ACCURATE BEAM DYNAMICS EQUATIONS IN PROTON LINEAR ACCELERATORS

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(Presented by P. M. Lapostolle)

1. INTRODUCTION

Interest in the knowledge of beam dynamics in the Alvarez Structure of proton linear accelerators has been renewed with the advent of several projects for injectors into circular machines, etc. Programmes using multiple integrations per cell, in which the fields have been determined by mesh calculation or approximate analytic formulae, have been in existence for some time, but have the disadvantage of being time consuming. Thus there is still a need for accurate difference equations to describe the beam motion. Previously (1, 2), equations of motion, in the axial plane in particular, have made use of the socalled Panofsky equations, viz.

$$W_{n+1} = W_n \, eE_o gT\,(W) \cos\varphi \qquad [1]$$

$$\varphi_{n+1} = \varphi_n + 2\pi\left(\frac{\beta n,s}{\beta n} - 1\right) \qquad [2]$$

(the phase convention here is to use $\varphi_s = -26°$, for example). These equations are in fact incorrect, since phase space area in the (W, φ) plane varies, that is Liouville's theorem is violated. This error was first realized by J. S. Bell (3), who pointed out that although the equations allow for a velocity dependence into the energy gain, they do not allow for a phase dependence into the time taken for the particle to cross the cell. It is the purpose in the first part of this paper to derive a correction term to the phase equation [2] in such a way that phase space area is conserved. In the second part, accurate general relations are derived for radial, as well as axial, motion along the linac, which includes coupling terms, and such that the requirements of Liouville's theorem are satisfiied.[3] Finally, by way of example, axial motions using the new equations and a multi-integration routine are described and compared.

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[3] More details on the analytic treatment in this can be found in ref. (4) and (5).

2. CORRECTION TO THE AXIAL EQUATIONS OF MOTION

Of the two equations above, the first admits a velocity dependence in the energy gain through the transit time factor T, whilst the second can be regarded as the finite difference form of the usual equation for phase with respect to r.f.:

$$\frac{d\varphi}{dz} = k\left(\frac{1}{\beta_p} - \frac{1}{\beta_w}\right) \rightarrow \Delta\varphi = k\int\left(\frac{1}{\beta_p} - \frac{1}{\beta_w}\right) dz,$$

$$k = \frac{2\pi}{\lambda} \qquad [3]$$

where β_p, β_w are velocities w.r.t.c. of the particle and wave respectively. If, indeed, there is no velocity change, then for gap spacing $L = \beta_w \lambda$

$$\varphi_{n+1} = \varphi_n + 2\pi\left(\frac{L}{\beta_p \lambda} - 1\right) \qquad [4]$$

When there is a velocity change, there is a phase change which depends on the acceleration in the gap, which can be denoted by an additional term $f(W_n, \varphi_n)$ in [4]. Since W, φ are canonical variables (the independent variable being z), transformation of W, φ from one position to another has unit Jacobian, that is, if across a gap

$$\begin{aligned} W_{n+1} &= W_n + e\,E_o\,gT\,(W_n)\cos\varphi_n \\ \varphi_{n+1} &= \varphi_n + f\,(W_n, \varphi_n) \end{aligned} \qquad [5]$$

Then

$$J\,\frac{W_{n+1}, \varphi_{n+1}}{W_n, \varphi_n} = \begin{vmatrix} 1 + e\,E_o g \dfrac{dT}{dW}\cos\varphi_n & \dfrac{\partial f}{\partial W} \\ - e\,E_o gT \sin\varphi_n & 1 + \dfrac{\partial f}{\partial W} \end{vmatrix} = 1\,.$$

This is satisfied to the first order by $f = -e E_o g(dT/dW) \sin\varphi + K$. New, phisically, f is a measure of the difference of velocity change in the two halves of the gap. If the particle arrives at the gap centre at the peak of the r.f. ($\varphi = 0$), there is no difference: hence f, and K, are zero. The solution f to give a unit Jacobian ensures that Liouville's a theorem is satisfied:

$$\iint d(\Delta W_{n+1})\, d(\Delta\varphi_{n+1}) = \iint d(\Delta W_n)\, d(\Delta\varphi_n)\, J\left(\frac{W_{n+1}, \varphi_{n+1}}{W_n, \varphi_n}\right) = \iint d(\Delta W_n)\, d(\Delta\varphi_n). \quad [6]$$

Finally, an interesting alternative point of view comes from the definition that W and φ are canonical, i.e. H. exists, and

$$\frac{\partial}{\partial W}(dW/dz) = -\frac{\partial}{\partial\varphi}(d\varphi/dz).$$

If $dW/dz = e\,ET\cos\varphi$: then if T is independent oif W, $d\varphi/dz$ is independent of φ and has the form [3] above, giving the well-known equations for motion on a travelling wave (which work particularly well when the linac parameters are slowly warying); or if T is a function of W, then $d\varphi/dz = f + g(W)$, here f is defined as above, and g(W) has the form of [3] above.

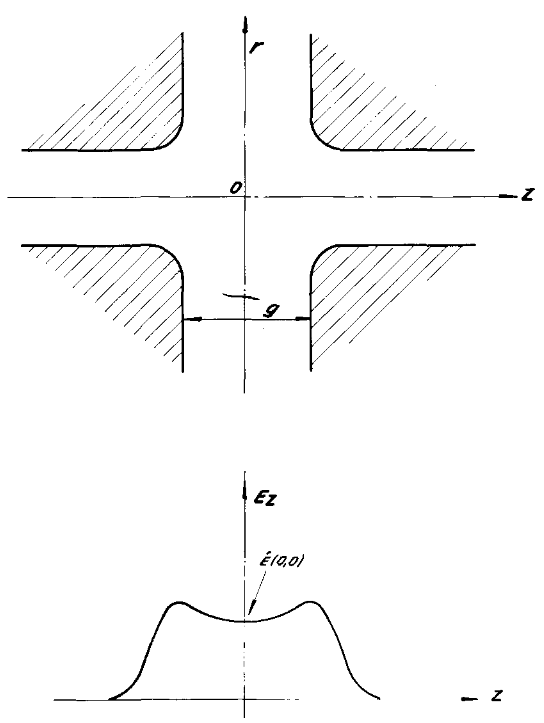

Fig. 1 - Accelerating gap and axial field E_z distribution.

3. GENERAL EQUATIONS OF MOTION IN A LINAC

a) Expressions for the fields

Coming now to the general case, consider an accelerating gap in which the longitudinal, E_z field distribution along the axis has been determined, either by computation or by model measurement (Fig. 1). For the sake of simplicity, the gap is assumed symmetrical about the mid-plane $z = 0$, and is circularly symmetrical about the z-axis. In the co-ordinate system (z, r, t), if $E_z(z, 0, t)$ is the axial field, taking

$$\int_{-\infty}^{+\infty} E_z(z, o, t) = V_o \cos(\omega t + \varphi) \quad [7]$$

and

$$\int_{-\infty}^{+\infty} E_z(z, o, t)\cos k_z z\, dz = V_o\cos(\omega t + \varphi)\, T_o(k_z) \quad [8]$$

where V_o i sthe gap voltage $k_z = 2\pi/\beta\lambda$, and $T_o(k_z)$ is called the axial transit time factor. Fourier transform relations give

$$E_z(z, r, t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} T_o(k_z)\, I_o(k_r r)\cos k_z z\, dk_z \cdot V_o\cos(\omega t + \varphi) \quad [9]$$

where

$$k_r^2 = k_z^2 - \omega^2/c^2 \quad [10]$$

and I is the Modified Bessel Function. Similarly

$$E_r(z, r, t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} T_o(k_z)\frac{k_z}{k_r} I_1(k_r r)\sin k_z z\, dk_z \cdot V_o\cos(\omega t + \varphi)$$

and [11]

$$Z_o H\varphi(z, r, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty +} T_o(k_z)\frac{\omega}{ck_r} I_1(k_r r)\cos k_z z\, dk_z \cdot V_o\sin(\omega t + \varphi).$$

b) Energy gain: 1st approximation

With the use of equation [9], the energy gain of a particle crossing the gap can be obtained, by integrating

$$d(mv_z) = e E_z(z. r, t) = e E_z(z, r, t)\frac{dz}{v_z} \quad [12]$$

where to a first approximation v_z equals a some mean velocity v, so that

$$\omega t = \omega z/v \quad [13]$$

(and it will be seen later that v is the actual velocity in the mid-plane $z = 0$). For a trajectory at a distance r parallel to the axis, the energy gain is given by

$$\Delta W = e \lim_{l \to \infty} \int_{-l}^{l} E_z \left(z, r, \frac{z}{v} \right) dz \qquad [14]$$

with equation [9], this gives

$$\Delta W = \lim_{l \to \infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} e V_o T_o (k_z) I_o (k_r r) \cdot \left[\frac{\sin (\omega/v + k_z) l}{\omega/v + k_z} + \frac{\sin (\omega/v - k_z) l}{\omega/v - k_z} \right] \cos \varphi \, dk_z \qquad [15]$$

This is recognised immediately as the Fourier-Direchlet integral, and gives (for k_z continuous)

$$\Delta W = e V_o T (k) I_o (k_r r) \cos \varphi \qquad [16]$$

where $k = \omega/v = 2\pi/\beta\lambda$, and, now, $k_r^2 = k^2 - \omega^2/c^2$.

c) Energy gain: 2nd approximation

As pointed out in section 2, a phase error $\Delta\varphi$ exists, which in the notation of the present section, can be obtained by integration of $\omega\Delta (1/v)$, where $\Delta (1/v)$ equals $(1/v) - (1/V)$ and V is velocity before or after the gap (V is constant for $z < 0$, or $z > 0$). With the relation $\Delta (1/v) = (- \Delta W/W) (1/v) (c^2/v^2 - 1)$, where W is the total energy, the integral can be evaluated as beofre and the result is exactly equal to f of section 2.

Now to improve the accuracy of the computation of energy gain a phase error of this type is introduced, and equation [13] is replaced by

$$\omega t = \frac{\omega t}{v} + \delta\varphi$$

but where $\delta\varphi$ is taken to be zero in the mid-plane $z = 0$. [Nevertheless, in the integral of the correcting term it is still legitimate to use [13]]. This computation leads to a Taylor expansion for the energy gain, tre first two terms of which are

$$\Delta W = \Delta W_1 - \Delta W_1 \frac{k e \hat{E}_z (0, r)}{4w T_o (K) I_o (k_r r)} \frac{d^2}{dk^2} \left[T_o (k) I_o (k_r r) \right] \sin \varphi \qquad [17]$$

where $E_z (0, r)$ is the peak E-field (see Fig. 1), and w is the kinetic energy in the mid-plane, where are also taken k, and φ. The term ΔW_1 is that given by [16] where v and φ are true velocity and phase in the mid-plane. Numerical evaluation shows that the second term is always less than one per cent of ΔW_1, and can be neglected. Hence equation [16] gives the energy gain to a high order of accuracy when the true velocity and phase are used in the mid-plane.

d) Other relations

In a similar way the following can also be computed:

1) Energy gain for a non-parallel trajectory, by replacing r by $r + r' z$ in [14], and expanding the I_o term.

TABLE I

$$W_+ - W_- = eV_o T_o I_o (k_r r) \cos \varphi + eV_o \frac{d}{dk} [T_o k_r I_1 (k_r r)] r'' \sin \varphi$$

$$\varphi_1 - \varphi_- = \frac{eV_o k}{2w} \frac{d}{dk} [T_o I_o (k_r r] \sin \varphi - \frac{eV_o k}{2w} \frac{d^2}{dk^2} [T_o k_r I_1 (k_r r) r' \cos \varphi$$

$$r'_+ - r'_- = \frac{eV_o}{2w} \frac{k_r}{k} T_o I_1 (k_r r) \sin \varphi + \frac{eV_o}{2w} \frac{d}{dk} \left[T_o \frac{k_r^2}{k} I'_r (k_r r) \right] r' \cos \varphi$$

$$r_+ - r_- = - \frac{eV_o}{2w} \frac{d}{dk} \left[T_o \frac{k_r}{k} I_1 (k_r r) \right] \cos \varphi - \frac{eV_o}{2w} \frac{d^2}{dk^2} \left[T_o \frac{k_r^2}{k} I_1 (k_r r) \right] r' \sin \varphi$$

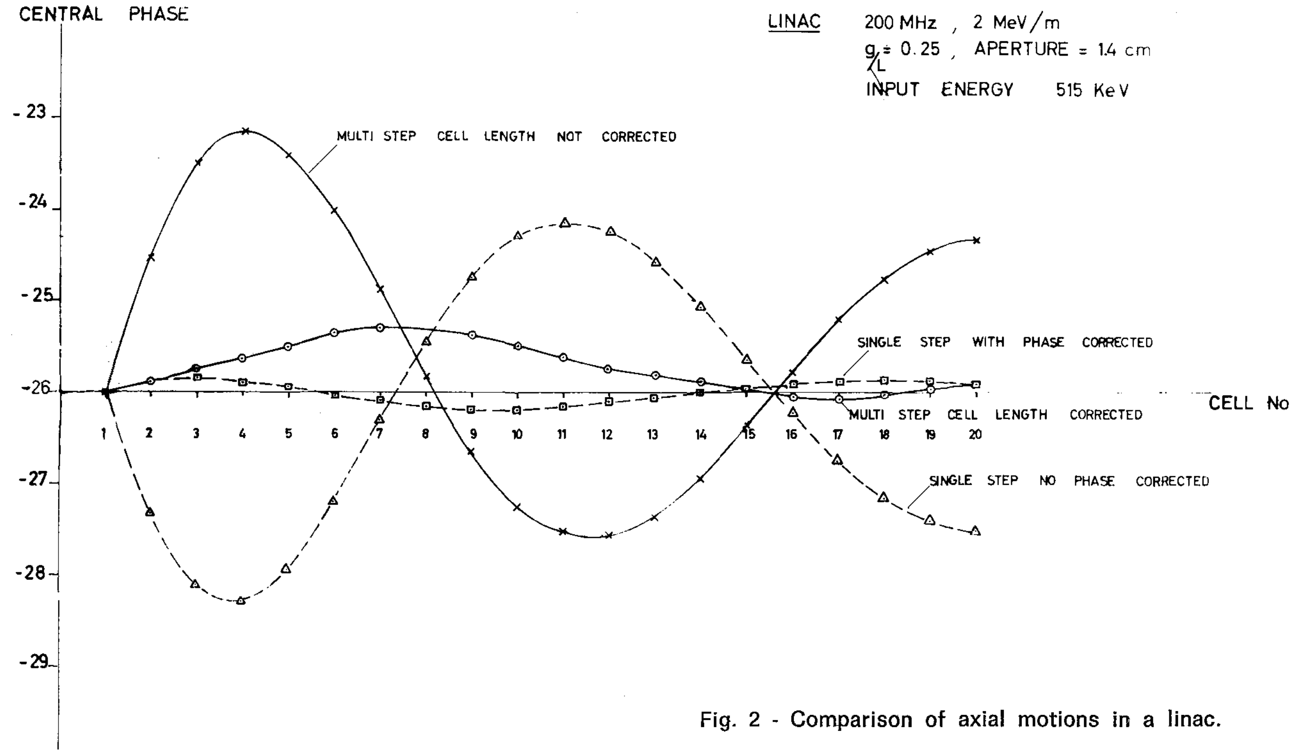

Fig. 2 - Comparison of axial motions in a linac.

2) Transverse motion, with the help of equations [11].

Table I summarizes the various beam dynamics relations, denoting by the indices − or + the values "before" and "after" the mid-plane in the single step computation. All parameters on the right hand sides refer to values in the mid-plane. The first two expressions are of particular interest: the first being the Panofsky formula 1), whilst the second is the phase correction f, or $\Delta\varphi$ already discussed. Furthermore, all the expressions of this table represent a set of transformations which can be easily seen to satisfy Liouville's theorem.

e) Mid-plane valeus and practical expressions

The mid-plane values are not known, and generally are not the average between "before" and "after". To compute them, integrals of the type [14] must be used, but with the upper limit $z = 0$: the Fourier integral cannot now be used and the integration becomes impossible. To obtain simple expressions, a function $S_o(k_z)$ is defined, analogously to $T_o(k_z)$, by

$$S_o(k_z) = 2\int_0^\infty E_z(z, 0, t)\sin(k_z z)\,dz \;/\; V_o\cos(\omega t + \varphi)\,. \qquad [18]$$

By replacing $T_o(k_z)\cos k_z z$ in [9] by $^1/_2$ [$T_o \cdot (k_z)\cos k_z z - S_o(k_z)\sin k_z z$], it becomes possible to write a field E_z in the form of a Fourier expansion which is the same as [9] for $z < 0$ and is zero for $z > 0$. Similar expression apply to the other components given in [11]. The computations now proceed as before.

Nevertheless, mid-gap values given by intrinsic equations are difficult to obtain, and even the expressions of Table I are rather complicated. So it is of interest to find approximate, but simpler forms. Now, in beam dynamics studies it is generally necessary to compute many trajectories in a given structure. But for all these trajectories the field distributions are the same, and the trajectories differ only in radial position and momentum, phase φ, and velocity v (or kinetic energy w) which however must always remain close to a certain value v_o (or w_o). Thus, it seems reasonable to introduce instead of absolute values of velocity (or energy) relative values:

$$\Delta W = W - W_o \qquad [19]$$

and try to express all the relations in the form of simple expansions around w_o, for which coefficients can be computed. A treatment similar to that already described can be used to compute mid-plane values.

Table II gives a list of coefficients to compute in each gap for a given "synchronous" energy. Table III gives expression for mid-plane values as computed from the "before" values (and here the more general co-ordinates x and y have been used instead of r). Finally, Table IV shows practical expressions giving the values "after", and these last values can be used to compute the "before" values for the following gap.

4. COMPARISON OF AXIAL MOTION WITH SINGLE STEP AND MULTIPLE STEP INTEGRATIONS

A programme has been written to compare axial motion by the equations given here, and a 32-integrations-per-cell routine (this routine and its use have been discussed in references (6) and (7), and will not be described here). The linac is first defined by using a nominal acceleration rate K, which may or may not be a function of β_s. The change in β_s per cycle (≡ per cell) is then closely $(1-\beta_{s,i}^2)^{3/2} K\lambda/W_o$, where $\beta_{s,i}$ is the

TABLE II

$$V_o \cos(\omega t + \varphi) = \int_{-\infty}^{+\infty} E_z(z, c, t)\, dz .$$

$$T_o(k_o) V_o \cos(\omega t + \varphi) = \int_{-\infty}^{+\infty} E_z(z, o, t) \cos k_o z\, dz$$

$$S_o(k_o) V_o \cos(\omega t + \varphi) = \int_{o}^{\infty} E_z(z, o, t) \sin k_o z\, dz$$

$$V_1 = \frac{V_o}{w_o} \qquad V_2 = \frac{V_o k_o}{w_o}$$

$$\Theta_o = T_o(k_o) k_{ro}^2 \qquad \Theta_1 = T_o(k_o) \frac{k_{ro}^2}{k_o}$$

$$\Theta_2 = T_o'(k_o) \frac{k_o}{w_o} \qquad \Theta_3 = T_o'(k_o) \frac{k_o k_{ro}^2}{w_o}$$

$$\Sigma_o = S_o(k_o) k_{ro}^2 \qquad \Sigma_1 = S_o(k_o) \frac{k_{ro}^2}{k_o}$$

$$\Sigma_2 = S_o'(k_o) \frac{k_o}{w_o} \qquad \Sigma_3 = S_o'(k_o) \frac{k_o k_{ro}^2}{w_o}$$

All the derivatives are taken relative to $k = \omega/v$ for $k = k_o$ with $k_{ro}^2 - \omega^2/c^2$, all values with index $_o$ corresponding to synchronous kinetic energy w_o. One has for instance:

$$T'(k_o) V_o \cos(\omega t + \varphi) = -\int_{-\infty}^{+\infty} z E_z(z, o, t) \sin k_o z\, dz.$$

Non-relativistic equations are used and in several relations β^2 is omitted with resplect to 1; but also kr_o can be replaced by k_o without appreciable error.

input value of β_s. Regarding each cell separately gives the energy change over the whole cell and half cell respectively:

$$\left.\begin{aligned} \Delta W &= eVT(\beta_c)\cos\varphi_s \\ \Delta W_i &= \frac{1}{2}\Delta W + \frac{1}{2} eVS(\beta_c)\sin\varphi_s \end{aligned}\right\} V = E_o g = \bar{E}L . \qquad [20]$$

At low energies the gap fields can be assumed uniform, so that the axial parts of T and S are respectively $\sin\vartheta/\vartheta$, $(1-\cos\vartheta)/\vartheta$, where $\vartheta = \pi g/\beta_c\lambda$. To solve equations [20], a definition of L is required (and hence g, the two being related through the resonant frequency). If the length of each half cell is such that the phase shift of the synchronous particle is exactly π, the resulting cell length is given to a good approximation by

$$L = \bar{\beta}\lambda\, [1 - \Delta\varphi(\beta_c)/2\pi] \qquad [21]$$

where $\Delta\varphi(\beta_c)$ is the phase correction term introduced in section 2, and $\bar{\beta}$ is the mean value of β_s in the cell. The distance between gap centres is now (to small error)

$$l = \lambda\beta_o\, [1 - \frac{1}{4\pi}(\Delta\varphi_{n+1} + \Delta\varphi_n)] \qquad [22]$$

where β_o is the output β_s from the cell. The phase shift between gap centres n, n + 1 is 2π, and for the general particle (suffixed p) is approximately

$$\varphi_{n+1} \approx \varphi_n + 2\pi\left(\frac{l}{\beta_p\lambda} - 1\right) + \Delta\varphi(\beta_{c,p}) \qquad [23]$$

as described in section 2. With equations [20] to [23] the motion of the general particle can be found.

Fig. 2 compares the axial motions with the two routines of a 20-cell linac with parameters as shown. With the multi-step routine, modification of cell length from $\bar{\beta}\lambda$ to that of equation [21], (and resulting modification to gaps and fields), reduces the amplitude of phase oscillation of the reference particle from 3° to about ½°. With the single step (and much faster) routine, the amplitude of the phase oscillation is reduced from 2½° to almost zero in the modified linac. That the phase oscillations still have a small amplitude is because a) omission of the correcting term in Eq. [17] in the modified linac induces a slight change in the synchronous phase angle with the multi-step routine; and b) with the single step routine Eq. [23] is slightly less accurate than Eq. [22]. Nevertheless, in their present form, the equations give a very sound basis for general study of linac beam motion.

TABLE III

$$r = (r_-)\left[1 - \frac{eV_1}{8}\left[\theta'_1 \cos(\varphi_-) + \Sigma'_1 \sin(\varphi_-)\right]\right] - (r'_-)\frac{eV_1}{8}\left[\theta''_1 \sin(\varphi_-) - \Sigma''_1 \cos(\varphi_-)\right]$$

$$r' = -(r_-)\frac{eV_1}{8}\left[\Theta_1 \sin(\varphi_-) - \Sigma_1 \cos(\varphi_-)\right] + (r'_-)\left[1 + \frac{eV_1}{8}\left[\theta'_1 \cos(\varphi_-) + \Sigma'_1 \sin(\varphi_-)\right]\right]$$

$$\operatorname{tg}\varphi = \frac{\sin(\varphi_-) - \frac{eV_2}{4}\left(S'_0 + \Sigma'_0\frac{r^2}{4}\right) + \frac{eV_2}{8}\left(\Sigma'_2 + \Sigma'_3\frac{r^2}{4}\right)\left[(\delta W_-) + \frac{eV_0}{2}\left[T_0\cos(\varphi_-) + S_0\sin(\varphi_-)\right]\right]}{\cos(\varphi_-) - \frac{eV_2}{4}\left(T'_0 + \Theta'_0\frac{r^2}{4}\right) + \frac{eV_2}{8}\left(\Theta'_2 + \Theta'_3\frac{r^2}{4}\right)\left[(\delta W_-) + \frac{eV_0}{2}\left[T_0\cos(\varphi_-) + S_0\sin(\varphi_-)\right]\right]}$$

$$\delta W = \frac{(\delta W_-) + \frac{eV_0}{2}\left(T_0 + \Theta_0\frac{r^2}{4}\right)\cos\varphi + \frac{eV_0}{2}\left(S_0 + \Sigma_0\frac{r^2}{4}\right)\sin\varphi}{1 + \frac{eV_0}{4}\left(\Theta_2 + \Theta_3\frac{r^2}{4}\right)\cos\varphi + \frac{eV_0}{4}\left(\Sigma_2 + \Sigma_3\frac{r^2}{4}\right)\sin\varphi}$$

TABLE IV

$$\delta W_+ = \delta W_- + eV_0\left(T_0 + \Theta_0\frac{x^2+y^2}{4}\right)\cos\varphi - \frac{eV_0}{2}\left(\Theta_2 + \Theta_3\frac{x^2+y^2}{4}\right)\delta W\cos\varphi + \frac{eV_0}{2}\Theta'_0(xx' + yy')\sin\varphi$$

$$\varphi_+ = \varphi_- + \frac{eV_2}{2}\left(T'_0 + \Theta'_0\frac{x^2+y^2}{4}\right)\sin\varphi - \frac{eV_2}{4}\left(\Theta'_2 + \Theta'_3\frac{x^2+y^2}{4}\right)dW\cos\varphi - \frac{eV_2}{4}\Theta''_0(xx' + yy')$$

$$x_+ = x_- - \frac{eV_1}{4}\Theta'_1 x\cos\varphi + \frac{eV_2}{8}\frac{\delta W}{w_0}\Theta''_1 x\cos\varphi - \frac{eV_1}{4}\Theta''_1 x'\sin\varphi$$

$$y_+ = y_- - \frac{eV_1}{4}\Theta'_1 y\cos\varphi + \frac{eV_2}{8}\frac{\delta W}{w_0}\Theta''_1 y\cos\varphi - \frac{eV_1}{4}\Theta''_2 y'\sin\varphi$$

$$x'_+ = x'_- - \frac{eV_1}{4}\Theta_1 x\sin\varphi + \frac{eV_2}{8}\frac{\delta W}{w_0}\Theta'_1 x\sin\varphi + \frac{eV_1}{4}\Theta'_1 x'\cos\varphi$$

$$y'_+ = y'_- - \frac{eV_1}{4}\Theta y\sin\varphi + \frac{eV_2}{8}\frac{\delta W}{w_0}\Theta'_2 y\sin\varphi - \frac{eV_1}{4}\Theta'_2 y'\cos\varphi$$

REFERENCES

(1) W. K. H. Panofsky: UCRL 1216 (1951).
(2) H. G. Hereward and K. Johnson: Proc. CERN Symposium, (1956) vol. I, p. 167.
(3) J. S. Bell, AERE T/M 114 (1954).
(4) P. M. Lapostolle: CERN Report AR/Int. SC/65-11 (1965).
(5) M. Promé: CEA Saclay No. SEFS TD 65/12 (1965).
(6) R. Taylor: AERE - R3012 (1959).
(7) K. Batchelor, A. Carne, J. M. Dickson and D. J. Warner: Int. Conf. on High Energy Accelerators, Dubna, 1963.

DISCUSSION

CROWLEY-MILLING: Have these computations been compared with the original computations mode of the time of the design of the Harwell and CERN linear accelerators using an analogue computer fed with the measured field conditions in the drift tube gaps?

LAPOSTOLLE: No, not that particular one, comparisons are in program at present but have only been done yet with a more recent elaborate programme developed at the Rutherford Laboratory.

NISHIKAWA: But already in my paper on the normal mode analysis.