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STRING FIELD THEORY IN MINIMAL MODEL BACKGROUNDS AND NON-PERTURBATIVE 2-DIMENSIONAL GRAVITY

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ABSTRACT

The classical phase space of free closed string field theory in the background of (p,q) minimal models is studied. It is shown that in the limit $q \to \infty$ for fixed p, this becomes the phase space of $p-1$ massless chiral bosons on a two-dimensional target space, twisted by Z_p. It is argued that in the interacting theory, the bosons remain free and massless in the limit, but the non-linear gauge symmetries of string field theory require the imposition of W_p-algebra conditions on the Hilbert space, allowing a single physical state. The wave function for this state is the KdV τ-function associated to non-perturbative two-dimensional gravity in the matrix-model approach.

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1. Introduction

The study of a double scaling limit of large-N matrix models has led to non-perturbative results[1]-[5] that are believed to correspond to the solution of two-dimensional gravity coupled to $c \leq 1$ conformal field theories[6]. These results are summarised in the "string equation" and generalised KdV flows, which have recently been reformulated in an elegant way using loop equations[7].

For the one-matrix models, which are identified with $(2, q)$ minimal models coupled to gravity, the loop equations imply a linear set of constraints on the square root of the partition function, which appear as Virasoro conditions for a Z_2-twisted free chiral boson in two dimensions. It was conjectured[7] that for $p-1$ matrix models, corresponding to (p, q) minimal models, one finds W-algebra constraints for $p-1$ free bosons twisted by Z_p. These constraints are believed to uniquely determine a particular KdV τ-function in each case.

Starting from the formulation of closed-string field theory (CSFT)[8]-[10] in arbitrary backgrounds[11], Sen has recently investigated strings in minimal model backgrounds, by including the Liouville sector in the CSFT Hilbert space[12]. In particular, he obtains the scaling relations of two-dimensional gravity[13] [14] in this framework. He also shows that, for large values of the Liouville field (the region of weak Liouville coupling) a certain gauge symmetry of CSFT can be identified with the L_0 constraint of Ref.[7].

One may ask whether the $p-1$ free bosons and their W-symmetries have a physical interpretation in string theory, and whether the complete results coming from matrix models in the double scaling limit can be derived explicitly within the formalism of string field theory. In the present work we propose an approach to this problem, and initiate this programme by obtaining a spacetime interpretation for string field theory in minimal model backgrounds. We will find that the W-generators emerge as *spacetime* symmetries on a two-dimensional target space. We suggest how these spacetime symmetries could arise from the gauge symmetries of CSFT, following the proposals made in Refs.[7],[12].

Explicitly, we derive the classical phase space of free CSFT in (p, q) backgrounds. This corresponds to studying the quadratic semiclassical approximation around the given backgrounds. We find that in the limit $q \to \infty$ for fixed p, the classical phase

space becomes isomorphic to that of a collection of $p-1$ free massless bosons in a two-dimensional cylindrical target space. There are two unusual properties: the bosons are *chiral* on the target space, and *twisted* around the cylinder by elements of Z_p. The effective field theories of these bosons manifestly have the Virasoro generators and the higher spin currents $W_3, \cdots, W_p$ as their symmetries.

More precisely, the target spacetime in the $q \to \infty$ limit is a $1+1$ dimensional Minkowski cylinder, with the Liouville mode φ_L representing the time direction. The compact space direction is described by the free boson φ_M which appears in the Feigin-Fuchs representation of minimal models[15][16]. For (p, q) models with finite p and q, the allowed momenta of this boson are restricted to a finite set of values, but this becomes an infinite set in the limit $q \to \infty$, and it is then that the spacetime interpretation holds as described above. The fact that pairs of vertex operators with different momenta correspond to the same primary field implies that the spacetime modes are purely chiral, propagating only along one component of the light cone.

In the last section we give a qualitative sketch of how we expect to obtain the results of Ref.[7] entirely from string field theory. We argue, by analogy with the quantization of Chern-Simons gauge theory in 2+1 dimensions[17]-[20] that the classical phase space of interacting string field theory is *smaller* than that for the free case. Covariant interacting CSFT has been shown to have non-polynomial interactions and gauge symmetries. It is generally believed that for large values of the Liouville field, the interaction terms in the effective action are suppressed. We will argue that the same effect is achieved by taking $q \to \infty$, so that the $p-1$ bosons remain free and massless in the presence of interactions. However, the non-linear terms in the gauge transformation laws remain, and we suggest that these can be identified with Virasoro and higher W-algebra symmetries on the free bosons. These symmetries would then have to be imposed on the Hilbert space, leading to precisely the conditions obtained in Ref.[7].

This offers the interpretation that the classical phase space of CSFT in minimal backgrounds reduces to a single point, and upon quantization, the associated wave function is just the τ-function of Ref.[7]. Stated in this way, CSFT in minimal model backgrounds is a topological field theory. In this context, the fact that the limit $c \to -\infty$ contains all the information about the theory (about which we will have more to say in the final section) simply means that the semiclassical approximation is exact, a well-known feature of topological field theories. ($c \to -\infty$ is the semi-classical limit for Liouville theory, where c is the matter central charge[21]).

We mention here some recent papers which may be related to the results presented below. The relationship of strings in minimal model backgrounds to topological field theory has been discussed in [22]-[27]. String field theory in non-critical backgrounds is the subject of Refs.[12] and [28]-[32]. Liouville field theory is discussed in detail in Ref.[33].

2. Free CSFT in $(2,q)$ Backgrounds

Consider the Hilbert space $\mathcal{H} = \mathcal{H}_M \otimes \mathcal{H}_L \otimes \mathcal{H}_{gh}$, where $\mathcal{H}_M$ is the Hilbert space of a "matter" conformal field theory, $\mathcal{H}_L$ is the Hilbert space of the corresponding Liouville theory, and $\mathcal{H}_{gh}$ is the Hilbert space of the b, c ghost system.

Define the BRS charge

$$\begin{aligned} Q_B \equiv \oint : c(z)\left(T^{(M)}(z) + T^{(L)}(z) + \tfrac{1}{2}T^{(gh)}(z)\right) : \\ + \oint : \bar{c}(\bar{z})\left(\bar{T}^{(M)}(\bar{z}) + \bar{T}^{(L)}(\bar{z}) + \tfrac{1}{2}\bar{T}^{(gh)}(\bar{z})\right) : \end{aligned} \tag{1}$$

Here, $T^{(M,L,gh)}$ are the holomorphic energy-momentum tensors in the matter, Liouville and ghost sectors, and the barred objects are their anti-holomorphic counterparts.

The holomorphic ghost fields have mode expansions

$$\begin{aligned} c(z) &= \sum c_n z^{1-n} \\ b(z) &= \sum b_n z^{-2-n} \end{aligned} \tag{2}$$

We also define the combinations $c_0^{\pm} \equiv \frac{1}{\sqrt{2}}(c_0 \pm \bar{c}_0)$ and similarly for $b_0^{\pm}$.

In what follows, we will restrict ourselves to the subspace of $\mathcal{H}$ which satisfies the constraint $L_0 = \bar{L}_0$, and will use the same symbol $\mathcal{H}$ for this subspace.

The classical string field is a state $|\Psi\rangle$ of ghost number 3 in $\mathcal{H}$. The classical free field equation is

$$Q_B b_0^- |\Psi\rangle = 0 \tag{3}$$

which follows from the action[34]

$$S = \tfrac{1}{2}\langle\Psi| Q_B b_0^- |\Psi\rangle \tag{4}$$

This action is invariant under the gauge transformations

$$\delta(b_0^- |\Psi\rangle) = Q_B(b_0^- |\Lambda\rangle \tag{5}$$

where $|\Lambda\rangle$ is a state in $\mathcal{H}$ of ghost number 2.

The classical phase space is the space of solutions of the field equation modulo gauge transformations. Because of the b_0^- in (3), this is the cohomology space $H^2(Q_B)$ of Q_B on states of ghost number 2 not containing c_0^-. Evidently, a class of representatives of this cohomology space is

$$b_0^- |\Psi\rangle = |\text{primary}\rangle_{(M+L)} \otimes c_1\bar{c}_1 |0\rangle_{gh} \tag{6}$$

where the matter+Liouville primary satisfies $L_0 + \bar{L}_0 = 0$. One can convince oneself that this in fact spans $H^2(Q_B)$. This is equivalent to the statement that any representative of an element in $H^2(Q_B)$ can be brought to the ground state in the ghost sector by a BRS transformation.

It will be convenient in what follows to expand the string field in terms of the c-ghost zero modes:

$$|\Psi\rangle = c_0^- |\psi^-\rangle + c_0^- c_0^+ |\psi^{-+}\rangle + |\psi\rangle + c_0^+ |\psi^+\rangle \tag{7}$$

The presence of b_0^- in the action implies that the last two terms are irrelevant, and we will henceforth discard them.

We now gauge fix the theory in the Siegel gauge

$$b_0^+ |\Psi\rangle = 0, \quad \text{i.e.,} \quad |\psi^{-+}\rangle = 0 \tag{8}$$

In this gauge, the field equation (3) can be decomposed into an equation of motion

$$(L_0 + \bar{L}_0)|\psi^-\rangle = 0 \tag{9}$$

which follows from the gauge-fixed action

$$S_{\text{fixed}} = \frac{1}{\sqrt{2}}\langle\psi^-| c_0^- c_0^+ (L_0 + \bar{L}_0)|\psi^-\rangle \tag{10}$$

and a constraint equation

$$\tilde{Q}_B |\psi^-\rangle = 0 \tag{11}$$

where

$$\tilde{Q}_B \equiv Q_B - \sqrt{2}c_0^+(L_0 + \bar{L}_0) \tag{12}$$

One can alternatively describe the classical phase space in the following way. Consider the gauge-fixed action restricted to the space of field configurations which satisfy the constraint equation (11), modulo those gauge transformations which preserve the gauge. We call this space $\mathcal{H}_c$. Elements of $\mathcal{H}_c$ are represented by the "off-shell" primaries, namely, states of the form (6), but not necessarily satisfying $L_0 + \bar{L}_0 = 0$. The classical phase space is then given by the solutions of the field equation of the gauge-fixed action (10).

Let $\varphi_L(z,\bar{z})$ be the Liouville field, and $\varphi_M(z,\bar{z})$ be the scalar field in terms of which the minimal models can be described via the Feigin-Fuchs representation. The analytic stress-energy tensors of the two theories are usually written[16],[33]

$$\begin{aligned} T^{(L)} &= -\tfrac{1}{2}\partial\varphi_L\partial\varphi_L - \frac{Q_L}{2}\partial^2\varphi_L \\ T^{(M)} &= -\tfrac{1}{2}\partial\varphi_M\partial\varphi_M + i\frac{Q_M}{2}\partial^2\varphi_M \end{aligned} \tag{13}$$

where the coefficients Q_L, Q_M of the anomalous terms are given in terms of the minimal model central charge c by

$$\begin{aligned} Q_L &= \sqrt{\frac{25-c}{3}} \\ Q_M &= \sqrt{\frac{1-c}{3}} \end{aligned} \tag{14}$$

The corresponding Virasoro operators $L_0^{(L)}$, $L_0^{(M)}$ have the expressions

$$\begin{aligned} L_0^{(L)} &= \tfrac{1}{2}P_L(P_L - iQ_L) \ + \ \text{oscillators} \\ L_0^{(M)} &= \tfrac{1}{2}P_M(P_M - Q_M) \ + \ \text{oscillators} \end{aligned} \tag{15}$$

A standard difficulty in Liouville theory coupled to $c < 1$ matter is that the vertex operators relevant for the construction of physical fields of conformal dimension $(1,1)$ have imaginary momentum P_L, leading to non-normalizable states. In this situation the momentum operator is effectively anti-Hermitian, and one has to resort to various prescriptions (see Refs.[33],[12] and references therein) to make sense of the amplitudes.

The above problems are closely associated with the fact that the Liouville mode is usually quantized as a spacelike string coordinate for $c < 25$. For minimal model

backgrounds, this leads to the additional difficulty that there is no available timelike coordinate on the target space, in which case BRS quantization would not have its usual interpretation, of implementing the decoupling of negative-norm states which originate from the Minkowski nature of spacetime.

In a study of the BRS cohomology in minimal model backgrounds, Itoh[35] has in fact proposed that the Liouville mode should be taken to be *timelike*. He then demonstrates that the BRS quantization procedure works in the usual way to remove unphysical states.

Since covariant string field theory is crucially based on BRS quantization, we will follow this approach here. Making the transformation $\varphi_L \to -i\varphi_L$, we find

$$T^{(L)}(z) = \tfrac{1}{2}\partial\varphi_L\partial\varphi_L + i\frac{Q_L}{2}\partial^2\varphi_L \tag{16}$$

whence

$$L_0^{(L)} = -\tfrac{1}{2}P_L(P_L - Q_L) \ + \ \text{oscillators} \tag{17}$$

and the Liouville vertex operators required to make physical fields now have real momenta, while the Liouville mode behaves as a timelike coordinate.

It should be stressed that the resulting formulae for anomalous dimensions are precisely the same as those obtained by treating the Liouville mode as Euclidean and with imaginary momenta. However, the correspondence between correlation functions computed in this approach (in which the Liouville momentum must satisfy a δ-function conservation law, taking into account the Liouville anomaly) and correlation functions in matrix models is somewhat subtle, and we will return to this point in Section 4.

Now define a two-dimensional spacetime coordinate $X^\mu(z,\bar{z}) = (\varphi_L(z,\bar{z}), \varphi_M(z,\bar{z}))$ and a two-vector $Q_\mu = (Q_L, Q_M)$. The combined Liouville-matter system then has a stress-energy tensor

$$T(z) = -\tfrac{1}{2}\eta_{\mu\nu}\partial X^\mu\partial X^\nu + \frac{i}{2}Q_\mu\partial^2 X^\mu \tag{18}$$

Seen in this way, non-critical string theory in the background of a minimal model resembles the critical string, but with an anomalous term in $T(z)$ proportional to a given vector Q_μ. (This was observed in Ref.[35] in a slightly different formalism). However, it is important to note an additional constraint on the spectrum of this theory, compared to that of the critical string. In the spacelike sector (the mode

$X^1 = \varphi_M$), not all vertex operators are allowed primaries, which means that not all momenta are present. The ones that are allowed correspond to the cohomology of the BRS-like screening charges defined by Felder[16], which is the set of primaries of the (p,q) minimal model. For finite (p,q) there are finitely many allowed momenta, and one does not expect the coordinate $X^1(z,\bar{z})$ to resemble a physical space dimension. To get a spacetime interpretation, we take p fixed and let q tend to infinity. There are infinitely many primaries in this limit, and one finds that X^1 corresponds to a compact space direction across which the physical particles have non-trivial boundary conditions.

To be more explicit, let us take the matter sector to be a $(2,q)$ minimal model. The central charge is

$$c = 1 - 3\frac{(q-2)^2}{q} \tag{19}$$

and, from Eq.(14), we find

$$Q_\mu = \left(\frac{q+2}{\sqrt{q}}, \frac{q-2}{\sqrt{q}}\right) \tag{20}$$

A general vertex operator

$$V_{\alpha_\mu} = :e^{i\alpha_\mu X^\mu}: \tag{21}$$

has conformal dimension

$$h_{\alpha_\mu} = \tfrac{1}{2}\alpha_\mu(\alpha^\mu - Q^\mu) \tag{22}$$

Restricting our attention to the matter sector, define $\alpha_\pm$ by

$$\tfrac{1}{2}\alpha_\pm(\alpha_\pm - Q_M) = 1 \tag{23}$$

which leads to

$$\alpha_- = -\frac{2}{\sqrt{q}}, \qquad \alpha_+ = \sqrt{q} \tag{24}$$

It is shown by Felder that the physical vertex operators for this theory are given by the cohomology of the BRS-like operator

$$J_+ = \oint dz V_{\alpha_+}(\varphi_M) \tag{25}$$

and the vertex operators in the cohomology are

$$\begin{aligned} V_{\alpha^{(j)}}(\varphi_M) &= :e^{i\alpha^{(j)}\varphi_M}: \\ \alpha^{(j)} &= \tfrac{1}{2}(1-j)\alpha_- = \frac{j-1}{\sqrt{q}} \\ j &= 1,2,\cdots q-1 \end{aligned} \tag{26}$$

The operator for momentum α^j is identified with that for α^{q-j}, so that only half the fields in the set are independent.

The Liouville vertex operators are defined so as to produce a field of dimension $(1,1)$ when multiplied with these matter vertex operators. Eq.(22) then shows that the combined vertex operators are

$$\begin{aligned} V_{\alpha_\mu^{(j)}} &= :e^{i\alpha_\mu^{(j)}X^\mu}: \\ \alpha_\mu^{(j)} &= \left(\frac{j+1}{\sqrt{q}}, \frac{j-1}{\sqrt{q}}\right), \quad \left(\frac{q+1-j}{\sqrt{q}}, \frac{j-1}{\sqrt{q}}\right) \end{aligned} \tag{27}$$

We take the independent degrees of freedom to correspond to $j > \frac{q}{2}$, and choose the first momentum vector above for the combined system. This will be seen to have the interpretation of a particle moving forward in time.

Consider now the gauge-fixed string field theory action, Eq.(10). Let us expand it in terms of a basis of primary states of the form

$$|\Phi^-(P)\rangle = |P_\mu\rangle_{L+M} \otimes c_1\bar{c}_1|0\rangle \tag{28}$$

We will treat P_μ for the time being as an arbitrary Minkowski two-momentum, on which the constraint Eq.(26) will ultimately be imposed. With this constraint, the states above are precisely the off-shell primaries of the Liouville-matter system, and the equations of motion of the gauge-fixed action will determine the time (Liouville) component of the two-momentum as an on-shell condition.

Expand an arbitrary configuration

$$|\psi^-\rangle = \int d^2P\, \chi(P)\,|\Phi^-(P)\rangle \tag{29}$$

Then

$$\begin{aligned} S_{\text{fixed}} &= \tfrac{1}{2}\langle\psi^-|\,c_0^- c_0^+(L_0 + \bar{L}_0)\,|\psi^-\rangle \\ &= \int d^2P\, d^2P'\, \langle\Phi^-(Q-P')|\,\chi^*(P')(P_\mu(P^\mu - Q^\mu) - 2)\,\chi(P)\,|\Phi^-(P)\rangle \end{aligned} \tag{30}$$

Reality of the string field corresponds to $\chi^*(P) = \chi(Q-P)$, and after Fourier-transforming

$$\chi(P) = \int d^2X\, e^{-iP.X}\chi(X) \tag{31}$$

the gauge-fixed action reduces to

$$S_{\rm fixed} = \tfrac{1}{2}\int d^2X\ e^{-iQ_\mu X^\mu}(\partial_\mu\chi\partial^\mu\chi - 2\chi^2) \tag{32}$$

But for the anomalous term in front of the Lagrangian, this would describe a tachyonic particle, subject to the constraints coming from the BRS screening charge. Note, however, that the conjugation property of $\chi(P)$ above implies that $\chi^*(X) = e^{-iQ.X}\chi(X)$, which means that $\tilde\chi(X) = e^{-i\frac{Q}{2}.X}\chi(X)$ is real. In terms of this new variable the action reduces to

$$\begin{aligned} S_{\rm fixed} &= \tfrac{1}{2}\int d^2X\ (\partial_\mu\tilde\chi\partial^\mu\tilde\chi + (\frac{Q^2}{4} - 2)\tilde\chi^2) \\ &= \tfrac{1}{2}\int d^2X\ \partial_\mu\tilde\chi\partial^\mu\tilde\chi \end{aligned} \tag{33}$$

since from Eq.(20) we have $Q^2 = 8$.

This action must now be subjected to the constraint that the space momentum modes are those appropriate to the $(2,q)$ minimal model. For finite q, this allows the momenta in Eq.(27), with $j > \frac{q}{2}$. In terms of the variable $\tilde\chi$ defined above, whose momentum is $\tilde P_\mu \equiv P_\mu - \frac{1}{2}Q_\mu$, this becomes

$$\tilde P_\mu = \frac{1}{\sqrt{q}}(j - \frac{q}{2}, j - \frac{q}{2}) \tag{34}$$

Defining an integer n by $n + \frac{1}{2} = j - \frac{q}{2}$, we find that the physical modes have

$$\begin{aligned} \tilde P_\mu &= \frac{1}{\sqrt{q}}(n + \tfrac{1}{2}, n + \tfrac{1}{2}) \\ 0 &\le n \le \frac{q-3}{2} \end{aligned} \tag{35}$$

The Minkowski-space action for the real field $\tilde\chi$ has particle solutions with these momenta, but also antiparticle solutions with the opposite momenta $-\tilde P_\mu$, which represent the same degrees of freedom, since the field is real. Thus

$$\tilde\chi(X^\mu) = \sum_{n=-\frac{q-1}{2}}^{\frac{q-3}{2}} b_{n+\frac{1}{2}}\, e^{\frac{i}{\sqrt{q}}(n+\frac{1}{2})X^+} \tag{36}$$

where $X^\pm = X^1 \pm X^0$ and $b_{n+\frac{1}{2}} = b^*_{-n-\frac{1}{2}}$.

Now take the limit $q \to \infty$. To obtain a sensible limit, one must first re-scale the coordinates and momenta:

$$\begin{aligned} X^\mu &= \sqrt{q}\hat X^\mu \\ P_\mu &= \frac{1}{\sqrt{q}}\hat P_\mu \end{aligned} \tag{37}$$

In these variables, the allowed values of the space component of the rescaled momentum become all half-integers, as can be seen by taking the limit in Eq.(36) above.

Note that the space coordinate X^1 in the Feigin-Fuchs representation is compact (we drop the hat on the re-scaled coordinates and momenta in the rest of this section). Thus the two-dimensional target space is, after rescaling, a Minkowski cylinder of unit radius, on which the boson is constrained to propagate chirally. The additional information is that, because the momenta are half-integer, the scalar is *antiperiodic* around the cylinder:

$$\tilde\chi(X^0, X^1 + 2\pi) = -\tilde\chi(X^0, X^1) \tag{38}$$

Thus the classical phase space of free string field theory in the background of $(2,\infty)$ conformal matter is isomorphic to the phase space of a Z_2-twisted free chiral scalar.

Evidently, these two dimensions represent the target space for the "effective" theory. The time dimension has its origin in the continuous momentum mode of Liouville theory. The compact spatial dimension is contributed by the infinitely many primaries of the conformal matter as $q \to \infty$. Note that although in this limit the matter central charge $c \to -\infty$ from Eq.(19), the "effective" central charge in a non-unitary theory should be taken to be $c_{\rm eff} = c - 24h_{\rm min}$ where $h_{\rm min}$ is the most negative conformal dimension in the theory. In the large-q limit, $c_{\rm eff} \to 1$, which seems to be the physical origin of this extra dimension.

This leads one to ask if in some sense string field theory in minimal model backgrounds is related to the $c = 1$ string[36]-[42]. To understand this, consider again the combined matter-Liouville stress-energy tensor of Eq.(18). We can perform a Lorentz transformation on the two-dimensional coordinate X^μ:

$$\begin{aligned} X'^0 &= \frac{1}{\sqrt{8}}Q_\mu X^\mu \\ X'^1 &= \frac{1}{\sqrt{8}}\epsilon_\mu X^\mu \end{aligned} \tag{39}$$

where $\epsilon_\mu = (Q_1, Q_0)$, so that $\epsilon_\mu Q^\mu = 0$. The stress-energy tensor then has the same form as before, but with a new anomaly two-vector

$$Q'_\mu = (\sqrt{8}, 0) \tag{40}$$

But this just means that the new coordinate X'^1 is non-anomalous and describes a $c = 1$ theory, while the Liouville anomaly is precisely that appropriate to $c = 1$ matter. The only unusual thing about the system, viewed in this way, is that the coordinates have very bizarre boundary conditions (the combination of the new space and time coordinates which equals the old space coordinate is a periodic variable). For finite q, there are also constraints on a combination of the space and time components of the momenta, coming from Felder's BRS charge, although this restriction ultimately just leads to chirality in the target space. Thus we see that in the limit, string theory in minimal model backgrounds is just the same as string theory in a $c = 1$ background with unusual boundary conditions on the world-sheet coordinates. It would be interesting to know if this is also related to the suggestion made in Ref.[12] that different minimal models correspond to different boundary conditions on the string field.

3. (p,q) Backgrounds

For (p,q) backgrounds, the anomaly 2-vector is

$$Q_\mu = \sqrt{\frac{2}{pq}}\,(q+p,\ q-p) \tag{41}$$

and the numbers $\alpha_\pm$ defined in Eq.(23) are

$$\alpha_- = -\sqrt{\frac{2p}{q}}, \quad \alpha_+ = \sqrt{\frac{2q}{p}} \tag{42}$$

In the matter sector, the BRS cohomology of Felder allows the vertex operators

$$\begin{aligned} V_{\alpha^{(j,k)}} &= :e^{i\alpha^{(j,k)}\varphi_M}: \\ \alpha^{(j,k)} &= \tfrac{1}{2}(1-j)\alpha_- + \tfrac{1}{2}(1-k)\alpha_+ \\ j &= 1,2,\cdots,q-1 \\ k &= 1,2,\cdots,p-1 \end{aligned} \tag{43}$$

The vertex operators for $\alpha^{(j,k)}$ and $\alpha^{(q-j,p-k)}$ are identified. We represent the independent physical degrees of freedom as those with $pj > qk$.

The combined matter-Liouville vertex operators are then

$$\begin{aligned} V_{\alpha_\mu^{(j,k)}} &= :e^{i\alpha_\mu^{(j,k)}X^\mu}: \\ \alpha_\mu^{(j,k)} &= \sqrt{\frac{2}{pq}}\left(\frac{p}{2}(j+1) + \frac{q}{2}(1-k), \frac{p}{2}(j-1) + \frac{q}{2}(1-k)\right) \end{aligned} \tag{44}$$

Defining

$$\begin{aligned} n &= j - [\frac{qk}{p}] - 1 \\ r &= qk \bmod p \end{aligned} \tag{45}$$

where $[x]$ denotes the integer part of x, we can rewrite these momenta as

$$\alpha_\mu^{(r,n)} = \tfrac{1}{2}Q_\mu + \sqrt{\frac{p}{2q}}\left(n+1-\frac{r}{p}, n+1-\frac{r}{p}\right) \tag{46}$$

with

$$\begin{aligned} r &= 1,\cdots,p-1 \\ n &= 0,\cdots,q-2-[\frac{qk}{p}] \end{aligned} \tag{47}$$

Repeating the steps from Eq.(28) to Eq.(33) for this case, we find that the physical degrees of freedom in the limit $q \to \infty$ are a collection of $p-1$ real bosons , whose action can be conveniently written in terms of $\frac{p-1}{2}$ complex fields if p is odd:

$$S_{fixed} = \sum_{r=1}^{\frac{p-1}{2}} \int d^2X\ \partial_\mu \tilde{\chi}^{(r)*} \partial^\mu \tilde{\chi}^{(r)} \tag{48}$$

After re-scaling

$$\begin{aligned} X^\mu &\to \sqrt{\frac{q}{2p}}\ X^\mu \\ P_\mu &\to \sqrt{\frac{2p}{q}}\ P_\mu \end{aligned} \tag{49}$$

we find

$$\tilde{\chi}^{(r)} = \sum_{n\in Z} b_{n-\frac{r}{p}} e^{i(n-\frac{r}{p})X^+} \tag{50}$$

with b_n complex, from which it follows that the boundary condition around the cylinder is

$$\tilde{\chi}^{(r)}(X^0, X^1 + 2\pi) = e^{-2\pi i \frac{r}{p}} \tilde{\chi}^{(r)}(X^0, X^1) \tag{51}$$

For even p there are $\frac{p-2}{2}$ complex fields of this type and an extra real boson which is antiperiodic around the cylinder.

Taking p odd and defining $m=\frac{p-1}{2}$, the symmetries of the effective action Eq.(48) are the Virasoro symmetry generated by

$$T_{++}\sim\sum_{i=1}^{m}\partial_+\tilde{\chi}^{(m-i)}\partial_+\tilde{\chi}^{(i)} \tag{52}$$

and the higher-spin symmetries generated by

$$W^{(N)}_{++\cdots+}\sim\sum_{i_1=1}^{m}\cdots\sum_{i_N=1}^{m}\partial_+\tilde{\chi}^{(i_1)}\cdots\partial_+\tilde{\chi}^{(i_N)}\delta(i_1+\cdots i_N) \tag{53}$$

with $N=3,\cdots,m$. It is easy to see that these generators are all single valued across the cylinder.

Let us also examine the limit of the above results when $p\to\infty$. We label the bosons above by the phase $\frac{r}{p}$ that they acquire on going around the cylinder. In the limit, this label becomes a continuous variable between 0 and 1, which we denote y. Then the effective action is

$$S_{fixed}=\int d^2X\int_0^1 dy\quad\partial_\mu\tilde{\chi}(X^\mu,1-y)\partial^\mu\tilde{\chi}(X^\mu,y) \tag{54}$$

with the boundary conditions

$$\begin{aligned}\tilde{\chi}(X^0,X^1+2\pi,y)&=e^{-2\pi iy}\tilde{\chi}(X^0,X^1,y)\\ \tilde{\chi}(X^0,X^1,y+1)&=\tilde{\chi}(X^0,X^1,y)\end{aligned} \tag{55}$$

This seems to be an unusual realisation (because of the boundary conditions above) of the W_∞ algebra[43]. The effective action begins to look three-dimensional, but there are no derivatives in the third direction. It would be interesting to know if this arises from some Chern-Simons-like action in three spacetime dimensions.

Thus we have shown that the classical phase space of free closed string field theory in the background of (p,q) minimal models, in the limit $q\to\infty$ for fixed p, is that of a set of $p-1$ real massless chiral bosons, each twisted by a different element of Z_p. The action for these bosons manifestly has conformal symmetry and higher-spin symmetries $W^{(3)}\ldots W^{(p)}$.

4. Nonlinear Gauge Symmetries and Topological Invariance

So far we have dealt with free CSFT. This corresponds to studying the quadratic approximation of the full theory about the given backgrounds. The exact theory has an infinite set of interaction terms[8]-[10]. To define these, we introduce the appropriate N-string vertex. Given N states $|A_1\rangle,\cdots,|A_N\rangle$ in the Hilbert space $\mathcal{H}$, let $A_1(z,\bar{z}),\cdots A_N(z,\bar{z})$ be the fields which create these states. Then the N-string vertex

$$\{A_1\cdots A_N\}:\mathcal{H}^{\otimes N}\to C \tag{56}$$

is given by

$$\begin{aligned}\{A_1\cdots A_N\}\equiv-\int\prod_{i=1}^{2N-6}(d\tau_i)\Big\langle\Big(\prod_{i=1}^{2N-6}\Big(\int(\eta_{i\bar{z}}{}^{z}b(z)+\bar{\eta}_{iz}^{\bar{z}}\bar{b}(\bar{z}))d^2z\Big)\Big)&\\ f_1^{(N)}\circ(b_0^-A_1(0))f_2^{(N)}\circ(b_0^-A_2(0))\cdots f_N^{(N)}\circ(b_0^-A_N(0))\Big\rangle&\end{aligned} \tag{57}$$

Here $f_i^{(N)}$ is a given set of conformal maps which define the Strebel[44] representation of the N-punctured sphere, $f\circ A$ denotes the conformal transform of the field A under f, $\eta_{i\bar{z}}{}^{z}$ are the Beltrami differentials and τ_i the moduli for the N-punctured sphere.

We also introduce the gluing operator

$$|[A_2\cdots A_N]\rangle:\mathcal{H}^{\otimes N-1}\to\mathcal{H} \tag{58}$$

given by

$$\{A_1\cdots A_N\}=(-1)^{n_1+1}\langle A_1|\,|[A_2\cdots A_N]\rangle \tag{59}$$

where n_1 is the ghost number of A_1, $\langle A_1|\equiv\langle 0|I\circ A_1(0)$ and I is the $SL(2,C)$ conformal map $I(z)=-1/z$.

With these definitions the classical action of interacting CSFT is

$$S=\tfrac{1}{2}\langle\Psi|Q_Bb_0^-|\Psi\rangle+\sum_{N=3}^{\infty}\frac{g^{N-2}}{N!}\{\Psi^N\} \tag{60}$$

and the field equation is

$$Q_Bb_0^-|\Psi\rangle+\sum_{N=3}^{\infty}\frac{g^{N-2}}{(N-1)!}|[\Psi^{N-1}]\rangle=0 \tag{61}$$

These are invariant under the non-linear gauge transformation

$$\delta(b_0^- |\Psi\rangle) = Q_B(b_0^- |\Lambda\rangle) + \sum_{N=3}^{\infty} \frac{g^{N-2}}{(N-2)!} |[\Psi^{N-2}\Lambda]\rangle \tag{62}$$

where $|\Lambda\rangle$ is a gauge transformation parameter of ghost number 2. The gauge-invariant classical phase space is the space of solutions of Eq.(61) modulo the gauge transformations (62). Clearly it is a complicated problem to find this space in general. However, some remarkable simplifications occur in the $q \to \infty$ limit of minimal models, as we now discuss.

Our goal is a canonical quantization of the interacting CSFT in Eq.(60). The conventional approach to the Hamiltonian quantization of a gauge theory is to pick a gauge - in our case the Siegel gauge - quantize the resulting field theory, and then impose the constraint equations, associated to the gauge trasformations which preserve the gauge, as operator equations. This selects the subspace of physical states. However, for topological gauge theories, an alternative quantization scheme has proven to be more useful[17]-[20],[45]. In this, one first imposes the gauge constraints at the classical level, and obtains a gauge invariant classical phase space which is drastically smaller than the original one, since most of the degrees of freedom have been eliminated. The quantization of this gauge invariant phase space in many cases turns out to be a relatively simple problem.

We have already seen in the previous sections that this is exactly the case for free CSFT in $(2, q)$ backgrounds. The classical gauge invariant phase space in the limit $q \to \infty$ is a linear space spanned by the solutions of the equation $\partial_- \tilde{\chi}(X^0, X^1) = 0$ for a two dimensional field $\tilde{\chi}$ on $S^1 \times R^1$ which is Z_2 twisted over S^1. The quantization of this phase space leads to the corresponding Fock space. One concludes that the scalar bosonic field appearing in Ref.[7] is indeed the *gauge invariant* second-quantized string field, as was conjectured there. Moreover, there is a close analogy with abelian Chern-Simons gauge theory, where the equation of motion and gauge invariance are

$$\begin{aligned} dA &= 0 \\ \delta A &= d\epsilon \end{aligned} \tag{63}$$

and the cohomology is finite dimensional.

Consider now the interacting string field theory. Although we lack a clear geometric interpretation of Eq.(61), it is conceivable that it expresses a "flatness" condition

for some gauge field on an infinite-dimensional space. The gauge invariant phase space would therefore correspond to the moduli space of appropriate flat connections, in the same way as the classical gauge invariant phase space of non-abelian Chern-Simons gauge theory in 3D is the moduli space of flat non-abelian gauge connections. In the latter case, the equation of motion and gauge invariance are

$$\begin{aligned} dA + gA \wedge A &= 0 \\ \delta A &= d\epsilon + g[\epsilon, A] \end{aligned} \tag{64}$$

where this time A and ϵ are matrix-valued.

Now, in the Chern-Simons case a non abelian flat connection can be gauge transformed to an abelian one, namely, a constant gauge field in the Cartan torus. Hence the solutions of the linearized problem do correspond to solutions of the non-linear equations. However the enlarged group of gauge trasformations allows one to identify different "abelian solutions"[17]-[20],[45].

For the critical string, there does not appear to be a close analogy to this statement. However, in (p, q) minimal model backgrounds in the limit $q \to \infty$ the following simplification occurs. The classical phase space in the free theory has been shown to correspond to the fields

$$c\bar{c} : e^{i\frac{Q_\mu}{2}X^\mu} e^{i\sqrt{\frac{2p}{q}}(n-\frac{r}{p})X^+} : \tag{65}$$

In the limit $q \to \infty$, these configurations have a momentum which goes like

$$P_\mu \sim \tfrac{1}{2}Q_\mu + \mathcal{O}(q^{-\frac{1}{2}}) \tag{66}$$

Now insert such configurations in the full equation of motion, Eq.(61). The first term vanishes by construction, but the remaining terms do not. However, the Nth term is a state with momentum

$$P_\mu \sim \tfrac{1}{2}NQ_\mu + \mathcal{O}(q^{-\frac{1}{2}}) \tag{67}$$

By conservation of momentum, these states are orthogonal to the states in Eq.(65), and the corresponding fields have vanishing correlators amongst themselves. Thus in a certain sense they are null states. Another way to see this is that the bracket $\{\Psi^N\}$ defined in Eq.(57) vanishes, for $N \geq 3$ on the configurations Eq.(65) for large q. (This reasoning is not strictly valid for the space component of the momenta, due to the presence of screening charges in minimal models. However, there are no charges in the Liouville sector that can absorb the time component of the momentum, which gives the result above.)

Thus the interaction terms effectively vanish in the limit $q \to \infty$. This is the analogue of the statement[28][12] that, for finite q but large values of the Liouville coordinate X^0, the string field theory effective action is free.

Consider now the gauge transformation law, Eq.(62). Suppose we take a parameter $|\Lambda\rangle$ which is closed but not exact:

$$\begin{aligned} &Q_B(b_0^- |\Lambda\rangle) = 0 \\ &b_0^- |\Lambda\rangle \neq Q_B(b_0^- |\Xi\rangle) \end{aligned} \tag{68}$$

At the free level, such transformations act trivially on the classical phase space. However at first order in g,

$$\delta(b_0^- |\Psi\rangle) \sim g\, |[\Psi\Lambda]\rangle \tag{69}$$

which is a linear homogeneous transformation of $|\Psi\rangle$. This projects down to a linear homogeneous transformation of the cohomology space of Q_B, that is the classical free phase space, thanks to Eq.(68). But we have just argued that solutions of the free classical field equations are in fact solutions of the full interacting equations of motion. Therefore solutions connected by gauge transformations Eq.(69) should be identified in the non-linear theory. Since gauge transformations map the space of classical solutions into itself, they should be a symmetry of the classical action Eq.(33). They should therefore act as a subset of Virasoro transformations, the only linear symmetries of this action, and should be imposed as quantum constraints on the Hilbert space of the chiral bosons. The maximal anomaly-free set that can arise in this way is $L_{-1}, L_0, \cdots$. The results of Ref.[7] indicate that one should expect precisely this set.

In practice this is not so simple, for the following reason. The discussion above would be valid if there were gauge parameters which produced only variations of the form of Eq.(69) without producing any of the other terms in Eq.(66). However, Sen[12] has shown, working at finite q, that in particular the parameter

$$|\Lambda\rangle = c_0^-(c_1 a_{-1}^0 - \bar{c}_1 \bar{a}_{-1}^0)\,|0\rangle \tag{70}$$

(where a_n^μ are the oscillator modes of the X^μ) indeed produces a Virasoro-like transformation (corresponding to L_0) from Eq.(69), but also gives a BRS variation from the

first term in Eq.(66). This is interpreted as a variation in the zero-momentum dilaton field. Although this field is BRS exact and hence a trivial element of the cohomology, the fact that it is produced by the parameter $|\Lambda\rangle$ above is important in providing a specific inhomogeneous term in the Virasoro action on the free scalar field. For finite q, this is needed to correctly reproduce the multicritical matrix model corresponding to a $(2, q)$ background.

In a certain sense, however, the limit $q \to \infty$ is more appropriate to make contact with Ref[7]. The results of these papers amount to saying that, for one-matrix models, for example, there is a state in the Fock space of the quantized chiral boson which contains information about all $(2, q)$ models and their deformations. On such a Fock space the Virasoro trasformations do indeed act homogeneously, and one would therefore expect them to come from gauge transformations of string field theory satisfying Eq.(68). However, this "universal" state selected by the Virasoro constraints is not normalizable in the usual scalar product for the bosonic Fock space. One can make this state normalizable by introducing a *new* scalar product in the Fock space: this involves the choice of the "expectation values" of the modes of the scalar fields, corresponding to a shift in the t_n of Ref.[7] by fixed values $\bar{t}_n$, and each of these choices describes a fixed multicritical point or deformation around one. This situation is reminiscent, once again, of that in Chern-Simons theory. There it has been realized that, in the framework in which the constraints are imposed at classical level, there is an ambiguity in the choice of the scalar product, leading to different inequivalent quantum realizations of the same classical theory[45]. One might speculate that a similar phenomenon occurs in the theory at hand, and that this corresponds to "spontaneous symmetry breaking" for string field theory. Then it should be possible to derive the "universal" state from string field theory in the formalism described above, by defining a suitable scalar product. The matrix-model correlation functions would be obtained by differentiating the wave function of this state with respect to the parameters t_n, and evaluating this at $t_n = \bar{t}_n$. We intend to report on this in the future.

Irrespective of the actual details, it is physically evident, from the spacetime interpretation for large q, that gauge parameters of the form

$$|\Lambda\rangle = c_0^-(c_1 a_{-1}^- - \bar{c}_1 \bar{a}_{-1}^-)\,|k_\mu\rangle \tag{71}$$

where $k_\mu = (k,k)$ with k integer, and a^μ_n are the oscillator modes of X^μ, are the Virasoro generators L_k*.

An analogous situation is expected to hold for the (p,∞) case with $p \geq 3$. The corresponding effective action now also has non-linear W-symmetries. From the point of view of string field theory, this means that higher-order terms in the gauge transformation law will act as W-symmetries on the classical phase space, where the term of order g^{N-2} can produce W_{N-1} generators, with $N = 1,2,\cdots,p+1$. This would relate the non-linear symmetries obtained in matrix models in connection with generalized KdV hierarchies, with the nonlinear gauge invariance of closed-string field theory, providing rather non-trivial confirmation of two ideas with very different origins and motivations.

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* We learned from Ashoke Sen that he has independently arrived at a similar conclusion.

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