

A Lecture on the Hierarchy Problem and Gravity

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Abstract

In this lecture we shall briefly review some motivations for physics beyond the Standard Model. We focus our attention on the hierarchy problem and discuss the role of gravity in defining and solving this problem.

1 Motivation for beyond the Standard Model physics

Fundamental physics is about understanding the laws of Nature at different length scales. Let me give examples of some of the important ones. The Hubble length, $L_H \sim 10^{28}$ cm, which represents the size (and the age) of our observable Universe, is, by default, the largest observed length in Nature. Another important length, $L_{\text{exp.gr}} \sim 0.2$ mm, is the shortest length down to which Newton's law of gravity has been experimentally tested [1]. Then comes a so-called quantum chromodynamics (QCD) length, $L_{\text{QCD}} \sim 10^{-14}$ cm, at which the strong interaction changes regime and from the theory of composite hadrons (such as pions, protons, neutrons) becomes a theory of quarks and gluons. Then there is a weak interaction length, $L_W \sim 10^{-16}$ cm, set by the Compton wavelengths of W and Z bosons. Not very far from the weak scale, there is a length scale, $L_{\text{LHC}} \sim 10^{-17}$ cm, that will be probed by the Large Hadron Collider (LHC) experiments. An extremely important role in gravitational interactions is played by the Planck length, $L_P \sim 10^{-33}$ cm. Finally, let me introduce the notion of the shortest length scale of Nature, which I shall denote by L_* . The physical meaning of this length is that it marks an absolute bound on the resolution capability of the most powerful microscope. As we shall see, assuming that our understanding of large-distance Einsteinian gravity is correct, the absolute lower bound on L_* is set by L_P . That is, in any consistent theory of gravity that in the far-infrared flows to Einstein's general theory of relativity, the minimal length L_* cannot be below L_P , but can be much longer.

Currently, it is well accepted that the physics of known elementary particles at distances of the order of or larger than the weak length is extremely well described by the Standard Model (SM) of strong and electroweak interactions, which is based on the gauge group $SU(3)_C \times SU(2)_L \times U(1)_Y$. Moreover, at distances $L > L_{\text{exp.gr}}$, there is no evidence for any departure from Einstein's general relativity (GR).

So, then, why do we need any physics beyond the Standard Model (BSM)? The motivation for physics beyond the Standard Model is that the Standard Model contains certain mysteries. These mysteries can be expressed in the following sets of questions/issues that we shall group in different categories.

(A) Naturalness questions

- (1) The origin and hierarchy of the fermion families and of the hierarchy of their masses. The Standard Model contains three generations of quarks and leptons, with identical gauge quantum numbers, but very different masses that change (if we count neutrinos) within 11–12 orders of magnitude. This triplication of families and the origin of their mass hierarchy is a big mystery.
- (2) The strong CP-problem [2]. Why does the strong interaction not violate CP maximally? It has every right to do so because of the existence of the so-called θ term, $\theta \text{Tr } \tilde{F}F$, which for some mysterious reason appears to be very small in our vacuum, $\theta < 10^{-9}$.
- (3) The hierarchy problem. We shall pay special attention to this problem in our lecture.

(B) Cosmological/astrophysical mysteries

The second set of questions appears when we consider the Standard Model in the cosmological and astrophysical context. These are:

- (1) the origin of the dark matter in the Universe;
- (2) the origin of the baryon asymmetry; and
- (3) the origin of the inflation.

Note that the inflationary paradigm, according to which the Universe underwent a quasi-de Sitter stage of expansion, makes it impossible to attribute the origin of the baryon asymmetry to some suitable initial conditions. Within the inflationary framework, baryogenesis requires a dynamical explanation, and the Standard Model alone cannot provide it.

(C) Unification principle

The third set of questions has to do with the Unification principle. This principle has proven to be extremely successful in the past and suggests that the Standard Model cannot be a final theory. An extremely important question in this direction is unification with gravity.

2 The hierarchy problem

The main focus of this lecture is the hierarchy problem and the role of gravity in it. Among all the questions listed above, the hierarchy problem is the only one that directly motivates the existence of some new physics at LHC distances. Even the dark matter gives a direct motivation for new physics around L_{LHC} only when embedded within the context of particular solutions of the hierarchy problem. Therefore, we shall consider the hierarchy problem in more detail.

The hierarchy problem is sometimes posed as a problem of a very small number, the ratio of gravitational (Newton) and weak (Fermi) interaction constants,

$$\frac{G_{\text{N}}}{G_{\text{F}}} \sim 10^{-34}. \quad (1)$$

In the units $\hbar = 1$, the Newton and Fermi constants can be expressed in terms of length scales as $G_{\text{N}} = L_{\text{P}}^2$ and $G_{\text{F}} = L_{\text{W}}^2$ respectively. The problem, however, is not a big or a small number *per se*, but its quantum sensitivity to the size of a *black box*, the cut-off of the low-energy effective field theory. As we shall explain, in the absence of any new physics, the size of this black box is controlled by gravity. Thus, gravity plays a central role in posing the problem. Therefore, let us elaborate on the nature of gravity and its role in the hierarchy problem.

2.1 What is gravity?

Einstein's gravity is a theory of a dynamical metric, $g_{\mu\nu}(x)$, which, depending on a particular situation, can be represented as a background metric $\langle g_{\mu\nu} \rangle$ plus a small perturbation, $\delta g_{\mu\nu}$,

$$g_{\mu\nu} = \langle g_{\mu\nu} \rangle + \delta g_{\mu\nu}. \quad (2)$$

Obviously, $g_{\mu\nu}$ is a frame-dependent quantity. In studying the gravitational properties of a given localized source, it would be useful for us to go into the reference frame in which the centre of mass of the source is static. That is, we choose a frame in which we are freely falling together with (but not relative to) the source in a gravitational field created by the rest of the Universe. In this way we shall eliminate the influence of the external sources.

In fact, this is how we probe the gravitational attraction of the Earth in our everyday life. We exclude the influence of the rest of the Universe by freely falling together with the Earth in the Universe's gravitational field.

In such a case, the metric created by most of the sources within our Universe is Minkowskian to a very good approximation, and thus can be presented as

$$g_{\mu\nu} = \eta_{\mu\nu} + \delta g_{\mu\nu}, \quad (3)$$

where $\delta g_{\mu\nu} \ll 1$. For example, for the gravitational field on the surface of the Earth, we have $\delta g_{00} \sim 10^{-8}$. In such a case, $\delta g_{\mu\nu}$ can be written in terms of a linear perturbation of a canonically normalized massless spin-2 field, the graviton,

$$\delta g_{\mu\nu} = \frac{h_{\mu\nu}}{M_{\text{P}}}. \quad (4)$$

In a semiclassical treatment, $h_{\mu\nu}$ describes a massless spin-2 particle with two propagating degrees of freedom. As said above, for most sources (in the coordinate frame specified above) we can assume that $\delta g_{\mu\nu} \ll 1$, which in the language of the graviton translates as $h_{\mu\nu} \ll M_{\text{P}}$. The exception to this rule is provided by black holes and objects that are close to becoming black holes (e.g. neutron stars). If we approach a black hole from infinity, near the horizon we get $\delta g_{\mu\nu} \sim 1$ and the linear approximation breaks down.

Note that the same is true about the entire Universe. For example, we can randomly fix an origin of the coordinate system and move radially. Then $\delta g_{\mu\nu}(r)$ will be determined by the mass inside the sphere of radius r . The departure from the Minkowski metric will become of order one for $r \sim L_{\text{H}}$. This is not surprising, since the entire mass of the Universe within its Hubble size is almost exactly equal to the mass of the same (Hubble) size black hole. From this point of view, we can say that the Universe is the largest black hole we know!

In our treatment, we shall never need to go beyond the horizon, and we shall always stop short of the place where the linear approximation breaks down. All our conclusions will be derived from the observations that we can reliably make in the linearized limit. As we shall see, these conclusions will allow us to go surprisingly far in our understanding of gravity.

The linear theory of gravity can be derived by taking a linearized limit of Einstein's GR described by the Einstein–Hilbert action (numerical factors of order one will be ignored),

$$S_{\text{EH}} = \int dx^4 \sqrt{-g} (L_{\text{P}}^{-2} \mathcal{R} + \text{SM Lagrangian}). \quad (5)$$

Before going to linearized theory, let me make some remarks. I have written the theory in terms of L_{P} . If we restore dependence on \hbar , the Planck length is defined as

$$L_{\text{P}}^2 = \hbar G_{\text{N}}. \quad (6)$$

From the above expression, it is obvious that the Planck length is intrinsically quantum in its nature. This is already apparent from the fact that it vanishes in the limit $\hbar \rightarrow 0$. We shall explore the significance of the Planck length in great detail below.

Another comment we would like to make is that the Einstein–Hilbert action is unique in the sense that it is the only action that on any small-curvature (sub-Planckian curvature) classical background propagates two degrees of freedom of a weakly coupled massless spin-2 particle. For example, the addition of other invariants (other than possibly boundary terms) results in propagation of additional degrees of freedom.

The linearized equation for small metric perturbation about the Minkowski space has the form

$$\mathcal{G}_{\mu\nu}^{(L)} = L_{\text{P}} T_{\mu\nu}, \quad (7)$$

where $T_{\mu\nu}$ is a conserved energy–momentum tensor that at the linear level we shall treat as an external source, and $\mathcal{G}_{\mu\nu}^{(L)}$ is a linearized Einstein tensor, which in terms of a canonically normalized graviton $h_{\mu\nu}$ takes the following form:

$$\mathcal{G}_{\mu\nu}^{(L)} = \square h_{\mu\nu} - \eta_{\mu\nu} \square h - \partial_\mu \partial^\alpha h_{\alpha\nu} - \partial_\nu \partial^\alpha h_{\alpha\mu} + \partial_\mu \partial_\nu h + \eta_{\mu\nu} \partial^\alpha \partial^\beta h_{\alpha\beta}. \quad (8)$$

In fact, equation (7) is also unique. It is a unique ghost-free linear equation describing propagation of a massless spin-2 field on a Minkowski background. It exhibits a gauge freedom under the shift of a graviton by a symmetrized derivative of an arbitrary vector, ξ_μ ,

$$h_{\mu\nu} \rightarrow h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu. \quad (9)$$

Note that gauge symmetry (or, to be more precise, gauge redundancy) is not imposed, but rather is a consequence of the consistency requirement. Any other form of the kinetic term would result in propagation of unwanted degrees of freedom, for which it would be impossible to impose a simultaneous positivity of the energy and the norm.

We now wish to understand how the graviton responds to the sources. That is, we shall introduce an external source $T_{\mu\nu}$ and find a corresponding $h_{\mu\nu}$. This task is seemingly complicated by the fact that the differential operator acting on $h_{\mu\nu}$ does not look easily invertible. But this technicality can be easily circumvented by using the gauge redundancy. We shall choose $\partial^\mu h_{\mu\nu} = \frac{1}{2} \partial_\nu h$. Using this gauge and taking into account the relations obtained by tracing equation (7), we can rewrite it in a nice invertible form,

$$\square h_{\mu\nu} = L_{\text{P}}(T_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} T). \quad (10)$$

We shall be interested in static sources of mass M , which can be approximated by a spherical uniform mass distribution of size R ,

$$T_{\mu\nu} = \delta_\mu^0 \delta_\nu^0 \frac{3M}{4\pi R^3} \theta(R - r).$$

The graviton field produced by such a source outside of the mass distribution is

$$\delta g_{00} = \frac{h_{00}}{M_{\text{P}}} = \frac{2MG_{\text{N}}}{r}. \quad (11)$$

The quantity in the numerator has the dimension of a length, which we shall denote by $r_g \equiv 2G_{\text{N}}M$. This length scale has a special name, and is called the gravitational (Schwarzschild) radius of the source. It is an extremely important length scale. In order to understand its physical meaning, let us rewrite (11) as

$$\delta g_{00} = \frac{h_{00}}{M_{\text{P}}} = \frac{r_g}{r}. \quad (12)$$

This equation indicates that r_g sets a distance at which the metric perturbation created by the source becomes of order one.

Let us imagine now that the source of interest is an elementary particle of mass M or momentum p . Then the following new *quantum* length scales enter the problem. These are the well-known Compton and de Broglie wavelengths of the source of mass M or momentum p ,

$$L_{\text{C}} \equiv \frac{\hbar}{M} \quad \text{or} \quad L_{\text{dB}} \equiv \frac{\hbar}{p}. \quad (13)$$

The physical meaning of L_{C} (L_{dB}) is to set the scale at which the energy of the quantum fluctuations ($E = \hbar/L_{\text{C}}$) would become comparable to the energy of the source.

Both lengths, L_{P} and L_{C} , as well as the Planck length, L_{P} , vanish in the limit $\hbar \rightarrow 0$ when G_{N} and M are kept fixed. One of the consequences of this fact is that in classical GR ($\hbar = 0$) one can have black holes of arbitrarily small size. In reality, however, \hbar and G_{N} are the fixed constant and the only

parameter we can vary is the mass of the source M . The classicality is then achieved when we increase M so that $r_g \gg L_P, L_C$.

Consider now an elementary particle with rest mass $M \ll M_P \equiv \hbar/L_P$. For such a source $r_g \ll L_P \ll L_C$. Approaching this source from infinity, we shall encounter strong quantum effects way before we can probe its gravitational radius. So although r_g can be formally defined and the particle is point-like, it cannot be called a black hole.

The situation will change if we start increasing the mass of a particle. With increasing mass, r_g increases and the Compton wavelength decreases. The two will cross at L_P when the mass of an elementary particle becomes equal to M_P .

Further increase of M will create a situation when $r_g \gg L_P \gg L_C$. The roles of r_g and L_C have now been exchanged. Approaching such a source from infinity, we shall encounter a strong classical gravitational effect ($\delta g_{00} \sim 1$) way before we have any chance to probe the quantum nature of the source. To call such a source an elementary particle is a meaningless statement. Instead, it is a classical black hole.

Another way to give precise meaning to the quantum-to-classical transition described above, without referring to the geometric properties, is through the concept of the occupation number of gravitons created by the source.

Following Ref. [3], we shall now introduce this concept. Consider again a spherical source of uniform density and physical radius R well above its gravitational radius $R \gg r_g$. For such a source, the approximation of linear gravity is valid everywhere, and the gravitational field produced by it can be easily found from (10). For example, the Newtonian component of the metric perturbation about the flat space outside the source is given by (12) and falls off as r^2 for $r < R$. From the quantum field theory point of view, the above linearized metric represents a superposition of gravitons. The level of classicality is measured by their occupation number. These gravitons are non-propagating longitudinal gravitons, but this is unimportant for characterizing the classical properties of the fields. In a certain sense, we can think of these gravitons as representing a Bose condensate. The only peculiarity of this condensate is that, as long as $R \gg r_g$, the condensate cannot self-sustain. In order to exist, it requires an external source.

The measure of classicality of this field is the occupation number of gravitons in it, N . This number can be estimated easily as

$$N = \frac{1}{\hbar} M r_g. \quad (14)$$

The physical meaning of the number N becomes transparent from the following reasoning. The gravitational part of the energy is

$$E_{\text{grav}} \sim \frac{M r_g}{R}. \quad (15)$$

We should think of this energy as being the sum of the energies of the individual gravitons with wavelengths λ and occupation numbers N_λ ,

$$E_{\text{grav}} \sim \sum_{\lambda} N_{\lambda} \hbar \lambda^{-1}. \quad (16)$$

The reason why the total gravitational energy is extremely well approximated by a simple sum of the energies of the individual quanta is the following. First, the peak of the distribution is at $\lambda = R$. The contribution from shorter wavelengths is exponentially suppressed and can be ignored. Thus, for $R \gg L_P$, the gravitons contributing to the energy are of very long wavelengths and thus are weakly interacting. Thus, for the purpose of our estimate, the interaction between the gravitons can be ignored. Note that, for $R \gg r_g$, not only can the interactions between the individual gravitons be ignored, but also the interaction between any individual graviton and the entire collective gravitational energy. In other

words, for $R \gg r_g$ we can ignore gravitational self-sourcing. This is why in this regime the condensate cannot be self-sustained.

Then we easily obtain the occupation number of gravitons by dividing the total gravitational energy by the characteristic energy of a single quantum, $N \sim N_R \sim E_g/(\hbar R^{-1})$, which gives us (14).

For the future, it is very important to note that, even when $R \sim r_g$, the interactions among the individual gravitons continue to be negligible. However, the gravitational energy becomes of the order of the energy of the source. At this point the self-sourcing by the collective gravitational energy becomes important and the condensate becomes self-sustained. However, interactions among the individual gravitons are still negligible as long as $r_g \gg L_P$. So the occupation number of gravitons can still be safely estimated as given by (14).

Since, by default, the physical size of the source cannot become less than r_g , the occupation number of gravitons for an arbitrary source is thus universally given by (14).

The criterion of classicality then is

$$N \gg 1. \quad (17)$$

The above criterion has a clear physical meaning. A given configuration is classical when there are many gravitons in it. We can rewrite N in the following equivalent forms:

$$N = \frac{L_P^2}{L_C^2} = \frac{M^2}{M_P^2} = \frac{r_g^2}{L_P^2}. \quad (18)$$

The quantity N diverges for $\hbar \rightarrow 0$, as it should, since in the classical limit the number of quanta in any field configuration is infinite.

The fact that N is a good measure of classicality can be seen from the fact that, for any elementary particle lighter than M_P , it is less than one. For example, for an electron, $N = (m_e^2/M_P^2) \sim 10^{-44}$! This is why an electron can never be regarded as a classical gravitating object despite the fact that it does create a Newtonian gravitational field. The Newtonian gravity produced by a non-relativistic electron does not contain even a single quantum of graviton.

Alternatively, any source for which $N \gg 1$ is classical with a good approximation. Such sources decouple from the low-energy effective theory and their quantum effects are exponentially suppressed. The probability of such sources decaying into two (or few) particle states is suppressed by $\exp(-N)$.

Summarizing the above findings, we have reached the following conclusions. Quantum elementary particles exist as long as their Compton wavelength dominates over their gravitational radius – or, equivalently, as long as at $r = L_C$, the metric perturbation created by the source is much less than one, $\delta g_{00}(r \sim L_C) \ll 1$. Whenever this is not the case, we are no longer dealing with an elementary particle but a classical object. The quantum effects of such objects are exponentially suppressed. In Einstein gravity this crossover happens at $M = M_P$, and thus the Planck scale is an upper bound on the mass of any elementary particle.

The above makes the hierarchy problem real. Without gravity, one could argue that the problem is artificial. After all, the weak scale has to be somewhere and just happened to be around 100 GeV. But with gravity the problem becomes real, since now a universal regulator scale exists in the form of the Planck mass M_P , and one needs to explain why the weak scale is so much below it. Indeed, if the Higgs mass were not much below M_P , nobody would ask the question why it is not even higher. We know that it cannot be higher, since in such a case Higgs would no longer exist as a quantum particle.

So what keeps the Higgs light? In this lecture we shall focus on one possible approach to the problem, in which the solution is provided by gravity itself. We will not discuss other approaches, such as supersymmetry or technicolor [4].

3 The role of gravity

Our discussion has already prepared the basis for a most straightforward solution of the hierarchy problem. We have seen that a universal regulator scale is set by a distance at which the particle Compton wavelength and its gravitational radius cross. In Einstein's GR this happens at L_P . Following Ref. [5], let us ask, what if the crossover scale is much longer?

Let us denote this new generalized strong gravity scale by L_* . If L_* were around 10^{-17} cm, the hierarchy problem would be nullified. What is the physical meaning for such a large L_* ? This means that for a source of mass \sim TeV, the metric perturbation must become 10^{32} times stronger than the analogous perturbation in Einstein gravity. What is the underlying physical mechanism for such enhancement? It is very simple. The only way to achieve this is to make gravity 10^{32} times stronger (more attractive) at distances L_* than what it is in Einstein. Field-theoretically this translates as gravitational force being mediated by more messengers. That is, the metric fluctuation should propagate additional spin-2 degrees of freedom. In order not to mess up the predictions of GR at large distances, these new degrees of freedom must be massive, but with Compton wavelengths larger than or of the order of L_* . The upper bound on their Compton wavelengths is phenomenological and comes from non-observation of any deviation from Newtonian gravity in table-top measurements that have been conducted down to 0.2 mm [1].

The most general expansion of the *linearized* metric perturbation in terms of new degrees of freedom has the form

$$\delta g_{\mu\nu} = \frac{1}{M_P} \sum_m g_m^{(2)} h_{\mu\nu}^{(m)} + \frac{1}{M_P} \sum_m g_m^{(0)} \eta_{\mu\nu} \phi^{(m)}, \quad (19)$$

where $h_{\mu\nu}^{(0)}$ and $\phi^{(m)}$ stand for spin-2 and spin-0 degrees of freedom, respectively. The $g_m^{(2)}$ and $g_m^{(0)}$ parametrize the strength of their coupling relative to a zero-mode Einsteinian graviton, for which we have $g_0^{(2)} = 1$. Note that, since we couple metric perturbation only to the conserved energy-momentum tensor, the potential contribution from the derivatively coupled scalars of the form $\partial_\mu \partial_\nu \phi^{(m)}$, as well as contributions from spin-1 states, vanish.

At the linear level a ghost-free equation of motion satisfied by a massive spin-2 field is unique, and is given by the Pauli-Fierz form [6]

$$\mathcal{G}_{\mu\nu}^{(L)} - m^2 (h_{\mu\nu}^{(m)} - \eta_{\mu\nu} h^{(m)}) = \frac{g_m^{(2)}}{M_P} T_{\mu\nu}, \quad (20)$$

where the first term is the same as in massless theory and is given by (8).

This equation (20) shows that $h_{\mu\nu}$ propagates five degrees of freedom. Let us see how this counting goes. First, let us note that the presence of the mass term promotes $h_{\mu\nu}^{(m)}$ into a gauge-observable. In other words, in terms of this quantity, there is no longer a gauge redundancy of the form (9). For this reason, sometimes it is said that the mass term *breaks* the gauge symmetry explicitly. This is, however, the wrong terminology, because the original redundancy (9), existing at the massless level, is still there and is realized at the level of the components of the massive field $h_{\mu\nu}^{(m)}$, which contains three extra degrees of freedom in comparison with the massless one, $h_{\mu\nu}$. We shall uncover this redundancy in a moment. Let us come back to the counting of the degrees of freedom. Since $h_{\mu\nu}^{(m)}$ is a gauge-invariant symmetric tensor, the maximal number of components is 10. We wish to find out how many of these correspond to the independent propagating degrees of freedom. In order to see this, let us note that by taking the divergence of the equation (20) and by taking into the account the conservation of the source, we arrive at the following constraint:

$$\partial^\mu h_{\mu\nu}^{(m)} = \partial_\nu h^{(m)}. \quad (21)$$

This eliminates four out of the 10 potential degrees of freedom. Next, by taking the trace of the equation (20) and by taking into account the constraint (21), we arrive at the following equation:

$$h^{(m)} = \frac{1}{3m^2} \frac{g_m^{(2)}}{M_P} T. \quad (22)$$

This equation shows that the trace is fully determined by the source. This eliminates one more potential independent degree of freedom, and we are left with five. A further reduction in the number of degrees of freedom is impossible. It is useful to group these five degrees of freedom into irreducible representations of the Poincaré group corresponding to the massless case. This decomposition has the form

$$h_{\mu\nu}^{(m)} = h_{\mu\nu} + \partial_\mu A_\nu + \partial_\nu A_\mu + \frac{1}{6}\eta_{\mu\nu}\chi + \frac{1}{3}\frac{\partial_\mu\partial_\nu}{m^2}\chi, \quad (23)$$

where $h_{\mu\nu}$, A_μ and χ stand for the tensor, vector and scalar helicities, respectively. The physical meaning of these representations is that they diagonalize the kinetic terms, and only mix through the mass term. Thus, in the $m \rightarrow 0$ limit they would disintegrate into the *independent* representations of the Poincaré group, corresponding to massless spin-2, spin-1 and spin-0 particles, respectively.

The form (23) makes it obvious that the original gauge redundancy of the massless spin-2 case (9) is maintained fully intact. This is because the gauge shift of $h_{\mu\nu}$ is compensated by the vector component

$$h_{\mu\nu} \rightarrow h_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu, \quad A_\mu \rightarrow A_\mu - \xi_\mu. \quad (24)$$

We now wish to invert the equation (20) in order to find out what field is caused by a given source. This is easy to do. By using the constraint (21) and expressing the trace through the source via Eq. (22), we can easily invert the equation in the following way:

$$h_{\mu\nu}^{(m)} = \frac{g_m^{(2)}}{M_{\text{P}}^2} \frac{1}{\square - m^2} \left[T_{\mu\nu} - \frac{1}{3} \left(\eta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{m^2} \right) T \right]. \quad (25)$$

Note that the physical effect of this contribution to the metric perturbation is measured by convoluting it with a probe conserved source $t_{\mu\nu}$,

$$\int d^4x t^{\mu\nu} h_{\mu\nu}^{(m)}. \quad (26)$$

As a result, the derivative terms in (25) will vanish. So, we shall ignore them.

Consider a localized static source $T_{\mu\nu} = \delta_\mu^0\delta_\nu^0 M\delta(r)$ with Schwarzschild radius r_g . For such a source, the equation (25) can be easily solved. In particular, for the Newtonian component $h_{00}^{(m)}$ we get

$$h_{00}^{(m)} = \frac{4}{3} \frac{g_m^{(2)}}{M_{\text{P}}^2} \frac{M}{r} e^{-mr}. \quad (27)$$

Substituting this expression into (19) we get for the total metric perturbation the expression

$$\delta g_{00} = \frac{r_g}{r} \sum_m \rho_m e^{-mr}, \quad (28)$$

where $r_g = 2G_{\text{N}}M$ is the usual Einsteinian gravitational radius, and ρ_m parametrizes the relative strengths of massive spin-2 states with respect to the massless graviton. That is, $\rho_m \equiv \frac{4}{3}(g_m^{(2)})^2$, whereas $\rho_0 = 1$.

In our parametrization the criterion of gravity becoming strong at the scale L_* is that a source of the Compton wavelength L_* creates an order-one metric perturbation at distance $r \sim L_*$. That is, $\delta g_{00}(r = l_*) = 1$.

Since each gravitational degree of freedom of mass m contributes only at distances $r < 1/m$, at any distance r we are allowed to take into account only modes that are not heavier than $1/r$. Thus we can truncate the sum at $m = 1/r$. At the same time we can approximate the exponential factors for all the light modes by one. The sum then simplifies to

$$\delta g_{00}(r) = \frac{r_g}{r} \sum_{m=0}^{m=1/r} \rho_m. \quad (29)$$

Now the criterion of making gravity strong at L_* reads

$$\delta g_{00}(r = L_*) = \frac{L_P^2}{L_*^2} \sum_{m=0}^{m=1/L_*} \rho_m = 1. \quad (30)$$

Our task is now very simple. If we want to solve the hierarchy problem by making gravity strong at TeV energies, or equivalently at $L_* \sim 10^{-17}$ cm distances, the new spin-2 degrees of freedom must satisfy

$$\sum_{m=0}^{m=1/L_*} \rho_m \sim 10^{32}. \quad (31)$$

The physical meaning of the above equation is very simple. If we want to increase the fundamental length from $L_P \sim 10^{-33}$ cm all the way to $L_* \sim 10^{-17}$ cm, then at distance L_* we have to make gravity 10^{32} times stronger. The cumulative strength of all the new degrees of freedom at this scale must be 10^{32} . This can be achieved in different ways. We introduce either many weakly coupled states or alternatively a few strongly coupled ones.

If we postulate that all the new degrees of freedom have the same strength of couplings as the zero-mode graviton, $\rho_m = 1$, then we are left with a single parameter, the number of new graviton species, N_{species} , which is fixed at

$$N_{\text{species}} = L_P^2 / L_*^2. \quad (32)$$

For $L_* \sim 10^{-17}$ cm, this gives $N_{\text{species}} \sim 10^{32}$.

3.1 Origin of species

We shall now discuss how the gravitational species can originate from extra dimensions. For simplicity, we shall discuss the story with one extra dimension. Let x_μ be our four space-time coordinates, whereas y is an extra (fifth) space dimension. We shall denote the five-dimensional space-time index by capital latin letters, A, B, \dots . Let us consider two observers in this five-dimensional space. One of them, Alice, is a five-dimensional observer, whereas the other observer, Bob, is a four-dimensional one. For definiteness, let us assume that Bob is localized at $y = 0$ slice (brane). At the moment let us assume that the five-dimensional space is flat and non-compact.

Let us consider a five-dimensional massless particle, as seen by the two observers. This particle satisfies a five-dimensional massless dispersion relation,

$$p_A p^A = p_\mu p^\mu - p_y^2 = 0, \quad (33)$$

where p_μ and p_y are four-dimensional and extra dimensional projections of the momentum, respectively. Both Alice and Bob observe the same physical picture, but they describe it in two different languages.

From the point of view of Alice, she deals with a single massless particle that can assume different momenta. Since space is non-compact, the momenta are not quantized.

From the point of view of Bob, who can only measure the four-dimensional projection of the momentum, the same object represents a continuum of particles with different masses. From his perspective, the fifth momentum is just a constant entering the dispersion relation in the same way that mass would enter. Thus, Bob thinks that he sees a continuum of particles with masses $m^2 = p_y^2$.

Let us assume now that the particle in question is a five-dimensional massless graviton h_{AB} , which satisfies a five-dimensional linearized Einstein's equation,

$$\mathcal{G}_{AB}^{(L)} = L_5^{3/2} T_{AB}, \quad (34)$$

where L_5 is a five-dimensional Planck length, T_{AB} is a conserved energy–momentum tensor, and $\mathcal{G}_{AB}^{(L)}$ is a linearized Einstein tensor, which, in terms of a canonically normalized graviton h_{AB} , takes the following form:

$$\mathcal{G}_{AB}^{(L)} = \square h_{AB} - \eta_{AB} \square h - \partial_A \partial^C h_{CB} - \partial_B \partial^C h_{CA} + \partial_A \partial_B h + \eta_{AB} \partial^C \partial^D h_{CD}. \quad (35)$$

The above equation exhibits a five-dimensional gauge freedom,

$$h_{AB} \rightarrow h_{AB} + \partial_A \xi_B + \partial_B \xi_A, \quad (36)$$

where ξ_A is an arbitrary five-dimensional vector.

Just as in the case of an above-considered generic particle, from the point of view of Alice, there is a single five-dimensionally massless graviton propagating in a five-dimensional Minkowski space. The same graviton is seen by Bob as a continuum of four-dimensional massive gravitons. This can be easily seen by a standard dimensional reduction. For this we can rewrite our theory by introducing the ansatz

$$h_{\mu\nu}(x, y) = \int dm b_m(y) \tilde{h}_{\mu\nu}^{(m)}, \quad (37)$$

$$h_{\mu 5}(x, y) = \int dm b_m(y)' A_\mu^{(m)}, \quad (38)$$

$$h_{55}(x, y) = \int dm b_m(y)'' \phi^{(m)}, \quad (39)$$

where the prime stands for a y derivative and $b_m(y)$ represent a complete set of harmonic functions satisfying $b_m'' = -m^2 b_m$. Thus, the expressions (37)–(39) represent a Fourier expansion of the graviton components in plane waves with respect to the fifth coordinate. Introducing the notation

$$h_{\mu\nu}^{(m)} = \tilde{h}_{\mu\nu}^{(m)} + \partial_\mu (A_\nu^{(m)} + \partial_\nu \phi^{(m)}/2) + \partial_\nu (A_\mu^{(m)} + \partial_\mu \phi^{(m)}/2), \quad (40)$$

it is easy to see that each $h_{\mu\nu}^{(m)}$ satisfies the four-dimensional equation (20). We thus end up with a continuum of massive gravitons, a so-called Kaluza–Klein (KK) tower.

The norm of each KK graviton is given by the integral $\int_{-\infty}^{+\infty} dy |b_m(y)|^2$ and, correspondingly, the coupling to a four-dimensional energy–momentum source $T_{AB} = \delta(y) \delta_A^\mu \delta_B^\nu T_{\mu\nu}$, localized, say, at $y = 0$, is given by

$$g_m = \frac{|b_m(0)|}{\sqrt{\int_{-\infty}^{+\infty} dy |b_m(y)|^2}}. \quad (41)$$

Of course, in the non-compact limit the norm is divergent and consequently the coupling constant of each individual member of the continuum is infinitely small.

Let us now compactify the fifth coordinate on a circle of radius R . The effect of this compactification is that now the p_y momentum is quantized in units of $1/R$, and correspondingly the continuum of the massive KK tower becomes discrete. The norm becomes finite, $\int_0^{2\pi R} dy |b_m(y)|^2 = 2\pi R$, and each member now couples with the strength $L_5^{3/2}/\sqrt{2\pi R}$. Translating this in terms of the four-dimensional Planck length, we get the relation

$$L_P^2 = L_5^3/(2\pi R). \quad (42)$$

Noting that the number of KK species with Compton wavelengths below L_5 is $N_{\text{species}} = (R/L_5)$, we see that (42) exactly reproduces (32).

Our analysis can be straightforwardly generalized to more than one extra dimension. For d extra dimensions, the geometric relation between the four-dimensional and fundamental Planck lengths reads [5]

$$L_{\text{P}}^2 = L_*^2 (2\pi R L_*^{-1})^d. \quad (43)$$

However, in terms of the number of species, the relation is universal and is given by (32).

We can now apply the above findings to the solution of the hierarchy problem. As said above, having $L_* \sim 10^{-17}$ cm requires $N_{\text{species}} \sim 10^{32}$. Expressed in geometric terms, this translates as the radius of extra dimensions being

$$R \sim 10^{(32/d)-17} \text{ cm}. \quad (44)$$

This rules out the possibility of a single flat extra dimension, since in this case the radius has to be of Solar System size, in obvious contradiction with observations. However, $d = 2$ or higher is within the current limits of gravity measurements [1].

We see that extra dimensions provide a tower of new gravitational degrees of freedom, with the coefficients ρ_m in (30) being set by the values of the KK wave functions $\rho_m = |b_m(0)|^2$ evaluated at the position of the source in the y coordinate. For the flat extra dimensions [5] the KK modes are just plane waves and $\rho_m = 1$.

The situation can be different if extra dimensions are warped [7] or if the graviton wave function is strongly distorted due to the interactions with the branes [8]. In such a case $b_m(y)$ no longer satisfy a simple wave equation, but a more complicated equation in an effective y -dependent background potential. The form of the potential depends on the precise geometry of the extra space. So, in such a case ρ_m can be a strongly non-uniform function of m .

In particular, if warping is strong (as in Ref. [7]), already the first few KK excitations can become coupled 10^{32} times stronger than the Einstein's zero-mode graviton. In any case, the effective four-dimensional picture reduces down to the equation (30). So the defining property of any extra dimensional model from the point of view of a four-dimensional observer is the m dependence of parameter ρ_m . By changing this dependence, one can scan the entire landscape of extra dimensional models. Such scanning is beyond the scope of the present lecture.

4 Summary

The purpose of this lecture was to discuss the hierarchy problem, as the main motivation for BSM physics around TeV energies, and to stress the role of gravity in defining the problem, as well as in its potential solution. This role originates from the fact that gravity provides a universal regulator scale, the Planck length (or the Planck mass), which marks the boundary between the quantum elementary particles and the (semi)classical black holes. A straightforward solution offered by gravity then is to postulate that the fundamental length is around 10^{-17} cm. We have discussed in a model-independent way what it takes to have such an increase of the fundamental length and have shown that this requires the introduction of new graviton species. Their number and the coupling strengths are model-dependent, but the relation (31) that they must satisfy is universal. We have discussed how such new graviton species originate from extra dimensions. A universal model-independent prediction of the above class of theories is a production of new graviton species [9, 10] and production of micro-black holes at LHC [5, 11]. The latter signature follows from the fact that the fundamental length marks the crossover to the black hole regime. Any attempt at probing shorter distances results in black hole formation. However, the transition is gradual and the lightest black holes are simply quantum resonances, not much different from ordinary particles. Their precise properties can only be predicted qualitatively, and one needs more experimental input to understand their characteristics at a more quantitative level.

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