4)}\left(\tfrac{m_c^2}{Q^2}, \tfrac{m_b^2}{Q^2}\right) \otimes f_i^{(4)}(Q^2) \tag{4.3}$$

where now $q' = d, u, s, c$, and we have included the theoretical possibility of intrinsic beauty through the addition of a b-quark PDF. In this expression all the large logarithms $\log(m_c^2/Q^2)$ have been resummed, but at large Q^2 the potentially large logarithms

JHEP11(2015)122

$\log(m_b^2/Q^2)$ remain unresummed in the coefficient functions $C_i^{(4)}(m_c^2/Q^2, m_b^2/Q^2)$. The 4FS PDF evolution eq. (2.7) can be trivially extended to include the bottom quark PDF as

$$f_i^{(4)}(x,Q^2) = \sum_{j=g,q',\bar{q}',b,\bar{b}} \bar{\Gamma}_{ij}^{(4)}\left(Q^2,Q_0^2\right) \otimes f_j^{(4)}(Q_0^2); \tag{4.4}$$

where

$$\bar{\Gamma}_{ij}^{(4)}(Q^2,Q_0^2) = \begin{cases} \Gamma_{ij}^{(4)}(Q^2,Q_0^2) & i,j = g,q',\bar{q}' \\ \delta_{ij}, & i,j = b,\bar{b} \\ 0, & \text{otherwise.} \end{cases} \tag{4.5}$$

The Q^2 dependence of the b contribution to the structure function is all in the coefficient function, so $f_{b,\bar{b}}^{(4)}$ is independent of Q^2. Note that it is precisely the fact that the b quarks do not mix with the lighter partons in the 4FS that makes the extension of the previous formalism to include bottom quark effects trivial. If we wish to assume that there is no intrinsic beauty, we can simply take

$$f_b^{(4)} = f_{\bar{b}}^{(4)} = 0, \tag{4.6}$$

in analogy with eq. (3.9) for no intrinsic charm.

At high scales $Q^2 \gg m_b^2$, the logarithms of m_b^2/Q^2 in $C_i^{(4)}$ become large and need to be resummed. We must therefore factorize the large logarithms due to the bottom quark into the PDF, just as we did for the charm, leading naturally to a 5 flavor scheme (5FS). PDFs in the 5FS evolve as

$$f_i^{(5)}(Q^2) = \sum_{j=g,q',\bar{q}',b,\bar{b}} \Gamma_{ij}^{(5)}\left(Q^2,Q_0^2\right) \otimes f_j^{(5)}(Q_0^2). \tag{4.7}$$

and are related to 4FS PDFs by matching conditions analogous to eq. (2.12),

$$f_i^{(5)}(\mu_b^2) = \sum_{j=g,q',\bar{q}',b,\bar{b}} K_{ij}^{(5)}\left(\tfrac{m_b^2}{\mu_b^2}\right) \otimes f_j^{(4)}(\mu_b^2), \tag{4.8}$$

where we have introduced new matching functions $K_{ij}^{(5)}$, using a label $^{(5)}$ to distinguish them from the previous $K_{ij} \equiv K_{ij}^{(4)}$: in practice they are the same quantities, except that there is one more active flavour, and the 'heavy' index is now b. The scale $\mu_b \sim m_b$ is the threshold scale at which the 4FS PDFs are converted into 5FS PDF, during perturbative evolution.

In analogy with the ACOT expression eq. (2.25), equivalent to FONLL when there is intrinsic beauty, the resummed result can be thus be written as

$$\begin{aligned} F_{\text{ACOT}}(Q^2,m_c^2,m_b^2) &= \sum_{i,j=g,q',\bar{q}',b,\bar{b}} C_i^{(4)}\left(\tfrac{m_c^2}{Q^2},\tfrac{m_b^2}{Q^2}\right) \otimes K_{ij}^{(5)^{-1}}\left(\tfrac{m_b^2}{Q^2}\right) \otimes f_j^{(5)}(Q^2) \\ &= \sum_{i,j=g,q',\bar{q}',b,\bar{b}} C_i^{(5)}\left(\tfrac{m_c^2}{Q^2},\tfrac{m_b^2}{Q^2}\right) \otimes f_j^{(5)}(Q^2), \end{aligned} \tag{4.9}$$

JHEP11(2015)122

or equivalently, using the FONLL construction eqs. (3.19), (3.29), as

$$
\begin{aligned}
&F_{\mathrm{FONLL}}(Q^2, m_c^2, m_b^2) \\
&\quad = \sum_{i=g,q',\bar{q}'} \left[B_i^{(5)}\left(\tfrac{m_c^2}{Q^2}, \tfrac{m_b^2}{Q^2}\right) - B_i^{(5,0)}\left(0, \tfrac{m_b^2}{Q^2}\right) + C_i^{(5)}(0,0) \right] \otimes f_i^{(5)}(Q^2) \\
&\qquad + \sum_{i=b,\bar{b}} C_i^{(5)}(0,0) \otimes f_i^{(5)}(Q^2) \\
&\qquad + \sum_{i,j=b,\bar{b}} \Bigg[C_i^{(4)}\left(\tfrac{m_c^2}{Q^2}, \tfrac{m_b^2}{Q^2}\right) - C_i^{(4,0)}\left(0, \tfrac{m_b^2}{Q^2}\right) \\
&\qquad\qquad - \sum_{m=g,q',\bar{q}'} \left(B_m^{(5)}\left(\tfrac{m_c^2}{Q^2}, \tfrac{m_b^2}{Q^2}\right) - B_m^{(5,0)}\left(0, \tfrac{m_b^2}{Q^2}\right) \right) \otimes K_{mi}^{(5)}\left(\tfrac{m_b^2}{Q^2}\right) \Bigg] \\
&\qquad \otimes {K_{ij}^{(5)}}^{-1}\left(\tfrac{m_b^2}{Q^2}\right) \otimes \left[f_j^{(5)}(Q^2) - \sum_{k,l=g,q',\bar{q}'} K_{jk}^{(5)}\left(\tfrac{m_b^2}{Q^2}\right) \otimes {\tilde{K}_{kl}^{(5)}}^{-1}\left(\tfrac{m_b^2}{Q^2}\right) \otimes f_l^{(5)}(Q^2) \right],
\end{aligned}
\tag{4.10}
$$

where ${\tilde{K}_{ji}^{(5)}}^{-1}$ is the inverse of $K_{ij}^{(5)}$ restricted to the subspace of $i, j = g, q', \bar{q}'$. Note that although the massive coefficient functions now depend on both m_c^2 and m_b^2, all the matching matrices depend only on m_b^2/Q^2, since all the large logarithms of m_c^2/Q^2 were resummed in the previous step. Furthermore in the massless terms, we set both m_b and m_c to zero, since $m_b^2/Q^2 > m_c^2/Q^2$: the corresponding definitions of the massless subtractions are thus

$$
B_i^{(5)}\left(\tfrac{m_c^2}{Q^2}, \tfrac{m_b^2}{Q^2}\right) = \sum_{j=g,q',\bar{q}'} C_j^{(4)}\left(\tfrac{m_c^2}{Q^2}, \tfrac{m_b^2}{Q^2}\right) \otimes {\tilde{K}_{ji}^{(5)}}^{-1}\left(\tfrac{m_b^2}{Q^2}\right) \tag{4.11}
$$

$$
B_i^{(5,0)}\left(0, \tfrac{m_b^2}{Q^2}\right) = \sum_{j=g,q',\bar{q}'} C_j^{(4,0)}\left(0, \tfrac{m_b^2}{Q^2}\right) \otimes {\tilde{K}_{ji}^{(5)}}^{-1}\left(\tfrac{m_b^2}{Q^2}\right) \tag{4.12}
$$

where $C_j^{(4,0)}(0, m_b^2/Q^2)$ is the singular massless limit of $C_j^{(4)}(0, m_b^2/Q^2)$, which can be written as

$$
C_j^{(4,0)}\left(0, \tfrac{m_b^2}{Q^2}\right) = \sum_{j=g,q',\bar{q}',b,\bar{b}} C_j^{(5)}(0,0) \otimes K_{ji}^{(5)}\left(\tfrac{m_b^2}{Q^2}\right). \tag{4.13}
$$

This last equation then gives the correct definition of the matching coefficients $K_{ij}^{(5)}(m_b^2/Q^2)$, corresponding to eq. (2.14). Note that the subtraction of $B_i^{(5,0)}(0, m_b^2/Q^2)$ from the massive $B_i^{(5)}(m_c^2/Q^2, m_b^2/Q^2)$, while removing all the large logarithms $\log(m_b^2/Q^2)$, leaves untouched m_c dependent terms: this is fine since at large Q^2 such terms are always suppressed by m_c^2/Q^2, so can never become large even if enhanced by large logarithms $\log(m_b^2/Q^2)$. Note that if there were no intrinsic charm, $C_j^{(4)}(m_c^2/Q^2, m_b^2/Q^2)$ in eq. (4.11) would be replaced by $\bar{C}_j^{(4)}(m_c^2/Q^2, m_b^2/Q^2)$.

Of course the intrinsic beauty distribution must be very small indeed, suppressed by roughly m_c^2/m_b^2 compared to the intrinsic charm distribution. We can set it to zero by hand by taking as a boundary condition $f_b^{(4)} = f_{\bar{b}}^{(4)} = 0$, or in the massless scheme with

JHEP11(2015)122

five active flavours at $\mu_b^2 \sim m_b^2$ taking

$$f_b^{(5)}(\mu_b^2) = \sum_{j,k=g,q',\bar{q}'} K_{bj}^{(5)}\left(\tfrac{m_b^2}{\mu_b^2}\right) \otimes \tilde{K}_{jk}^{(5)\,-1}\left(\tfrac{m_b^2}{\mu_b^2}\right) \otimes f_k^{(5)}(\mu_b^2); \tag{4.14}$$

the last three lines in eq. (4.10) can then be dropped as they are subleading. Just as in eq. (3.37) we can write them in the compact form

$$\begin{aligned}\Delta F_{\text{FONLL}}\left(Q^2, m_c^2, m_b^2\right) = \sum_{i=b,\bar{b}} \left[C_i^{(5)}\left(\tfrac{m_c^2}{Q^2}, \tfrac{m_b^2}{Q^2}\right) - C_i^{(5)}\left(0,0\right)\right] \\ \otimes \left[f_i^{(5)} - \sum_{k,l=q',\bar{q}',g} K_{ik}^{(5)} \otimes \tilde{K}_{kl}^{(5)\,-1} \otimes f_l^{(5)}\right].\end{aligned} \tag{4.15}$$

From this we see that when there is no intrinsic beauty, in coefficient functions $C_{b,\bar{b}}^{(5)}$ with an incoming bottom quark we can ignore both charm mass dependence and bottom mass dependence, treating both quarks as massless in these diagrams. We then get the S-ACOT expression for the structure function, corresponding to just the first two lines of eq. (4.10):

$$F_{\text{S-ACOT}}(Q^2, m_c^2, m_b^2) = \sum_{i=g,q',\bar{q}',b,\bar{b}} \bar{C}_i^{(5)}\left(\tfrac{m_c^2}{Q^2}, \tfrac{m_c^2}{Q^2}\right) \otimes f_i^{(5)}(Q^2) \tag{4.16}$$

where

$$\bar{C}_i^{(5)}\left(\tfrac{m_c^2}{Q^2}, \tfrac{m_b^2}{Q^2}\right) = \begin{cases} B_i^{(5)}\left(\tfrac{m_c^2}{Q^2}, \tfrac{m_b^2}{Q^2}\right) - B_i^{(5,0)}\left(0, \tfrac{m_b^2}{Q^2}\right) + C_i^{(5)}(0,0), & i = g, q', \bar{q}', \\ C_i^{(5)}(0,0), & i = b, \bar{b}. \end{cases} \tag{4.17}$$

This is in contrast to the charm case: when there is no intrinsic charm, we can ignore the charm mass dependence in the coefficient functions $C_{c,\bar{c}}^{(4)}$ with incoming charm quark but not in principle the bottom mass dependence, arising through virtual loops.

4.2 The top quark

The whole procedure described in section 4.1 can be repeated at the top threshold: here of course it is clear that all the top quarks are generated perturbatively, but the necessity to resum large logarithms of m_t^2/Q^2 remains, and can be a performed by evolution of a top PDF in a 6FS. Here S-ACOT corresponds to setting $m_c = m_b = m_t = 0$ in all diagrams with an incoming top quark: writing $q'' = q, c, b$ and including explicit m_t dependence in the coefficient functions, then with the definitions

$$B_i^{(6)}\left(\tfrac{m_c^2}{Q^2}, \tfrac{m_b^2}{Q^2}, \tfrac{m_t^2}{Q^2}\right) = \sum_{j=g,q'',\bar{q}''} \bar{C}_j^{(5)}\left(\tfrac{m_c^2}{Q^2}, \tfrac{m_b^2}{Q^2}, \tfrac{m_t^2}{Q^2}\right) \otimes \tilde{K}_{ji}^{(6)\,-1}\left(\tfrac{m_t^2}{Q^2}\right) \tag{4.18}$$

$$B_i^{(6,0)}\left(0, 0, \tfrac{m_t^2}{Q^2}\right) = \sum_{j=g,q'',\bar{q}''} C_j^{(5,0)}\left(0, 0, \tfrac{m_t^2}{Q^2}\right) \otimes \tilde{K}_{ji}^{(6)\,-1}\left(\tfrac{m_t^2}{Q^2}\right) \tag{4.19}$$

$$C_j^{(5,0)}\left(0, 0, \tfrac{m_t^2}{Q^2}\right) = \sum_{j=g,q'',\bar{q}'',t,\bar{t}} C_j^{(6)}\left(0,0,0\right) \otimes K_{ji}^{(6)}\left(\tfrac{m_t^2}{Q^2}\right), \tag{4.20}$$

JHEP11(2015)122

we have

$$\bar{C}_i^{(6)}\left(\tfrac{m_c^2}{Q^2},\tfrac{m_b^2}{Q^2},\tfrac{m_t^2}{Q^2}\right)=\begin{cases}B_i^{(6)}\left(\tfrac{m_c^2}{Q^2},\tfrac{m_b^2}{Q^2},\tfrac{m_t^2}{Q^2}\right)-B_i^{(6,0)}\left(0,0,\tfrac{m_t^2}{Q^2}\right)+C_i^{(6)}(0,0,0), & i=g,q'',\bar{q}'',\\ C_i^{(6)}(0,0,0), & i=t,\bar{t}.\end{cases} \tag{4.21}$$

These are the coefficient functions which enter the structure functions

$$F_{\text{S-ACOT}}(Q^2,m_c^2,m_b^2,m_t^2)=\sum_{i=g,q'',\bar{q}'',t,\bar{t}}\bar{C}_i^{(6)}\left(\tfrac{m_c^2}{Q^2},\tfrac{m_c^2}{Q^2},\tfrac{m_t^2}{Q^2}\right)\otimes f_i^{(6)}(Q^2) \tag{4.22}$$

in the 6 flavour S-ACOT scheme.

4.3 Variable flavour number scheme

Putting all this together, we can now construct a variable flavour number scheme with charm, bottom and top quarks. For definiteness we will assume that both bottom and top are generated perturbatively, while charm may have an intrinsic component, which is the setup that is likely to be used when performing a PDF fit. The generic structure function can then be written equivalently in the various different n_f-flavour schemes, leading in principle to identical results (to all orders in α_s). In practice, at finite order, each of them is more appropriate for a specific range of scales. In particular, the 3FS is appropriate only for $Q\sim m_c$, the 4FS for $m_c\lesssim Q\lesssim m_b$ the 5FS for $m_b\lesssim Q\lesssim m_t$ and the 6FS for $Q\gtrsim m_t$. For scales above these ranges the results of a finite order calculation will be spoiled by large unresummed logarithms.

Therefore, one can construct a variable flavour number scheme using each result in its specific region of validity, using the heavy quark thresholds to switch from one result to another:

$$F(Q^2,m_c^2,m_b^2,m_t^2)=\begin{cases}\displaystyle\sum_{i=q,\bar{q},g,c,\bar{c}}C_i^{(3)}\left(\tfrac{m_c^2}{Q^2},\tfrac{m_b^2}{Q^2},\tfrac{m_t^2}{Q^2}\right)\otimes f_i^{(3)}(Q^2) & Q^2<\mu_c^2\\ \displaystyle\sum_{i=q,\bar{q},g,c,\bar{c}}C_i^{(4)}\left(\tfrac{m_c^2}{Q^2},\tfrac{m_b^2}{Q^2},\tfrac{m_t^2}{Q^2}\right)\otimes f_i^{(4)}(Q^2) & \mu_c^2\le Q^2<\mu_b^2\\ \displaystyle\sum_{i=q,\bar{q},g,c,\bar{c},b,\bar{b}}\bar{C}_i^{(5)}\left(\tfrac{m_c^2}{Q^2},\tfrac{m_b^2}{Q^2},\tfrac{m_t^2}{Q^2}\right)\otimes f_i^{(5)}(Q^2) & \mu_b^2\le Q^2<\mu_t^2\\ \displaystyle\sum_{i=q,\bar{q},g,c,\bar{c},b,\bar{b},t,\bar{t}}\bar{C}_i^{(6)}\left(\tfrac{m_c^2}{Q^2},\tfrac{m_b^2}{Q^2},\tfrac{m_t^2}{Q^2}\right)\otimes f_i^{(6)}(Q^2) & \mu_t^2\le Q^2\end{cases} \tag{4.23}$$

Notice that the sum in the 3FS runs also over the charm, to accomodate a possible intrinsic component, while the bottom and top PDFs are generated perturbatively and therefore appear in 5FS and 6FS (bottom) and 6FS (top) only; consistently, we have used the S-ACOT coefficient functions $\bar{C}_i^{(5)}$ and $\bar{C}_i^{(6)}$ in the 5FS and 6FS formulations. In principle the thresholds μ_i do not need to coincide with the analogous thresholds in the perturbative evolution of PDFs, as pointed out e.g. in refs. [43, 44], provided in both cases $\mu_i\sim m_i$, for $i=c,b,t$. Note that for all practical purposes the result in the 3FS can be ignored, since

JHEP11(2015)122

it is not needed to describe the structure function in the region of validity of perturbative QCD, $Q \gtrsim 1\,\text{GeV}$, in particular in the case of a fitted charm PDF.

It is useful to observe that in general discontinuities arise at the thresholds when switching from one scheme to another: these are formally higher order effects, and are ultimately due to interference between the coefficient functions and the matching functions K_{ij} in the perturbative evolution of the PDFs. If a strict expansion in α_s in all the contributions entering each formulation of eq. (4.23) is performed, these higher order interference is eliminated and the VFNS is continuous at threshold [32].

Keeping track of the mass dependence from all three heavy flavors in each coefficient function of eq. (4.23) is rather complicated in general, as one has to deal with calculations with three different masses. The simplest implementation of the VFNS, which can be seen as the minimally improved version of the zero-mass VFNS, is to deal with only one mass scale at a time: for instance in the 5FS the charm is considered massless and the top infinitely heavy. The advantage of this simpler VFNS, which is often used in practical applications,[4] is its simplicity: the problem with it is that there are uncontrolled approximations, with some of the terms that are dropped (for example the effect of bottom quark loops, appearing at $\mathcal{O}(\alpha_s^2)$, just below the bottom threshold, or charm mass effects at around the bottom threshold appearing already at LO with intrinsic charm, or $\mathcal{O}(\alpha_s)$ with perturbative charm) being potentially quite significant.

At low orders in α_s, including the full mass dependence is straightforward. At NNLO (i.e. $\mathcal{O}(\alpha_s^2)$) for contributions with an incoming light parton, and NLO (i.e. $\mathcal{O}(\alpha_s)$) for diagrams with an incoming heavy parton, the diagrams have at most one heavy quark, so the contributions from charm, bottom and top can simply be added. At the next order (N^3LO for contributions with an incoming light parton, and NNLO for diagrams with an incoming heavy parton) diagrams can contain two heavy quarks, possibly of different flavour, and so here the various combinations must be added. The expressions with two masses are already rather complicated (see e.g. [45–47]). Diagrams with three different masses only occur at N^4LO (N^3LO for an incoming heavy parton). Of course the dependence one the top quark mass at scales below or around the bottom mass, and on the charm mass at scales of order the top mass, must both be so small that they can be ignored for all practical purposes.

5 Summary and outlook

We have described the construction of a VFNS for inclusive deep-inelastic processes to any order in perturbation theory, assuming as a starting point the existence of the $\overline{\text{MS}}$ and decoupling (CWZ) renormalization schemes and the massless $\overline{\text{MS}}$ factorization scheme. We further assume the formal existence of all PDFs (whether corresponding to massless or massive partons) at all scales: thresholds are taken account of through the hard coefficient functions. The result we find is essentially unique to any given order in perturbation theory: in particular we obtain the same result from the ACOT procedure and from the FONLL procedure. The reason for this is clear: once the renormalization and factorization scheme

[4] With some observable-dependent exceptions: if one considers for instance bottom production in DIS, this could be described in a 4FS, but clearly keeping the exact bottom mass dependence.

JHEP11(2015)122

is fixed, and thus the PDFs and their evolution, the coefficient functions, containing all the dependence on quark masses, must also be fixed uniquely order by order in perturbation theory.

Of course assuming the formal existence of a charm (or indeed bottom or top) PDF at a given scale is not the same as assuming it is non-negligible, still less observable. The theoretical assumption that the PDFs vanish at threshold can always be implemented through imposition of a boundary condition on the perturbative evolution. This introduces a subleading ambiguity in the formalism, which (being subleading) is small but not resolvable at any given order in perturbation theory. It is only this ambiguity (and differences regarding the ordering of the perturbation expansion) that distinguishes phenomenologically the S-ACOT and FONLL schemes.

The way this ambiguity arises in the usual construction of a VFNS is through the addition of a charm quark PDF in the transition from the massive scheme (valid near threshold) to the massless scheme (valid far above it). This increases the size of the space of active partons: the matching matrix is then not square, and thus has no unique inverse. We have side stepped this ambiguity by adopting a slightly different procedure: we add the charm quark PDF by extending the space of light partons in the massive scheme (where the charm quark PDF does not mix with the light quark and gluon PDFs, and is thus decoupled), and only then match to the massless scheme. The matching matrix is then square, with a unique inverse, and the charm PDF below threshold can be interpreted as 'intrinsic' charm. The limit of no instrinsic charm can then be taken a posteriori, as a theoretical assumption, or instead the intrinsic charm can be determined empirically through a PDF fit - 'fitted charm'. The construction can be trivially extended to beauty and top.

Whether in practice one adopts an empirical procedure (determining the heavy quark distribution through a fit to data), or a theoretical prejudice (setting the heavy quark distribution in the massive scheme to zero) then depends critically on the heavy quark mass, and the precision of existing data. For charm, the charm mass is sufficiently low (at around 1.3 GeV), and the data are sufficiently precise (at the level of a few per cent), that the empirical approach may be necessary. For beauty it is probably at present still best to set the intrinsic distribution to zero, since measurements of a few per mille are out of reach. It is difficult to foresee the need for an intrinsic top distribution, since the top quark decays before there is time for nonperturbative effects to be significant. We hope to perform an empirical determination of intrinsic charm by fitting a charm PDF in the NNPDF formalism in the near future, as set out in ref. [41].

Acknowledgments

We thank Valerio Bertone, Stefano Forte and Juan Rojo for discussions. L. R. and M. B. are supported by an European Research Council Starting Grant 'PDF4BSM'. R. D. B. is funded by an STFC Consolidated Grant ST/J000329/1.

JHEP11(2015)122

A Inversion of matching matrices

Here we give various expressions useful for the inversion of block diagonal matrices, with particular application to the inversion of the matching matrix K when restricted to light and heavy subspaces.

Consider a block diagonal matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{A.1}$$

where A is $m \times m$, D is $n \times n$, and both are invertible with inverses A^{-1} and D^{-1}, while B is $m \times n$ and C is $n \times m$. Then the inverse of this matrix is

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} E^{-1} & -A^{-1}BF^{-1} \\ -D^{-1}CE^{-1} & F^{-1} \end{pmatrix}, \tag{A.2}$$

where

$$E = A - BD^{-1}C, \qquad F = D - CA^{-1}B. \tag{A.3}$$

The following results are also useful:

$$A^{-1}BF^{-1} = E^{-1}BD^{-1}, \qquad D^{-1}CE^{-1} = F^{-1}CA^{-1}, \tag{A.4}$$

and

$$E^{-1} = A^{-1} + A^{-1}BF^{-1}CA^{-1}, \qquad F^{-1} = D^{-1} + D^{-1}CE^{-1}BD^{-1}. \tag{A.5}$$

Applying these general results to the matching matrix K_{ij}, $i, j = g, q, \bar{q}, c, \bar{c}$, separated into light and heavy subspaces, thus with $m = 7$ and $n = 2$, we find that on the diagonal

$$K_{ij}^{-1} = \tilde{K}_{ij}^{-1} + \sum_{k,l=g,q,\bar{q}} \sum_{m,n=c,\bar{c}} \tilde{K}_{ik}^{-1} \otimes K_{km} \otimes K_{mn}^{-1} \otimes K_{nl} \otimes \tilde{K}_{lj}^{-1}, \qquad i, j = g, q, \bar{q} \tag{A.6}$$

$$K_{ij}^{-1} = \tilde{K}_{ij}^{-1} + \sum_{k,l=c,\bar{c}} \sum_{m,n=g,q,\bar{q}} \tilde{K}_{ik}^{-1} \otimes K_{km} \otimes K_{mn}^{-1} \otimes K_{nl} \otimes \tilde{K}_{lj}^{-1}, \qquad i, j = c, \bar{c} \tag{A.7}$$

while the off-diagonal mixing is given by

$$\begin{aligned} K_{ij}^{-1} &= -\sum_{k=g,q,\bar{q}} \sum_{l=c,\bar{c}} \tilde{K}_{ik}^{-1} \otimes K_{kl} \otimes K_{lj}^{-1} \\ &= -\sum_{k=g,q,\bar{q}} \sum_{l=c,\bar{c}} K_{ik}^{-1} \otimes K_{kl} \otimes \tilde{K}_{lj}^{-1}, \qquad i = g, q, \bar{q}, \qquad j = c, \bar{c} \end{aligned} \tag{A.8}$$

$$\begin{aligned} K_{ij}^{-1} &= -\sum_{k=c,\bar{c}} \sum_{l=g,q,\bar{q}} \tilde{K}_{ik}^{-1} \otimes K_{kl} \otimes K_{lj}^{-1} \\ &= -\sum_{k=c,\bar{c}} \sum_{l=g,q,\bar{q}} K_{ik}^{-1} \otimes K_{kl} \otimes \tilde{K}_{lj}^{-1}, \qquad i = c, \bar{c}, \qquad j = g, q, \bar{q}\,. \end{aligned} \tag{A.9}$$

JHEP11(2015)122

In all these expressions the inverses of the matrices $\tilde{K}_{ij}$ are taken in the light and heavy subspaces, while those of K_{ij} are taken in the full space:

$$\sum_{k=g,q,\bar{q},c,\bar{c}} K^{-1}_{ik}K_{kj} = \sum_{k=g,q,\bar{q},c,\bar{c}} K_{ik}K^{-1}_{kj} = \delta_{ij}, \qquad i,j = g,q,\bar{q},c,\bar{c} \tag{A.10}$$

$$\sum_{k=g,q,\bar{q}} \tilde{K}^{-1}_{ik}K_{kj} = \sum_{k=g,q,\bar{q}} K_{ik}\tilde{K}^{-1}_{kj} = \delta_{ij}, \qquad i,j = g,q,\bar{q} \tag{A.11}$$

$$\sum_{k=c,\bar{c}} \tilde{K}^{-1}_{ik}K_{kj} = \sum_{k=c,\bar{c}} K_{ik}\tilde{K}^{-1}_{kj} = \delta_{ij}, \qquad i,j = c,\bar{c}. \tag{A.12}$$

Note in particular that if we restrict to a subspace, K^{-1} is not the inverse of K, rather

$$\sum_{k=g,q,\bar{q}} K^{-1}_{ik}\big[K_{kj} - \sum_{m,n=c,\bar{c}} K_{km}\tilde{K}^{-1}_{mn}K_{mj}\big] = \delta_{ij}, \qquad i,j = g,q,\bar{q} \tag{A.13}$$

$$\sum_{k=c,\bar{c}} \tilde{K}^{-1}_{ik}\big[K_{kj} - \sum_{m,n=g,q,\bar{q}} K_{km}K^{-1}_{mn}K_{mj}\big] = \delta_{ij}, \qquad i,j = c,\bar{c}. \tag{A.14}$$

B Explicit results

We present in this appendix explicit expressions for neutral current DIS in terms of primary ingredients such as massive 3FS and massless 4FS coefficient functions. We restrict our study to the charm structure functions (defined for present purposes to be that part of the structure function in which the struck quark is a charm quark) $F_2^c(Q^2,m_c^2)$, $F_L^c(Q^2,m_c^2)$ and $F_3^c(Q^2,m_c^2)$ since this is where the effect of any intrinsic charm will be most visible: extension to other structure functions (both neutral and charged current) is straightforward.

B.1 F_2^c to NNLO

F_2^c receives contributions from incoming charm quarks (starting at LO), incoming gluons (starting at NLO), and incoming light quarks (starting at NNLO):

$$F_2^c(Q^2,m_c^2) = F_{2,c}^c(Q^2,m_c^2) + F_{2,g}^c(Q^2,m_c^2) + \sum_q F_{2,q}^c(Q^2,m_c^2). \tag{B.1}$$

As explained in section 3, each of these contributions can be further decomposed into the contribution from standard FONLL without intrinsic charm, computed using eq. (3.19), and what may be thought of as an intrinsic charm correction, computed using eq. (3.29):

$$F_{2,i}^c(Q^2,m_c^2) = F_{2,i}^c(Q^2,m_c^2)\big|_{\rm zic} + \Delta F_{2,i}^c(Q^2,m_c^2) \tag{B.2}$$

with $i = c,g,q$. The 'zero intrinsic charm' contributions are then given by (ignoring the damping factor discussed in section 3.7)

$$F_{2,c}^c(Q^2,m_c^2)\big|_{\rm zic} = C_{2,c}^{(4)}\big(0,\alpha_s\big) \otimes f_{c+}^{(4)}(Q^2), \tag{B.3}$$

$$F_{2,g}^c(Q^2,m_c^2)\big|_{\rm zic} = \left[B_{2,g}^{(4)}\left(\tfrac{m_c^2}{Q^2},\alpha_s\right) - B_{2,g}^{(4,0)}\left(\tfrac{m_c^2}{Q^2},\alpha_s\right) + C_{2,g}^{(4)}\big(0,\alpha_s\big)\right] \otimes f_g^{(4)}(Q^2), \tag{B.4}$$

$$F_{2,q}^c(Q^2,m_c^2)\big|_{\rm zic} = \left[B_{2,q}^{(4)}\left(\tfrac{m_c^2}{Q^2},\alpha_s\right) - B_{2,q}^{(4,0)}\left(\tfrac{m_c^2}{Q^2},\alpha_s\right) + C_{2,q}^{(4)}\big(0,\alpha_s\big)\right] \otimes f_{q+}^{(4)}(Q^2), \tag{B.5}$$

JHEP11(2015)122

where

$$f_{q+}^{(4)}(Q^2) = f_q^{(4)}(Q^2) + f_{\bar{q}}^{(4)}(Q^2), \qquad f_{c+}^{(4)}(Q^2) = f_c^{(4)}(Q^2) + f_{\bar{c}}^{(4)}(Q^2). \tag{B.6}$$

In what follows we will expand each of these contributions out to NNLO in $\alpha_s \equiv \alpha_s^{(4)}(Q^2)$, using eq. (2.11), and we will employ the following notation for the expansion of the various coefficient and matching functions:

$$C_i^{(4)}(0, \alpha_s) = \sum_{p=0}^{\infty} \left(\alpha_s^{(4)}(Q^2)\right)^p C_i^{(4),p}(0), \tag{B.7}$$

$$C_i^{(3)}\left(\tfrac{m_c^2}{Q^2}, \alpha_s\right) = \sum_{p=0}^{\infty} \left(\alpha_s^{(4)}(Q^2)\right)^p C_i^{(3),p}\left(\tfrac{m_c^2}{Q^2}\right), \tag{B.8}$$

$$C_i^{(3,0)}\left(\tfrac{m_c^2}{Q^2}, \alpha_s\right) = \sum_{p=0}^{\infty} \left(\alpha_s^{(4)}(Q^2)\right)^p C_i^{(3,0),p}\left(\tfrac{m_c^2}{Q^2}\right), \tag{B.9}$$

$$K_{ij}\left(\tfrac{m_c^2}{Q^2}\right) = \delta_{ij} + \sum_{p=1}^{\infty} \left(\alpha_s^{(4)}(Q^2)\right)^p K_{ij}^p\left(\tfrac{m_c^2}{Q^2}\right), \tag{B.10}$$

The expansions of K_{ij}^{-1} and $\tilde{K}_{ij}^{-1}$ can be straightforwardly related to the expansion of K_{ij}. Many of the $\mathcal{O}(\alpha_s)$ coefficients vanish,

$$K_{qq}^1 = K_{qg}^1 = K_{gq}^1 = K_{cq}^1 = K_{qc}^1 = 0, \tag{B.11}$$

(and all combinations with a quark replaced with an anti quark) further simplifying the expansions. For FONLL-A we need only the $\mathcal{O}(\alpha_s)$ contributions, while for FONLL-C (and B) we also need the $\mathcal{O}(\alpha_s^2)$ contributions for a full implementation. All of the coefficient functions and matching coefficients are known up to $\mathcal{O}(\alpha_s^2)$, except for those whose second index is a heavy quark which are known only up to $\mathcal{O}(\alpha_s)$. In the following, we will omit the argument m_c^2/Q^2 in the expansion coefficients of K_{ij}, while we will keep the argument in the coefficient functions for clarity; we will also use $\alpha_s \equiv \alpha_s^{(4)}(Q^2)$.

Expanding eqs. (B.3)–(B.5) to NNLO, we have

$$F_{2,c}^c(Q^2, m_c^2)\big|_{\text{zic}} = \left[C_{2,c}^{(4),0}(0) + \alpha_s C_{2,c}^{(4),1}(0) + \alpha_s^2 C_{2,c}^{(4),2}(0)\right] \otimes f_{c+}^{(4)}(Q^2), \tag{B.12}$$

$$\begin{aligned} F_{2,g}^c(Q^2, m_c^2)\big|_{\text{zic}} = {} & \alpha_s \left[C_{2,g}^{(3),1}\left(\tfrac{m_c^2}{Q^2}\right) - C_{2,g}^{(3,0),1}\left(\tfrac{m_c^2}{Q^2}\right) + C_{2,g}^{(4),1}(0)\right] \otimes f_g^{(4)}(Q^2) \\ & + \alpha_s^2 \Big[C_{2,g}^{(3),2}\left(\tfrac{m_c^2}{Q^2}\right) - C_{2,g}^{(3,0),2}\left(\tfrac{m_c^2}{Q^2}\right) + C_{2,g}^{(4),2}(0) \\ & \quad - \left(C_{2,g}^{(3),1}\left(\tfrac{m_c^2}{Q^2}\right) - C_{2,g}^{(3,0),1}\left(\tfrac{m_c^2}{Q^2}\right)\right) \otimes K_{gg}^1\Big] \otimes f_g^{(4)}(Q^2), \end{aligned} \tag{B.13}$$

$$F_{2,q}^c(Q^2, m_c^2)\big|_{\text{zic}} = \alpha_s^2 \left[C_{2,q}^{(3),2}\left(\tfrac{m_c^2}{Q^2}\right) - C_{2,q}^{(3,0),2}\left(\tfrac{m_c^2}{Q^2}\right) + C_{2,q}^{(4),2}(0)\right] \otimes f_{q+}^{(4)}(Q^2), \tag{B.14}$$

where we also expanded $B_{2,g}^{(4)}$ (and its massless limit) using the definition eq. (3.17).

JHEP11(2015)122

We now focus on the new terms $\Delta F^c_{2,i}(Q^2, m_c^2)$: using eq. (3.29), and dropping terms which do not contribute at NNLO, we have

$$\Delta F^c_{2,c}(Q^2,m_c^2) = \sum_{i=c,\bar{c}} \left[C^{(3)}_{2,i}\left(\tfrac{m_c^2}{Q^2},\alpha_s\right) - C^{(3,0)}_{2,i}\left(\tfrac{m_c^2}{Q^2},\alpha_s\right)\right] \otimes K^{-1}_{ic} \otimes f^{(4)}_{c+}(Q^2)$$
$$- \left[C^{(3)}_{2,g}\left(\tfrac{m_c^2}{Q^2},\alpha_s\right) - C^{(3,0)}_{2,g}\left(\tfrac{m_c^2}{Q^2},\alpha_s\right)\right] \otimes K_{gc} \otimes f^{(4)}_{c+}(Q^2) + \mathcal{O}(\alpha_s^3), \quad \text{(B.15)}$$

$$\Delta F^c_{2,g}(Q^2,m_c^2) = -2\Big[C^{(3)}_{2,c}\left(\tfrac{m_c^2}{Q^2},\alpha_s\right) - C^{(3,0)}_{2,c}\left(\tfrac{m_c^2}{Q^2},\alpha_s\right)\Big]$$
$$\otimes K^{-1}_{cc} \otimes K_{cg} \otimes K^{-1}_{gg} \otimes f^{(4)}_{g}(Q^2) + \mathcal{O}(\alpha_s^3), \quad \text{(B.16)}$$

$$\Delta F^c_{2,q}(Q^2,m_c^2) = -2\Big[C^{(3)}_{2,c}\left(\tfrac{m_c^2}{Q^2},\alpha_s\right) - C^{(3,0)}_{2,c}\left(\tfrac{m_c^2}{Q^2},\alpha_s\right)\Big] \otimes K_{cq} \otimes f^{(4)}_{q+}(Q^2) + \mathcal{O}(\alpha_s^3), \quad \text{(B.17)}$$

where the second line in eq. (B.15) comes from the B terms in eq. (3.29) which do not contribute at this order in the other two cases. Expanding to NNLO we find

$$\Delta F^c_{2,c}(Q^2,m_c^2) = \Big[C^{(3),0}_{2,c}\left(\tfrac{m_c^2}{Q^2}\right) - C^{(3,0),0}_{2,c}\left(\tfrac{m_c^2}{Q^2}\right)\Big] \otimes f^{(4)}_{c+}(Q^2)$$
$$+ \alpha_s\Big[C^{(3),1}_{2,c}\left(\tfrac{m_c^2}{Q^2}\right) - C^{(3,0),1}_{2,c}\left(\tfrac{m_c^2}{Q^2}\right)$$
$$- \left(C^{(3),0}_{2,c}\left(\tfrac{m_c^2}{Q^2}\right) - C^{(3,0),0}_{2,c}\left(\tfrac{m_c^2}{Q^2}\right)\right) \otimes K^1_{cc}\Big] \otimes f^{(4)}_{c+}(Q^2)$$
$$+ \alpha_s^2\Big[C^{(3),2}_{2,c}\left(\tfrac{m_c^2}{Q^2}\right) - C^{(3,0),2}_{2,c}\left(\tfrac{m_c^2}{Q^2}\right)$$
$$- \left(C^{(3),1}_{2,c}\left(\tfrac{m_c^2}{Q^2}\right) - C^{(3,0),1}_{2,c}\left(\tfrac{m_c^2}{Q^2}\right)\right) \otimes K^1_{cc}$$
$$- \left(C^{(3),0}_{2,c}\left(\tfrac{m_c^2}{Q^2}\right) - C^{(3,0),0}_{2,c}\left(\tfrac{m_c^2}{Q^2}\right)\right) \otimes \left(K^2_{cc} + K^2_{\bar{c}c} - 2K^1_{cg} \otimes K^1_{gc} - K^1_{cc} \otimes K^1_{cc}\right)$$
$$- \left(C^{(3),1}_{2,g}\left(\tfrac{m_c^2}{Q^2}\right) - C^{(3,0),1}_{2,g}\left(\tfrac{m_c^2}{Q^2}\right)\right) \otimes K^1_{gc}\Big] \otimes f^{(4)}_{c+}(Q^2), \quad \text{(B.18)}$$

$$\Delta F^c_{2,g}(Q^2,m_c^2) = -\alpha_s 2\Big[C^{(3),0}_{2,c}\left(\tfrac{m_c^2}{Q^2}\right) - C^{(3,0),0}_{2,c}\left(\tfrac{m_c^2}{Q^2}\right)\Big] \otimes K^1_{cg} \otimes f^{(4)}_{g}(Q^2)$$
$$- \alpha_s^2 2\Big[\left(C^{(3),1}_{2,c}\left(\tfrac{m_c^2}{Q^2}\right) - C^{(3,0),1}_{2,c}\left(\tfrac{m_c^2}{Q^2}\right)\right) \otimes K^1_{cg}$$
$$+ \left(C^{(3),0}_{2,c}\left(\tfrac{m_c^2}{Q^2}\right) - C^{(3,0),0}_{2,c}\left(\tfrac{m_c^2}{Q^2}\right)\right) \otimes \left(K^2_{cg} - K^1_{cg} \otimes K^1_{gg} - K^1_{cc} \otimes K^1_{cg}\right)\Big]$$
$$\otimes f^{(4)}_{g}(Q^2), \quad \text{(B.19)}$$

$$\Delta F^c_{2,q}(Q^2,m_c^2) = -\alpha_s^2 2\Big[C^{(3),0}_{2,c}\left(\tfrac{m_c^2}{Q^2}\right) - C^{(3,0),0}_{2,c}\left(\tfrac{m_c^2}{Q^2}\right)\Big] \otimes K^2_{cq} \otimes f^{(4)}_{q+}(Q^2). \quad \text{(B.20)}$$

Note that the LO coefficient functions are proportional to δ-functions, so their convolutions with the matching terms are trivial. While the LO and NLO terms can all be computed in full, many of the other terms at NNLO cannot yet be computed since the NNLO diagrams with an incoming heavy quark line have yet to be evaluated.

Using eq. (2.14), we can re-express the coefficients $C^{(3,0),k}_{2,i}$ appearing in the previous

JHEP11(2015)122

equations in terms of the expansion coefficients of $C_{2,i}^{(4)}$ and K_{ij}. Up to $\mathcal{O}(\alpha_s^2)$ we have

$$\begin{aligned} C_{2,c}^{(3,0)}\left(\tfrac{m_c^2}{Q^2}\right) &= C_{2,c}^{(4),0}(0) + \alpha_s\left[C_{2,c}^{(4),1}(0) + C_{2,c}^{(4),0}(0)\otimes K_{cc}^1\right] \\ &\quad + \alpha_s^2\left[C_{2,c}^{(4),2}(0) + C_{2,c}^{(4),1}(0)\otimes K_{cc}^1 + C_{2,g}^{(4),1}(0)\otimes K_{gc}^1 + C_{2,c}^{(4),0}(0)\otimes (K_{cc}^2 + K_{\bar{c}c}^2)\right] \\ &\quad + \mathcal{O}(\alpha_s^3), \end{aligned} \tag{B.21}$$

$$\begin{aligned} C_{2,g}^{(3,0)}\left(\tfrac{m_c^2}{Q^2}\right) &= \alpha_s\left[C_{2,g}^{(4),1}(0) + 2C_{2,c}^{(4),0}(0)\otimes K_{cg}^1\right] \\ &\quad + \alpha_s^2\left[C_{2,g}^{(4),2}(0) + C_{2,g}^{(4),1}(0)\otimes K_{gg}^1 + 2C_{2,c}^{(4),1}(0)\otimes K_{cg}^1 + 2C_{2,c}^{(4),0}(0)\otimes K_{cg}^2\right] \\ &\quad + \mathcal{O}(\alpha_s^3), \end{aligned} \tag{B.22}$$

$$C_{2,q}^{(3,0)}\left(\tfrac{m_c^2}{Q^2}\right) = \alpha_s^2\left[C_{2,q}^{(4),2}(0) + 2C_{2,c}^{(4),0}(0)\otimes K_{cq}^2\right] + \mathcal{O}(\alpha_s^3). \tag{B.23}$$

Substituting these into eqs. (B.12)–(B.20) we obtain expressions for $F_{2,i}^c\big|_{\rm zic}$ and $\Delta F_{2,i}^c$ in terms of the expansion coefficients of $C_i^{(3)}(m_c^2/Q^2)$, $C_i^{(4)}(0)$ and $K_{ij}(m_c^2/Q^2)$, which can be regarded as primary quantities.

When we evaluate the sum $F_{2,c}^c(Q^2,m_c^2)\big|_{\rm zic} + \Delta F_{2,c}^c(Q^2,m_c^2)$ there are considerable cancellations between the two terms: in fact we find using eq. (3.32)

$$\begin{aligned} F_{2,c}^c(Q^2,m_c^2) &= C_{2,c}^{(3),0}\left(\tfrac{m_c^2}{Q^2}\right)\otimes f_{c+}^{(4)}(Q^2) \\ &\quad + \alpha_s\left[C_{2,c}^{(3),1}\left(\tfrac{m_c^2}{Q^2}\right) - C_{2,c}^{(3),0}\left(\tfrac{m_c^2}{Q^2}\right)\otimes K_{cc}^1\right]\otimes f_{c+}^{(4)}(Q^2) \\ &\quad + \alpha_s^2\Big[C_{2,c}^{(3),2}\left(\tfrac{m_c^2}{Q^2}\right) - C_{2,c}^{(3),1}\left(\tfrac{m_c^2}{Q^2}\right)\otimes K_{cc}^1 \\ &\qquad - C_{2,c}^{(3),0}\left(\tfrac{m_c^2}{Q^2}\right)\otimes\left(K_{cc}^2 + K_{\bar{c}c}^2 - 2K_{cg}^1\otimes K_{gc}^1 - K_{cc}^1\otimes K_{cc}^1\right) \\ &\qquad - C_{2,g}^{(3),1}\left(\tfrac{m_c^2}{Q^2}\right)\otimes K_{gc}^1\Big]\otimes f_{c+}^{(4)}(Q^2), \end{aligned} \tag{B.24}$$

$$\begin{aligned} F_{2,g}^c(Q^2,m_c^2) &= \alpha_s\left[C_{2,g}^{(3),1}\left(\tfrac{m_c^2}{Q^2}\right) - 2C_{2,c}^{(3),0}\left(\tfrac{m_c^2}{Q^2}\right)\otimes K_{cg}^1\right]\otimes f_g^{(4)}(Q^2) \\ &\quad + \alpha_s^2\Big[C_{2,g}^{(3),2}\left(\tfrac{m_c^2}{Q^2}\right) - C_{2,g}^{(3),1}\left(\tfrac{m_c^2}{Q^2}\right)\otimes K_{gg}^1 - 2C_{2,c}^{(3),1}\left(\tfrac{m_c^2}{Q^2}\right)\otimes K_{cg}^1 \\ &\qquad - 2C_{2,c}^{(3),0}\left(\tfrac{m_c^2}{Q^2}\right)\otimes\left(K_{cg}^2 - K_{cg}^1\otimes K_{gg}^1 - K_{cc}^1\otimes K_{cg}^1\right)\Big]\otimes f_g^{(4)}(Q^2), \end{aligned} \tag{B.25}$$

$$F_{2,q}^c(Q^2,m_c^2) = \alpha_s^2\left[C_{2,q}^{(3),2}\left(\tfrac{m_c^2}{Q^2}\right) - 2C_{2,c}^{(3),0}\left(\tfrac{m_c^2}{Q^2}\right)\otimes K_{cq}^2\right]\otimes f_{q+}^{(4)}(Q^2). \tag{B.26}$$

Note that we can choose to order these expansions in different ways: the FONLL scheme naming FONLL-A includes all terms to $\mathcal{O}(\alpha_s)$, FONLL-C includes all terms to $\mathcal{O}(\alpha_s^2)$, while in FONLL-B one adds the logarithmic parts of the $O(\alpha_s^2)$ contributions not originated by a charm quark to FONLL-A. Note however that the massive NNLO coefficients $C_{2,c}^{(3),2}(m_c^2/Q^2)$ and the NNLO matching functions K_{ic}^2 are not known: for this reason FONLL-B and FONLL-C cannot be fully computed at present if one has to account for a possible intrinsic charm. The best option for going beyond NLO is to use full FONLL-A plus the 'zero intrinsic charm' contribution at higher orders, the ΔF terms being set to zero beyond $\mathcal{O}(\alpha_s)$. Alternatively, as proposed in ref. [42], it is possible to use a χ rescaling on the massless NNLO charm initiated coefficients to mimic the dominant charm mass effects.

JHEP11(2015)122

B.2 F_L^c to NNLO

Precisely the same arguments can be used for other structure functions, in particular F_L^c and F_3^c. Writing

$$F_L^c(Q^2,m_c^2) = F_{L,c}^c(Q^2,m_c^2) + F_{L,g}^c(Q^2,m_c^2) + \sum_q F_{L,q}^c(Q^2,m_c^2). \tag{B.27}$$

we find for the zero intrinsic charm contributions

$$F_{L,c}^c(Q^2,m_c^2)\big|_{\rm zic} = \left[\alpha_s C_{L,c}^{(4),1}(0) + \alpha_s^2 C_{L,c}^{(4),2}(0)\right] \otimes f_{c+}^{(4)}(Q^2), \tag{B.28}$$

$$\begin{aligned} F_{L,g}^c(Q^2,m_c^2)\big|_{\rm zic} &= \alpha_s\left[C_{L,g}^{(3),1}\left(\tfrac{m_c^2}{Q^2}\right) - C_{L,g}^{(3,0),1}\left(\tfrac{m_c^2}{Q^2}\right) + C_{L,g}^{(4),1}(0)\right] \otimes f_g^{(4)}(Q^2) \\ &+ \alpha_s^2\Big[C_{L,g}^{(3),2}\left(\tfrac{m_c^2}{Q^2}\right) - C_{L,g}^{(3,0),2}\left(\tfrac{m_c^2}{Q^2}\right) + C_{L,g}^{(4),2}(0) \\ &\qquad - \left(C_{L,g}^{(3),1}\left(\tfrac{m_c^2}{Q^2}\right) - C_{L,g}^{(3,0),1}\left(\tfrac{m_c^2}{Q^2}\right)\right) \otimes K_{gg}^1\Big] \otimes f_g^{(4)}(Q^2), \end{aligned} \tag{B.29}$$

$$F_{L,q}^c(Q^2,m_c^2)\big|_{\rm zic} = \alpha_s^2\left[C_{L,q}^{(3),2}\left(\tfrac{m_c^2}{Q^2}\right) - C_{L,q}^{(3,0),2}\left(\tfrac{m_c^2}{Q^2}\right) + C_{L,q}^{(4),2}(0)\right] \otimes f_{q+}^{(4)}(Q^2), \tag{B.30}$$

while

$$\begin{aligned} \Delta F_{L,c}^c(Q^2,m_c^2) &= C_{L,c}^{(3),0}\left(\tfrac{m_c^2}{Q^2}\right) \otimes f_{c+}^{(4)}(Q^2) \\ &+ \alpha_s\left[C_{L,c}^{(3),1}\left(\tfrac{m_c^2}{Q^2}\right) - C_{L,c}^{(3,0),1}\left(\tfrac{m_c^2}{Q^2}\right) - C_{L,c}^{(3),0}\left(\tfrac{m_c^2}{Q^2}\right) \otimes K_{cc}^1\right] \otimes f_{c+}^{(4)}(Q^2) \\ &+ \alpha_s^2\Big[C_{L,c}^{(3),2}\left(\tfrac{m_c^2}{Q^2}\right) - C_{L,c}^{(3,0),2}\left(\tfrac{m_c^2}{Q^2}\right) \\ &\quad - \left(C_{L,c}^{(3),1}\left(\tfrac{m_c^2}{Q^2}\right) - C_{L,c}^{(3,0),1}\left(\tfrac{m_c^2}{Q^2}\right)\right) \otimes K_{cc}^1 \\ &\quad - C_{L,c}^{(3),0}\left(\tfrac{m_c^2}{Q^2}\right) \otimes \left(K_{cc}^2 + K_{\bar{c}c}^2 - 2K_{cg}^1 \otimes K_{gc}^1 - K_{cc}^1 \otimes K_{cc}^1\right) \\ &\quad - \left(C_{L,g}^{(3),1}\left(\tfrac{m_c^2}{Q^2}\right) - C_{L,g}^{(3,0),1}\left(\tfrac{m_c^2}{Q^2}\right)\right) \otimes K_{gc}^1\Big] \otimes f_{c+}^{(4)}(Q^2), \end{aligned} \tag{B.31}$$

$$\begin{aligned} \Delta F_{L,g}^c(Q^2,m_c^2) &= -\alpha_s 2 C_{L,c}^{(3),0}\left(\tfrac{m_c^2}{Q^2}\right) \otimes K_{cg}^1 \otimes f_g^{(4)}(Q^2) \\ &- \alpha_s^2 2\Big[\left(C_{L,c}^{(3),1}\left(\tfrac{m_c^2}{Q^2}\right) - C_{L,c}^{(3,0),1}\left(\tfrac{m_c^2}{Q^2}\right)\right) \otimes K_{cg}^1 \\ &\quad + C_{L,c}^{(3),0}\left(\tfrac{m_c^2}{Q^2}\right) \otimes \left(K_{cg}^2 - K_{cg}^1 \otimes K_{gg}^1 - K_{cc}^1 \otimes K_{cg}^1\right)\Big] \otimes f_g^{(4)}(Q^2), \end{aligned} \tag{B.32}$$

$$\Delta F_{L,q}^c(Q^2,m_c^2) = -\alpha_s^2 2 C_{L,c}^{(3),0}\left(\tfrac{m_c^2}{Q^2}\right) \otimes K_{cq}^2 \otimes f_{q+}^{(4)}(Q^2). \tag{B.33}$$

Note that although $C_{L,c}^{(3,0),0} = 0$, $C_{L,c}^{(3),0}(m_c^2/Q^2)$ is nontrivial, though power suppressed. One can use expressions similar to eqs. (B.21)–(B.23) to re-express the structure functions above in terms of only $C_{L,i}^{(3)}$, $C_{L,i}^{(4)}$ and K_{ij}. When we evaluate the sum $F_{L,c}^c(Q^2,m_c^2)\big|_{\rm zic} +$

JHEP11(2015)122

$\Delta F^c_{L,c}(Q^2, m_c^2)$ we find

$$
\begin{aligned}
F^c_{L,c}(Q^2,m_c^2) &= C^{(3),0}_{L,c}\left(\tfrac{m_c^2}{Q^2}\right) \otimes f^{(4)}_{c+}(Q^2) \\
&\quad + \alpha_s\Big[C^{(3),1}_{L,c}\left(\tfrac{m_c^2}{Q^2}\right) - C^{(3),0}_{L,c}\left(\tfrac{m_c^2}{Q^2}\right) \otimes K^1_{cc}\Big] \otimes f^{(4)}_{c+}(Q^2) \\
&\quad + \alpha_s^2\Big[C^{(3),2}_{L,c}\left(\tfrac{m_c^2}{Q^2}\right) - C^{(3),1}_{L,c}\left(\tfrac{m_c^2}{Q^2}\right) \otimes K^1_{cc} \\
&\qquad - C^{(3),0}_{L,c}\left(\tfrac{m_c^2}{Q^2}\right) \otimes \left(K^2_{cc} + K^2_{\bar{c}c} - 2K^1_{cg} \otimes K^1_{gc} - K^1_{cc} \otimes K^1_{cc}\right) \\
&\qquad - C^{(3),1}_{L,g}\left(\tfrac{m_c^2}{Q^2}\right) \otimes K^1_{gc}\Big] \otimes f^{(4)}_{c+}(Q^2), \qquad \text{(B.34)} \\
F^c_{L,g}(Q^2,m_c^2) &= \alpha_s\Big[C^{(3),1}_{L,g}\left(\tfrac{m_c^2}{Q^2}\right) - 2C^{(3),0}_{L,c}\left(\tfrac{m_c^2}{Q^2}\right) \otimes K^1_{cg}\Big] \otimes f^{(4)}_{g}(Q^2) \\
&\quad + \alpha_s^2\Big[C^{(3),2}_{L,g}\left(\tfrac{m_c^2}{Q^2}\right) - C^{(3),1}_{L,g}\left(\tfrac{m_c^2}{Q^2}\right) \otimes K^1_{gg} - 2C^{(3),1}_{L,c}\left(\tfrac{m_c^2}{Q^2}\right) \otimes K^1_{cg} \\
&\qquad - 2C^{(3),0}_{L,c}\left(\tfrac{m_c^2}{Q^2}\right) \otimes \left(K^2_{cg} - K^1_{cg} \otimes K^1_{gg} - K^1_{cc} \otimes K^1_{cg}\right)\Big] \otimes f^{(4)}_{g}(Q^2), \qquad \text{(B.35)} \\
F^c_{L,q}(Q^2,m_c^2) &= \alpha_s^2\Big[C^{(3),2}_{L,q}\left(\tfrac{m_c^2}{Q^2}\right) - 2C^{(3),0}_{L,c}\left(\tfrac{m_c^2}{Q^2}\right) \otimes K^2_{cq}\Big] \otimes f^{(4)}_{q+}(Q^2). \qquad \text{(B.36)}
\end{aligned}
$$

B.3 F_3^c to NNLO

Similarly we write

$$F^c_3(Q^2,m_c^2) = F^c_{3,c}(Q^2,m_c^2) + \sum_q F^c_{3,q}(Q^2,m_c^2) \qquad \text{(B.37)}$$

and

$$f^{(4)}_{q-}(Q^2) = f^{(4)}_q(Q^2) - f^{(4)}_{\bar{q}}(Q^2), \qquad f^{(4)}_{c-}(Q^2) = f^{(4)}_c(Q^2) - f^{(4)}_{\bar{c}}(Q^2). \qquad \text{(B.38)}$$

At NLO the valence distributions $f^{(4)}_{c-}$ and $f^{(4)}_{q-}$ evolve independently, but at NNLO they mix. Thus even if there is no intrinsic charm, a small valence charm distribution $f^{(4)}_{c-}$ will be generated dynamically above threshold. We thus find that

$$
\begin{aligned}
F^c_{3,c}(Q^2,m_c^2)\big|_{\text{zic}} &= \Big[C^{(4),0}_{3,c}(0) + \alpha_s C^{(4),1}_{3,c}(0) + \alpha_s^2 C^{(4),2}_{3,c}(0)\Big] \otimes f^{(4)}_{c-}(Q^2), \qquad \text{(B.39)} \\
F^c_{3,q}(Q^2,m_c^2)\big|_{\text{zic}} &= \alpha_s^2\Big[C^{(3),2}_{3,q}\left(\tfrac{m_c^2}{Q^2}\right) - C^{(3,0),2}_{3,q}\left(\tfrac{m_c^2}{Q^2}\right) + C^{(4),2}_{3,q}(0)\Big] \otimes f^{(4)}_{q-}(Q^2), \qquad \text{(B.40)}
\end{aligned}
$$

JHEP11(2015)122

while

$$\begin{aligned}\Delta F_{3,c}^{c}(Q^2,m_c^2) &= \left[C_{3,c}^{(3),0}\left(\tfrac{m_c^2}{Q^2}\right) - C_{3,c}^{(3,0),0}\left(\tfrac{m_c^2}{Q^2}\right)\right]\otimes f_{c-}^{(4)}(Q^2)\\ &+\alpha_s\Big[C_{3,c}^{(3),1}\left(\tfrac{m_c^2}{Q^2}\right) - C_{3,c}^{(3,0),1}\left(\tfrac{m_c^2}{Q^2}\right)\\ &\quad - \left(C_{3,c}^{(3),0}\left(\tfrac{m_c^2}{Q^2}\right) - C_{3,c}^{(3,0),0}\left(\tfrac{m_c^2}{Q^2}\right)\right)\otimes K_{cc}^{1}\Big]\otimes f_{c-}^{(4)}(Q^2)\\ &+\alpha_s^2\Big[C_{3,c}^{(3),2}\left(\tfrac{m_c^2}{Q^2}\right) - C_{3,c}^{(3,0),2}\left(\tfrac{m_c^2}{Q^2}\right)\\ &\quad - \left(C_{3,c}^{(3),1}\left(\tfrac{m_c^2}{Q^2}\right) - C_{3,c}^{(3,0),1}\left(\tfrac{m_c^2}{Q^2}\right)\right)\otimes K_{cc}^{1}\\ &\quad - \left(C_{3,c}^{(3),0}\left(\tfrac{m_c^2}{Q^2}\right) - C_{3,c}^{(3,0),0}\left(\tfrac{m_c^2}{Q^2}\right)\right)\\ &\qquad \otimes (K_{cc}^{2}+K_{\bar{c}c}^{2}-2K_{cg}^{1}\otimes K_{gc}^{1}-K_{cc}^{1}\otimes K_{cc}^{1})\Big]\otimes f_{c-}^{(4)}(Q^2), \qquad \text{(B.41)}\end{aligned}$$

$$\Delta F_{3,q}^{c}(Q^2,m_c^2) = -2\alpha_s^2\left[C_{3,c}^{(3),0}\left(\tfrac{m_c^2}{Q^2}\right) - C_{3,c}^{(3,0),0}\left(\tfrac{m_c^2}{Q^2}\right)\right]\otimes K_{cq}^{2}\otimes f_{q-}^{(4)}(Q^2). \qquad \text{(B.42)}$$

Now when we evaluate the sum $F_{3,c}^{c}(Q^2,m_c^2|_{\rm zic} + \Delta F_{3,c}^{c}(Q^2,m_c^2)$ we have

$$\begin{aligned}F_{3,c}^{c}(Q^2,m_c^2) &= C_{3,c}^{(3),0}\left(\tfrac{m_c^2}{Q^2}\right)\otimes f_{c-}^{(4)}(Q^2)\\ &+\alpha_s\left[C_{3,c}^{(3),1}\left(\tfrac{m_c^2}{Q^2}\right) - C_{3,c}^{(3),0}\left(\tfrac{m_c^2}{Q^2}\right)\otimes K_{cc}^{1}\right]\otimes f_{c-}^{(4)}(Q^2)\\ &+\alpha_s^2\Big[C_{3,c}^{(3),2}\left(\tfrac{m_c^2}{Q^2}\right) - C_{3,c}^{(3),1}\left(\tfrac{m_c^2}{Q^2}\right)\otimes K_{cc}^{1}\\ &\quad - C_{3,c}^{(3),0}\left(\tfrac{m_c^2}{Q^2}\right)\otimes (K_{cc}^{2}+K_{\bar{c}c}^{2}-2K_{cg}^{1}\otimes K_{gc}^{1}-K_{cc}^{1}\otimes K_{cc}^{1})\Big]\otimes f_{c-}^{(4)}(Q^2),\end{aligned} \qquad \text{(B.43)}$$

$$F_{3,q}^{c}(Q^2,m_c^2) = \alpha_s^2\left[C_{3,q}^{(3),2}\left(\tfrac{m_c^2}{Q^2}\right) - 2C_{3,c}^{(3),0}\left(\tfrac{m_c^2}{Q^2}\right)\otimes K_{cq}^{2}\right]\otimes f_{q-}^{(4)}(Q^2). \qquad \text{(B.44)}$$

References

[1] NNPDF collaboration, R.D. Ball et al., *Parton distributions for the LHC Run II*, *JHEP* **04** (2015) 040 [arXiv:1410.8849] [INSPIRE].

[2] L.A. Harland-Lang, A.D. Martin, P. Motylinski and R.S. Thorne, *Parton distributions in the LHC era: MMHT 2014 PDFs*, *Eur. Phys. J.* **C 75** (2015) 204 [arXiv:1412.3989] [INSPIRE].

[3] P.M. Nadolsky et al., *Implications of CTEQ global analysis for collider observables*, *Phys. Rev.* **D 78** (2008) 013004 [arXiv:0802.0007] [INSPIRE].

[4] S. Alekhin, J. Blumlein and S. Moch, *The ABM parton distributions tuned to LHC data*, *Phys. Rev.* **D 89** (2014) 054028 [arXiv:1310.3059] [INSPIRE].

[5] R.D. Ball, *Global Parton Distributions for the LHC Run II*, arXiv:1507.07891 [INSPIRE].

[6] M.A.G. Aivazis, F.I. Olness and W.-K. Tung, *Leptoproduction of heavy quarks. 1. General formalism and kinematics of charged current and neutral current production processes*, *Phys. Rev.* **D 50** (1994) 3085 [hep-ph/9312318] [INSPIRE].

JHEP11(2015)122

[7] M.A.G. Aivazis, J.C. Collins, F.I. Olness and W.-K. Tung, *Leptoproduction of heavy quarks. 2. A Unified QCD formulation of charged and neutral current processes from fixed target to collider energies*, *Phys. Rev.* **D 50** (1994) 3102 [hep-ph/9312319] [INSPIRE].

[8] J.C. Collins, *Proof of factorization for diffractive hard scattering*, *Phys. Rev.* **D 57** (1998) 3051 [*Erratum ibid.* **D 61** (2000) 019902] [hep-ph/9709499] [INSPIRE].

[9] M. Krämer, F.I. Olness and D.E. Soper, *Treatment of heavy quarks in deeply inelastic scattering*, *Phys. Rev.* **D 62** (2000) 096007 [hep-ph/0003035] [INSPIRE].

[10] R.S. Thorne and R.G. Roberts, *An Ordered analysis of heavy flavor production in deep inelastic scattering*, *Phys. Rev.* **D 57** (1998) 6871 [hep-ph/9709442] [INSPIRE].

[11] R.S. Thorne, *A Variable-flavor number scheme for NNLO*, *Phys. Rev.* **D 73** (2006) 054019 [hep-ph/0601245] [INSPIRE].

[12] M. Buza, Y. Matiounine, J. Smith and W.L. van Neerven, *Charm electroproduction viewed in the variable flavor number scheme versus fixed order perturbation theory*, *Eur. Phys. J.* **C 1** (1998) 301 [hep-ph/9612398] [INSPIRE].

[13] M. Cacciari, M. Greco and P. Nason, *The p_T spectrum in heavy flavor hadroproduction*, *JHEP* **05** (1998) 007 [hep-ph/9803400] [INSPIRE].

[14] S. Forte, E. Laenen, P. Nason and J. Rojo, *Heavy quarks in deep-inelastic scattering*, *Nucl. Phys.* **B 834** (2010) 116 [arXiv:1001.2312] [INSPIRE].

[15] S.J. Brodsky, P. Hoyer, C. Peterson and N. Sakai, *The Intrinsic Charm of the Proton*, *Phys. Lett.* **B 93** (1980) 451 [INSPIRE].

[16] R. Vogt and S.J. Brodsky, *QCD and intrinsic heavy quark predictions for leading charm and beauty hadroproduction*, *Nucl. Phys.* **B 438** (1995) 261 [hep-ph/9405236] [INSPIRE].

[17] J. Pumplin, *Light-cone models for intrinsic charm and bottom*, *Phys. Rev.* **D 73** (2006) 114015 [hep-ph/0508184] [INSPIRE].

[18] European Muon collaboration, J.J. Aubert et al., *An Experimental Limit on the Intrinsic Charm Component of the Nucleon*, *Phys. Lett.* **B 110** (1982) 73 [INSPIRE].

[19] B.W. Harris, J. Smith and R. Vogt, *Reanalysis of the EMC charm production data with extrinsic and intrinsic charm at NLO*, *Nucl. Phys.* **B 461** (1996) 181 [hep-ph/9508403] [INSPIRE].

[20] S. Dulat et al., *Intrinsic Charm Parton Distribution Functions from CTEQ-TEA Global Analysis*, *Phys. Rev.* **D 89** (2014) 073004 [arXiv:1309.0025] [INSPIRE].

[21] P. Jimenez-Delgado, T.J. Hobbs, J.T. Londergan and W. Melnitchouk, *New limits on intrinsic charm in the nucleon from global analysis of parton distributions*, *Phys. Rev. Lett.* **114** (2015) 082002 [arXiv:1408.1708] [INSPIRE].

[22] J.C. Collins, *Hard scattering factorization with heavy quarks: A General treatment*, *Phys. Rev.* **D 58** (1998) 094002 [hep-ph/9806259] [INSPIRE].

[23] M. Guzzi, P.M. Nadolsky, H.-L. Lai and C.P. Yuan, *General-Mass Treatment for Deep Inelastic Scattering at Two-Loop Accuracy*, *Phys. Rev.* **D 86** (2012) 053005 [arXiv:1108.5112] [INSPIRE].

[24] W.-K. Tung, S. Kretzer and C. Schmidt, *Open heavy flavor production in QCD: Conceptual framework and implementation issues*, *J. Phys.* **G 28** (2002) 983 [hep-ph/0110247] [INSPIRE].

JHEP11(2015)122

[25] R.S. Thorne and W.K. Tung, *PQCD Formulations with Heavy Quark Masses and Global Analysis*, arXiv:0809.0714 [INSPIRE].

[26] J.C. Collins, F. Wilczek and A. Zee, *Low-Energy Manifestations of Heavy Particles: Application to the Neutral Current*, *Phys. Rev.* **D 18** (1978) 242 [INSPIRE].

[27] E. Laenen, S. Riemersma, J. Smith and W.L. van Neerven, *Complete O (α_s) corrections to heavy flavor structure functions in electroproduction*, *Nucl. Phys.* **B 392** (1993) 162 [INSPIRE].

[28] M. Buza, Y. Matiounine, J. Smith, R. Migneron and W.L. van Neerven, *Heavy quark coefficient functions at asymptotic values $Q^2 \gg m^2$*, *Nucl. Phys.* **B 472** (1996) 611 [hep-ph/9601302] [INSPIRE].

[29] E. Hoffmann and R. Moore, *Subleading Contributions to the Intrinsic Charm of the Nucleon*, *Z. Phys.* **C 20** (1983) 71 [INSPIRE].

[30] S. Kretzer and I. Schienbein, *Heavy quark initiated contributions to deep inelastic structure functions*, *Phys. Rev.* **D 58** (1998) 094035 [hep-ph/9805233] [INSPIRE].

[31] J.A.M. Vermaseren, A. Vogt and S. Moch, *The Third-order QCD corrections to deep-inelastic scattering by photon exchange*, *Nucl. Phys.* **B 724** (2005) 3 [hep-ph/0504242] [INSPIRE].

[32] M. Bonvini, A.S. Papanastasiou and F.J. Tackmann, *Resummation and Matching of b-quark Mass Effects in $b\bar{b}H$ Production*, arXiv:1508.03288 [INSPIRE].

[33] K.G. Chetyrkin, B.A. Kniehl and M. Steinhauser, *Strong coupling constant with flavor thresholds at four loops in the MS scheme*, *Phys. Rev. Lett.* **79** (1997) 2184 [hep-ph/9706430] [INSPIRE].

[34] I. Bierenbaum, J. Blumlein and S. Klein, *The Gluonic Operator Matrix Elements at $O(\alpha_s^2)$ for DIS Heavy Flavor Production*, *Phys. Lett.* **B 672** (2009) 401 [arXiv:0901.0669] [INSPIRE].

[35] J. Ablinger, J. Blumlein, S. Klein, C. Schneider and F. Wissbrock, *The $O(\alpha_s^3)$ Massive Operator Matrix Elements of $O(n_f)$ for the Structure Function $F_2(x, Q^2)$ and Transversity*, *Nucl. Phys.* **B 844** (2011) 26 [arXiv:1008.3347] [INSPIRE].

[36] J. Ablinger et al., *The Transition Matrix Element $A_{gq}(N)$ of the Variable Flavor Number Scheme at $O(\alpha_s^3)$*, *Nucl. Phys.* **B 882** (2014) 263 [arXiv:1402.0359] [INSPIRE].

[37] J. Ablinger et al., *The 3-Loop Non-Singlet Heavy Flavor Contributions and Anomalous Dimensions for the Structure Function $F_2(x, Q^2)$ and Transversity*, *Nucl. Phys.* **B 886** (2014) 733 [arXiv:1406.4654] [INSPIRE].

[38] J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. von Manteuffel and C. Schneider, *The 3-loop pure singlet heavy flavor contributions to the structure function $F_2(x, Q^2)$ and the anomalous dimension*, *Nucl. Phys.* **B 890** (2014) 48 [arXiv:1409.1135] [INSPIRE].

[39] B. Mele and P. Nason, *The Fragmentation function for heavy quarks in QCD*, *Nucl. Phys.* **B 361** (1991) 626 [INSPIRE].

[40] SM AND NLO MULTILEG WORKING GROUP collaboration, T. Binoth et al., *The SM and NLO Multileg Working Group: Summary report*, arXiv:1003.1241 [INSPIRE].

[41] R.D. Ball et al., *Intrinsic charm in a matched general-mass scheme*, arXiv:1510.00009 [INSPIRE].

JHEP11(2015)122

[42] T. Stavreva et al., *Heavy Quark Production in the ACOT Scheme at NNLO and N3LO*, *Phys. Rev.* **D 85** (2012) 114014 [arXiv:1203.0282] [INSPIRE].

[43] F. Olness and I. Schienbein, *Heavy Quarks: Lessons Learned from HERA and Tevatron*, *Nucl. Phys. Proc. Suppl.* **191** (2009) 44 [arXiv:0812.3371] [INSPIRE].

[44] A. Kusina et al., *Hybrid scheme for heavy flavors: Merging the fixed flavor number scheme and variable flavor number scheme*, *Phys. Rev.* **D 88** (2013) 074032 [arXiv:1306.6553] [INSPIRE].

[45] J. Ablinger, J. Blümlein, A. Hasselhuhn, S. Klein, C. Schneider and F. Wissbrock, *New Heavy Flavor Contributions to the DIS Structure Function $F_2(x, Q^2)$ at $\mathcal{O}(\alpha_s^3)$*, arXiv:1202.2700 [INSPIRE].

[46] J. Ablinger, J. Blümlein, S. Klein, C. Schneider and F. Wissbrock, *3-Loop Heavy Flavor Corrections to DIS with two Massive Fermion Lines*, arXiv:1106.5937 [INSPIRE].

[47] J. Blümlein, A. DeFreitas and C. Schneider, *Higher Order Heavy Quark Corrections to Deep-Inelastic Scattering*, *Nucl. Part. Phys. Proc.* **261-262** (2015) 185 [arXiv:1411.5669] [INSPIRE].

JHEP11(2015)122