CERN pbar-p UA1 collaboration

UA1/TN/82-24

21 June 1982

FAST TRAJECTORY FITTING IN UNIFORM FIELD

V. Karimaki

1. Introduction

Event reconstruction problem in the UA1 central detector is divided in the following three stages :

- pattern recognition
- trajectory fitting
- vertex finding

In this note we consider the trajectory fitting problem.

The UA1 magnetic field in the region of the central detector is uniform to a high extent so that pure helix fit is adequate at least for the part of the trajectory which is inside one chamber volume. Neglecting possible correlation between measurement of the current division coordinate (along field or Z-axis) and drift time coordinate (in XY-plane) the helix fit can be performed in two parts :

circle fit in XY-plane

straight line fit in SZ-plane

where S is the projected path length along the trajectory in XY-plane.

Since straight line fit is trivial we discuss here only the circle fitting problem. The central detector measures typically many thousands of points per event. It is therefore desirable to find fast fitting methods.

2. Circle fitting

2.1. Problem definition

We define the χ^2-function to be minimized as

$$\chi^2 = \sum w_i \varepsilon_i^2 \qquad (1)$$

where ε_i is a measurement residual normal to the trajectory :

$$\varepsilon_i = R-\sqrt{(X_i-A)^2 + (Y_i-B)^2}. \qquad (2)$$

R is the circle radius and (A,B) are coordinates of the circle centre. An exact solution of the minimization problem for the three parameters R,A,B involves a complicated iteration procedure and CPU time consuming computation of square roots for thousands of points per event. In the following we describe a method in which the problem is solved in explicit form in an approximation which leads to negligible systematic error in all practical cases.

2.2. Approximation

The approximation we make is based on the fact that we can always find a point whose distance from the circle is much smaller than the radius of curvature. The first measured point (X_1,Y_1) will do. Let us define the following quantities :

d = distance of the point (X_1,Y_1) from the circle $(=\varepsilon_1)$

φ = angle of tangent at the closest point of approach to point (X_1,Y_1)

$\rho = (R-d)^{-1}$

Then it is easy to show that equation (2) can be written in the form :

$$\varepsilon_i-d = [-r_i^2 + 2(R-d)x_i\sin\varphi - 2(R-d)y_i\cos\varphi]/(2R-d-\varepsilon_i) \qquad (2')$$

where $x_i=X_i-X_1$, $y_i=Y_i-Y_1$, $r_i^2=x_i^2+y_i^2$.

Since $d << R$, $\varepsilon_i << R$ we can approximate that $2R - d - \varepsilon_i \simeq 2(R-d)$ so that :

$$\varepsilon_i = -1/2\,\rho\, r_i^2 + d + x_i\sin\varphi - y_i\cos\varphi \qquad (3)$$

with a good accuracy. For example in the case $|\varepsilon_i|\simeq|d|\simeq 300\ \mu$ and $R \simeq 1$ m the error we make on ε_i is about $2\varepsilon_i^2/2R \simeq 0.1\ \mu$ which is highly negligible.

Notice that the residual (3) is now linear in the two parameters ρ and d and simple trigonometric expression in φ. The minimization problem of eq. (1) can be solved in this approximation in explicit form for parameters ρ, d and φ as we show in the following.

2.3. Explicit solution

To minimize (1) with residuals (3) we have to solve the following three simultaneous equations :

$$\sum w_i \varepsilon_i \partial \varepsilon_i / \partial \rho = 0$$

$$\sum w_i \varepsilon_i \partial \varepsilon_i / \partial d = 0$$

$$\sum w_i \varepsilon_i \partial \varepsilon_i / \partial \varphi = 0$$

The calculation is fairly lengthy but straight-forward and the solution reads as follows :

$$\varphi = 1/2 \ \mathrm{atan}(2q_1/q_2)$$

$$\rho = 2(\sin\varphi C_{13} - \cos\varphi C_{23})/C_{33} \qquad (4)$$

$$d = 1/2\rho\langle r_i^2\rangle - \sin\varphi\langle x_i\rangle + \cos\varphi\langle y_i\rangle \qquad \text{where}$$

$$q_1 = C_{33}C_{12} - C_{13}C_{23}$$

$$q_2 = C_{33}(C_{11}-C_{22}) - C_{13}^2 + C_{23}^2$$

and the coefficients $C_{k\ell}$ are the statistical covariances of the measurements $x_i = X_i - X_1$, $y_i = Y_i - Y_1$, $r_i^2 = x_i^2 + y_i^2$:

$$C_{11} = \langle x_i^2\rangle - \langle x_i\rangle^2$$

$$C_{12} = \langle x_i y_i\rangle - \langle x_i\rangle\langle y_i\rangle$$

$$C_{13} = \langle x_i r_i^2\rangle - \langle x_i\rangle \langle r_i^2\rangle$$

$$C_{22} = \langle y_i^2\rangle - \langle y_i\rangle^2$$

$$C_{23} = \langle y_i r_i^2\rangle - \langle y_i\rangle\langle r_i^2\rangle$$

$$C_{33} = \langle r_i^4\rangle - \langle r_i^2\rangle^2$$

The 180° direction ambiguity for the angle φ is solved so as to have $\rho = (R-d)^{-1}$ positive. Notice that d (shortest distance between the point (X_1,Y_1) and the circle) is then positive for (X_1,Y_1) inside the circle and negative outside the circle.

4. Covariance matrix for parameters ρ, φ, d

In order to simplify the notation let us identify the parameters ρ, φ and d with p_1, p_2, p_3. The covariance matrix is then

$$C_{ij} = cov(p_i,p_j) = (\sum \partial\varepsilon_i/\partial p_i \partial\varepsilon_j/\partial p_j)^{-1} \equiv (W_{ij})^{-1} \qquad (5)$$

that is the inverse of the symmetric matrix whose elements we give below in an explicit form :

$$W_{11} = \sum w_i r_i^4$$

$$W_{12} = -\cos\varphi \sum w_i x_i r_i^2 - \sin\varphi \sum w_i y_i r_i^2$$

$$W_{13} = \cos^2\varphi \sum w_i x_i^2 + \sin 2\varphi \sum w_i x_i y_i + \sin^2\varphi \sum W_i y_i^2$$

$$W_{22} = - \sum w_i r_i^2$$

$$W_{23} = \cos\varphi \sum w_i x_i + \sin\varphi \sum w_i y_i$$

$$W_{33} = \sum w_i$$

We emphasize here that the covariance matrix thus formulated is related to the reference point (X_1,Y_1).

2.5. Propagation of parameters and covariance matrix

In practical applications one is not usually interested in the parameters related to the first measured point but rather at the vertex of the particle origin. Below we give practical formulation for propagation of related quantities.

Suppose that given are the parameters ρ, φ and d and their covariance matrix C all related to a fixed point (X_1,Y_1). The problem is to propagate all these quantities to another fixed point (X,Y) (e.g. the primary vertex). The new parameters are :

$$\rho' = 1/\sqrt{(X-A)^2 + (Y-B)^2}$$

$$\varphi' = \operatorname{atan}\,[-(X-A)/(Y-B)] \tag{6}$$

$$d' = 1/\rho + d - 1/\rho$$

where (A,B) is the circle centre :

$$A = X_1 + S\,\sin\varphi/\rho$$

$$B = Y_1 - S\,\cos\varphi/\rho$$

and S is the sign of curvature.
Calculation of d' with the above formula leads, however, to truncation problems on the computer in the case of very small curvature tracks. A better formula reads :

$$d' = H[-((X-X_s)^2+(Y-Y_s)^2)/R + 2S(\sin\varphi(X-X_s)-\cos\varphi(Y-Y_s))]$$

where $H = R(R + 1/\rho')^{-1}$, $R = 1/\rho + d$

and (X_s,Y_s) is a point on the circle closest to the point (X_1,Y_1) :

$$X_s = X_1 - S\sin\varphi d$$

$$Y_s = Y_1 + S\cos\varphi d$$

Propagation of the covariance matrix is made by the matrix transformation :

$$C' = JCJ^T \tag{7}$$

where J is the Jacobian derivative matrix of the transformation equations (6). Below we list the elements of J in a form suitable for computations :

$$J_{11} = \partial\rho'/\partial\rho = \cos(\varphi'-\varphi)(\rho'/\rho)^2$$

$$J_{12} = \partial\rho'/\partial\varphi = -\sin(\varphi'-\varphi)\;\rho'^2/\rho$$

$$J_{13} = \partial\rho'/\partial d = 0$$

$$J_{21} = \partial\varphi'/\partial\rho = \sin(\varphi'-\varphi)\ \rho'/\rho^2$$

$$J_{22} = \partial\varphi'/\partial\varphi = \cos(\varphi'-\varphi)\ \rho'/\rho$$

$$J_{23} = \partial\varphi'/\partial d = 0$$

$$J_{31} = \partial d'/\partial\rho = [\cos(\varphi'-\varphi) - 1]/\rho^2$$

$$J_{32} = \partial d'/\partial\varphi = -\sin(\varphi'-\varphi)/\rho$$

$$J_{33} = \partial d'/\partial d = 1$$

2.6. Straight track limit

It is amusing to notice that in the straight track limit ≡ 0 the least squares solution (4) reduces to :

$$\varphi = 1/2\ \mathrm{atan}\ (2\ C_{12})/(C_{11}-C_{22})$$

$$d = -\sin\varphi\ \langle x_i\rangle + \cos\varphi\langle y_i\rangle \qquad (8)$$

This means that in principle one can use the same code with some additional tests for straight line fitting as for circle fitting.

3. Fitting code

A routine named CIRCLF is coded using the method described above. It can be found from the UA1 PAM-file IMA and is fully documented in line. It is attached with a routine (PNTCIR) which performs propagation of parameters and covariance matrix as described in section 2.5.

The code has been tested with Monte Carlo point generation and subsequent fitting. In the table below we give typical timing numbers on IBM 3081 (without point rejection).

Number of points	time/track (ms)	time/point (μs)
10	0.4	40
50	1.4	28
100	2.4	24
200	4.4	22

The error estimation code was also tested by comparing the fitted quantities and their estimated errors with time values used in MC generation. In Fig. 1a we plot the quantity :

$$[(1/R)_{true} - (1/R)_{fit}]/\Delta(1/R)_{fit}$$

where $$(1/R)_{fit} = 1/(1/\rho_{fit} + d_{fit})$$

and $\Delta(1/R)_{fit}$ is given by error propagation using covariance matrix of ρ_{fit}, d_{fit}.
Similarly in Fig. 1b and 1c we plot the quantities :

$$(\varphi_{true} - \varphi_{fit})/\Delta\varphi_{fit}$$

$$(d_{true} - d_{fit})/\Delta d_{fit}$$

respectively (recall that d is the distance of the first point from the circle and $1/\rho$ is the distance of the first point from the circle centre). All three plots follow normal distributions indicating that both fitting and error estimation perform well.

4. Summary

Fast helix-circle fitting method is described. Least squares circle fitting is solved in explicit form the solution involving computation of covariance matrix of the quantities x_i, y_i, $r_i^2 \equiv x_i^2 + y_i^2$ where $x_i=X_i-X_1$, $y_i=Y_i-Y_1$ and (X_i,Y_i) are the measured points. Error estimation and propagation is discussed. Monte Carlo test results on parameter fitting and error estimation are presented .

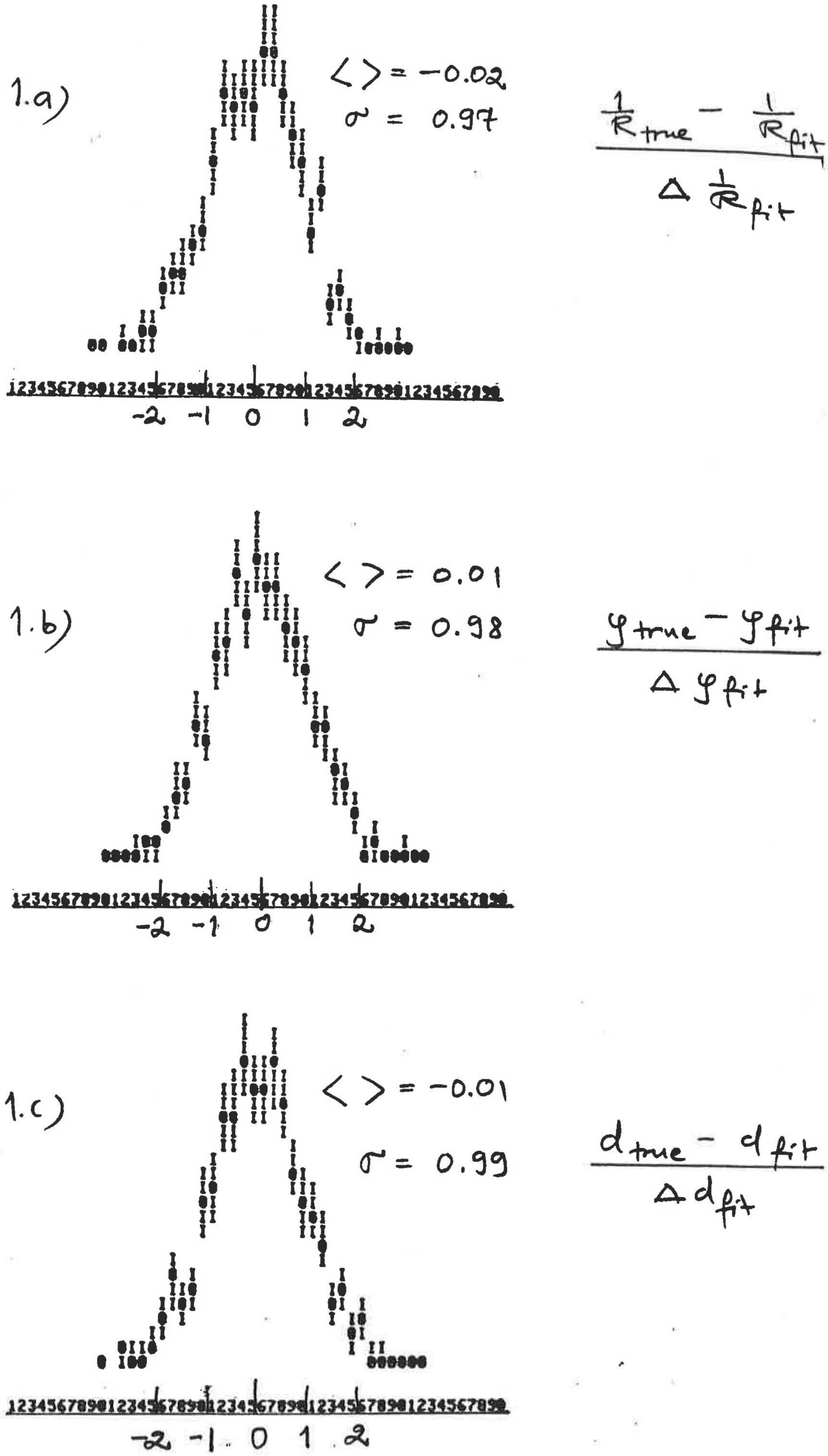

Fig.1. MC test for error estimation