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Chern-Simons Field Theory and Link Invariants

E. Guadagnini

Dipartimento di Fisica dell'Università di Pisa

Istituto Nazionale di Fisica Nucleare, Sezione di Pisa

M. Martellini*

CERN - Geneva

M. Mintchev

Istituto Nazionale di Fisica Nucleare, Sezione di Pisa

Dipartimento di Fisica dell'Università di Pisa

Abstract

The quantization of the non-Abelian Chern-Simons theory in three dimensions is performed in the framework of the BRS formalism. The perturbative analysis at two loops confirms that the model is finite. The vacuum expectation values of Wilson line operators $< W(L) >$ are computed to second order of perturbation theory. The meaning of the framing procedure for knots is analyzed in the context of the Chern-Simons field theory. The relation between $< W(L) >$ and the link invariant polynomials is discussed. We derive an explicit analytic expression for the second coefficient of the Alexander-Conway polynomial, which is related to the Arf and Casson invariants.

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* *On leave from Dipartimento di Fisica dell'Università di Milano and INFN, Sezione di Pavia.*

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1. Introduction

Since the discovery of solitons and instantons, topological methods have been successfully applied in field theory. Nowadays a possibility emerges to investigate some topological aspects of three- and four-manifolds by means of certain quantum field theories. The recently discovered topological field theories are examples of this kind. In particular, A. S. Schwarz [1] and Atiyah [2] have pointed out that the pure Chern-Simons (C-S) action, associated with a non-Abelian gauge group G, could be relevant for knot theory and might shed new light on the link invariants. A considerable development of this idea has been accomplished by Witten [3,4], who has shown that the physical states of the C-S model also describe the conformal blocks of the two-dimensional conformal invariant Wess-Zumino-Witten (WZW) model. Using this correspondence, Witten has argued that the vacuum expectation values of Wilson line operators $< W(L) >$ in the fundamental representation of G are related to a recently discovered link invariant, the HOMFLY polynomial [5]. The existence of a simple recursive procedure for the computation of this link invariant provides in turn a quick and elegant way for computing $< W(L) >$ in the fundamental representation. Summarizing, Witten has shown that there exists indeed an intrinsically three-dimensional definition of the new link invariants, the Jones [6] and HOMFLY polynomials, which was one of the open problems quoted in [2].

In this talk we report our recent results [7,8,9] concerning the consequences of the existence of a quantum field theory description of the link invariants. One of the main issues is the calculability of $< W(L) >$ in the three-dimensional field theory framework [9]. This problem plays a fundamental role in the program of connecting the C-S theory with the geometry of three-manifolds and knot theory. It turns out that $< W(L) >$ has a meaningful perturbative expansion in powers of the coupling constant. It is worth stressing that, differently from what one could naively expect, *no* divergences appear in the computation of $< W(L) >$ in the C-S theory. We report below the explicit calculations at second order of perturbation theory.

Another crucial problem that we will discuss is how to give an operative meaning to

the framing procedure for knots [10] in the context of quantum field theory. Contrary to a widespread belief, the framing has nothing to do with divergences of the expectation value $< W(L) >$. As already mentioned, there are no divergences in $< W(L) >$ at all. In the C-S model, the necessity of framing is related to the self-linking problem. As is well known to mathematicians [11,12], the framing procedure assigns a topological meaning to the self-linking for knots in R^3 by fixing the ambiguities due to a certain direction-dependent but finite limit.

We also show below that the perturbative expansion of $< W(L) >$ in the C-S model provides a systematic method for deriving explicit expressions of link and knot invariants. We give in particular an analytic expression of the second coefficient $a_2(C)$ of the Alexander-Conway polynomial [13] for a knot C. This coefficient is of particular interest because it is related to the Arf-invariant [14] of C and the Casson-invariant [15] associated with Dehn surgeries constructed with C.

Our analysis at second order in perturbation theory confirms that $< W(L) >$, computed in the fundamental representation of G, coincides with the S_L-polynomial, namely the most general regular isotopy invariant satisfying a linear skein relation. The S_L-polynomial has been introduced in [7] and generalizes both the Kauffman [16] and HOMFLY polynomials.

The C-S field theory also makes manifest the existence of relations between the coefficients of the HOMFLY polynomial. We explain how these relations can be derived by means of the perturbative expansion of $< W(C) >$. Afterwards we derive the general structure of these relations and prove their validity without referring to perturbation theory.

We shall not discuss below the relation of the C-S model to conformal field theory. The recent developments in that direction can be found in [17,18].

The talk is organized as follows. In sect.2 we recall some aspects of the covariant quantization of the C-S model and review some of its symmetry properties. In sect.3 we compute some correlation functions of the gauge potential and illustrate the finiteness of the C-S model at two loops. The explicit computation of $< W(L) >$ at second order in perturbation theory and for a generic link L with arbitrary representation of the gauge group G is described in sect.4. In sect.5 we recall the skein relations satisfied by the regular isotopy invariant S_L-polynomial and its relation with the HOMFLY polynomial. The connection between S_L and $< W(L) >$ is established, at second order in perturbation theory, in sect.6. Here we derive (to this order) also the values of the parameters entering the skein relation. In the last section we relate the knot invariants provided by the C-S theory with the HOMFLY coefficients and establish some general relations between the latter. Some technical details of the computations are reported in the Appendices. In particular, in Appendix B we discuss the framing-independence of the knot invariant appearing in $< W(L) >$ at second order in perturbation theory.

2. The Chern-Simons Action and its Symmetries

In this section, we recall some basic properties of the Chern-Simons model [19]. The expression of the classical action of the C-S model on a smooth three-manifold M^3 is

$$S_{CS} = \frac{k}{4\pi}\int_{M^3} d^3x\, \epsilon^{\mu\nu\rho}\, \mathrm{Tr}\,(A_\mu \partial_\nu A_\rho + i\frac{2}{3}A_\mu A_\nu A_\rho) \quad , \tag{2.1}$$

where $A_\mu = A_\mu^a T^a$, T^a being the generators in the defining representation of a compact simple Lie group G. The equations of motion following from (2.1) read

$$F^a_{\mu\nu} \equiv \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - f^{abc} A^b_\mu A^c_\nu = 0 \quad , \tag{2.2}$$

f^{abc} being the structure constants of G. Consider for example $G = SU(N)$ with

$$\mathrm{Tr}\, T^a T^b = \tfrac{1}{2}\,\delta^{ab} \quad . \tag{2.3}$$

Under a gauge transformation $\Omega(x) \in SU(N)$,

$$A_\mu(x) \mapsto A^\Omega_\mu(x) \equiv \Omega^{-1}(x) A_\mu(x) \Omega(x) - i\Omega^{-1}(x)\partial_\mu \Omega(x) \quad , \tag{2.4}$$

the action (2.1) transforms as

$$S_{CS}[A^\Omega] = S_{CS}[A] + 2\pi k\, S_{WZ}[\Omega] \quad , \tag{2.5}$$

where

$$S_{WZ}[\Omega] = \frac{1}{24\pi^2}\int_{M^3} d^3x\, \epsilon^{\mu\nu\rho}\, \mathrm{Tr}\,[\Omega^{-1}(x)\partial_\mu\Omega(x)\Omega^{-1}(x)\partial_\nu\Omega(x)\Omega^{-1}(x)\partial_\rho\Omega(x)] \quad . \tag{2.6}$$

The Wess-Zumino functional S_{WZ} takes integer values and therefore for integer k, the C-S model is invariant also under large (not connected to the identity) gauge transformations. In general, for any compact simple Lie group G, the C-S action can be normalized in such a way that the complete gauge invariance is preserved for integer values of k.

Another important property of the action (2.1) is its general covariance; being the integral of a three-form on a three-dimensional manifold, S_{CS} is independent of the metric that one can introduce on M^3. This property is essential in relating the C-S model with two-dimensional conformal field theory and knot theory [1,2,3].

Because of gauge invariance, in order to perform the quantization we adopt the standard Faddeev-Popov procedure. The total action is then

$$S = S_{CS} + S_{GF} + S_{FP} \quad , \tag{2.7}$$

where

$$S_{GF} = \frac{k}{4\pi}\int_{M^3} d^3x\, \sqrt{g} g^{\mu\nu}\, A^a_\mu \partial_\nu B^a \quad , \tag{2.8}$$

$$S_{FP} = -\int_{M^3} d^3x\, \sqrt{g} g^{\mu\nu}\, \partial_\mu \bar{c}^a (D_\nu c)^a \quad , \tag{2.9}$$

and

$$(D_\mu c)^a = \partial_\mu c^a - f^{abc} A^b_\mu c^c \quad . \tag{2.10}$$

In eqs.(2.8,9) $g_{\mu\nu}$ is some metric on M^3, B^a are Lagrange multipliers enforcing the gauge constraint $\partial_\mu(\sqrt{g} g^{\mu\nu} A^a_\nu) = 0$ and c and $\bar{c}$ are the usual Faddeev-Popov ghosts. The total action S is invariant under the BRS transformations [20]

$$\delta A^a_\mu = (D_\mu c)^a \quad , \quad \delta B^a = 0 \quad , \tag{2.11}$$

$$\delta c^a = \tfrac{1}{2} f^{abc} c^b c^c \quad , \quad \delta \bar{c}^a = \frac{k}{4\pi} B^a \quad . \tag{2.12}$$

The associated BRS charge operator Q is off-shell nilpotent at any order of perturbation theory because there are no gauge anomalies in three dimensions. The cohomology defined by Q determines in the usual way the physical space of the C-S model.

The term $S_{GF} + S_{FP}$ unavoidably spoils the general covariance. Since the metric dependence enters only in the gauge-fixing procedure however, general covariance is maintained on the physical space [21]. In fact, the symmetric energy-momentum tensor $\Theta_{\mu\nu}$ associated with (2.7) is

$$\Theta_{\mu\nu} = \frac{k}{4\pi}(A^a_{\{\mu}\partial_{\nu\}}B^a - g_{\mu\nu}A^a_\rho\partial^\rho B^a) - \partial_{\{\mu}\bar{c}^a(D_{\nu\}}c)^a + g_{\mu\nu}\partial_\rho\bar{c}^a(D^\rho c)^a \qquad (2.13)$$

and can be rewritten in the form

$$\Theta_{\mu\nu} = [Q\,,\,\partial_{\{\mu}\bar{c}^a A^a_{\nu\}} - g_{\mu\nu}\partial^\rho\bar{c}^a A^a_\rho] \quad . \qquad (2.14)$$

Since Q annihilates the physical states, the mean value of $\Theta_{\mu\nu}$ between physical states vanishes and general covariance is indeed preserved. Consequently, the expectation value of any gauge invariant and metric independent observable is a topological invariant in M^3 [1,2,3]. A typical example of such an observable is the Wilson line operator

$$W_R(C) = \mathrm{Tr}_R\,\mathrm{P}\,\exp\, i\oint_C A_\mu\, dx^\mu \quad , \qquad (2.15)$$

where the path ordering is performed along a given oriented closed path without self-intersections (knot) C and the trace is computed in the R-representation of G. More generally, we shall consider the expectation values of the type

$$< W(L) > \equiv < W_{R_1}(C_1)\cdots W_{R_m}(C_m) > \quad , \qquad (2.16)$$

where the link L is the union of non-intersecting knots C_i.

In order to get some insight into the nature of the physical states of the C-S model, we consider the equations of motion (2.2) following from the classical action (2.1). Eqs.(2.2) show one of the most striking features of the theory: roughly speaking, one has a gauge invariant theory in which even the transverse parts of A^a_μ are eliminated because of the equations of motion. So, what are the physical degrees of freedom of

this theory? In fact, there are no physical degrees of freedom in the usual sense of local field theory. For instance, in the canonical quantization on $M^3 = R\times S^2$ there is only one physical state - the vacuum, so the C-S model does not have a standard particle content. Yet the physical Hilbert space is not necessarily trivial. Consider for example the equations of motion (2.2) on the three-manifold $R\times\Sigma$ where Σ is a Riemann surface of non-trivial genus. The set $\{A^a_\mu\}$ of the solutions of eq.(2.2) modulo gauge transformations determines a non-trivial physical phase space [3,22].

Another interesting situation occurs in the presence of Wilson line operators $W(L)$. In this case, Σ may have punctures arising as intersection points of the link L with Σ. Consequently, the physical space is non-trivial even if $\Sigma = S^2$ [3,22] and one of the main problems is the explicit computation of the expectation value $< W(L) >$. At first sight, this problem seems to have a trivial solution. A naive use of eq.(2.2) would suggest that "on-shell" there is no curvature and therefore $< W(L) >$ is trivial. The failure of this argument can be seen in different ways. In the canonical approach, the R.H.S. of eq.(2.2) gets modified by pointlike source terms. This is precisely the origin of the punctures with a given representation of the group assigned to each of them [3]. In the functional integral approach, eq.(2.2) is not identically satisfied when $F^a_{\mu\nu}$ is inserted into the correlation functions. Let us elaborate on this point in some detail.

As is well known, an infinitesimal deformation of the path associated with a Wilson line operator leads to an $F_{\mu\nu}$-insertion. More precisely, one has

$$\frac{\delta U(1,2)}{\delta\sigma^{\mu\nu}(x)} = iU(1,x)R^a F^a_{\mu\nu}(x)U(x,2) \quad , \qquad (2.17)$$

where

$$U(1,2) = P\,\exp\, i\int_1^2 dx^\mu R^a A^a_\mu(x) \qquad (2.18)$$

is the holonomy operator between the points 1 and 2 in the R representation of the group G. Differently from the standard Yang-Mills theories, the $F_{\mu\nu}$-insertions are easily treated in the C-S model. In fact, since

$$F^a_{\mu\nu}(x) = \frac{4\pi}{k}\epsilon_{\mu\nu\rho}\frac{\delta S_{CS}}{\delta A^a_\rho(x)} \quad , \qquad (2.19)$$

a simple integration by parts in the Feynman path integral gives

$$\int DA\, \exp(iS_{CS})\, \mathrm{Tr}[U(1,x)R^a F^a_{\mu\nu}(x) U(x,1)] O_1 \cdots O_n =$$

$$= i\frac{4\pi}{k}\epsilon_{\mu\nu\rho} \int DA\, \exp(iS_{CS})\, \frac{\delta}{\delta A^a_\rho(x)} (\mathrm{Tr}[U(1,x)R^a U(x,1)] O_1 \cdots O_n) \quad , \qquad (2.20)$$

$O_1, ..., O_n$ being gauge invariant observables. Therefore, in the C-S model the $F_{\mu\nu}$-insertion (2.17) can be equivalently replaced by a functional derivative with respect to A_μ. The result (2.20) is correct, but needs a more rigorous argument. The point is that, because of gauge invariance, one has to take into account gauge fixing and ghost contributions. So, instead of equation (2.19) one has actually

$$F^a_{\mu\nu}(x) = \frac{4\pi}{k}\epsilon_{\mu\nu\rho}\frac{\delta S}{\delta A^a_\rho(x)} + f^a_{\mu\nu}(x) \quad , \qquad (2.21)$$

where

$$f^a_{\mu\nu}(x) = \frac{4\pi}{k}\epsilon_{\mu\nu\rho} f^{abc}\partial^\rho \bar{c}^b(x) c^c(x) - \epsilon_{\mu\nu\rho}\partial^\rho B^a(x) \quad . \qquad (2.22)$$

At this stage, in order to prove eq.(2.20), i.e.

$$< \mathrm{Tr}[U(1,x)R^a F^a_{\mu\nu}(x) U(x,1)]\, O_1 \cdots O_n >=$$

$$= i\frac{4\pi}{k}\epsilon_{\mu\nu\rho}\frac{\delta}{\delta A^a_\rho(x)} < \mathrm{Tr}[U(1,x)R^a U(x,1)]\, O_1 \cdots O_n > \quad , \qquad (2.23)$$

one must show that

$$< \mathrm{Tr}[U(1,x)R^a f^a_{\mu\nu}(x) U(x,1)]\, O_1 \cdots O_n >= 0 \quad . \qquad (2.24)$$

The L.H.S. of (2.24) can be rewritten in the form

$$\epsilon_{\mu\nu\rho} < \mathrm{Tr}[U(1,x)R^a \left(f^{abc}\partial^\rho \bar{c}^b(x) c^c(x) - [Q\,,\, \partial^\rho \bar{c}^a(x)]\right) U(x,1)]\, O_1 \cdots O_n > \quad . \qquad (2.25)$$

Because of the BRS invariance of the vacuum and of $O_1, ..., O_n$, the action of the BRS charge Q in (2.25) can be transferred on $U(1,x)$ and $U(x,1)$. Using the covariant properties of U under BRS tranformations, one finds that eq.(2.24) is indeed satisfied, which proves the validity of eq.(2.23). Eq.(2.23) plays an important role in examining the behaviour of $< W(L) >$ under deformation of the contour on which the path

ordering is defined. A variational method based on eq.(2.23) has been used in [7] for deriving a skein relation for $< W(L) >$ at order $\frac{1}{k}$. Below we confirm and extend the results of [7] by a different method.

We conclude this section by displaying a peculiar discrete symmetry of the action (2.7) in the flat case $g_{\mu\nu} = \delta_{\mu\nu}$. In addition to the standard C, P and T transformations defined for instance in [19], it is useful to introduce the inversion operator I acting as

$$\begin{aligned} A^a_\mu(x) &\mapsto -A^a_\mu(-x) \quad , \qquad B(x) \mapsto -B(-x) \quad , \\ c(x) &\mapsto c(-x) \quad , \qquad \bar{c}(x) \mapsto \bar{c}(-x) \quad , \\ k &\mapsto -k \quad . \end{aligned} \qquad (2.26)$$

It is easily seen that the total action (2.7) is invariant under CPT and I transformations separately. The inversion operator I will be of great importance for our analysis and has actually a simple geometrical meaning. The I-invariance states that a change in the sign of the coupling constant k is equivalent to (or can be compensated by) reversing the orientation of the three-manifold M^3. As we show now, the I-symmetry implies that $< W(L) >$ computed with the coupling constant k is equal to $< W(\widetilde{L}) >$ computed with the coupling constant $-k$, where the link $\widetilde{L}$ is the mirror image of the link L. In fact, consider the transformation of the Wilson line operator (2.15) under (2.26). One has

$$A_\mu(x)dx^\mu \mapsto -A_\mu(-x)dx^\mu = A_\mu(y)dy^\mu \quad , \qquad (2.27)$$

where

$$y^\mu = -x^\mu \quad . \qquad (2.28)$$

Note that in three dimensions the transformation (2.28) is equivalent, up to a π-rotation, to the inversion of one of the axes. Therefore, the change of variables (2.28) in (2.15) implies

$$W_R(C) \mapsto W_R(\widetilde{C}) \quad , \qquad (2.29)$$

where $\widetilde{C}$ is the mirror (and actually π-rotated) image of the knot C. Combining (2.29) with the rotational and I-invariance of the total action one finds

$$< W(L) > |_k = < W(\widetilde{L}) > |_{-k} \quad . \qquad (2.30)$$

Eq.(2.30) provides a non-trivial check on the computation of $< W(L) >$ and gives some restrictions on the k-dependence of the variables entering in the S_L polynomials. We shall return to this point in sect.5. Note that eq.(2.30) still holds on a general background $g_{\mu\nu}$ because of the metric-independence of $< W(L) >$.

It is worth mentioning that the C-S theory makes sense for both signs of k because the Hamiltonian vanishes on the physical space. In other words, with a given orientation of the three-manifold M^3, both the positive and negative values of k are permitted. These two possibilities are simply related by the exchange of the links, associated with the Wilson line operators, with their mirror images. For this reason, it is sufficient to consider positive values of k.

3. Correlation Functions in Perturbation Theory

In the present paper we will be mainly concerned with the computation of the expectation value $< W_R(C) >$. As is well known, this problem is twofold. First of all, one has to construct well-defined correlation functions $< A_{\mu_1}(x_1) \cdots A_{\mu_j}(x_j) >$. Second, one should give a definite meaning to $W_R(C)$ as a composite operator preserving, as much as possible, its geometrical meaning. In this section we address the first aspect of the problem [8] in the framework of standard perturbation theory. From now on we consider $M^3 = R^3$ (or S^3) with the flat metric $g_{\mu\nu} = \delta_{\mu\nu}$. We start by deriving the propagators following from the action (2.7). The quadratic part of S reads

$$S_{quadr} = \int d^3x \left(\frac{k}{8\pi} \epsilon^{\mu\nu\rho} A^a_\mu \partial_\nu A^a_\rho + \frac{k}{4\pi} A^a_\mu \partial^\mu B^a + \bar{c}^a \partial^\mu \partial_\mu c^a \right) \quad . \qquad (3.1)$$

The ghost propagator is immediately obtained

$$< c^a(x) \bar{c}^b(y) > = -\frac{i}{4\pi} \delta^{ab} \frac{1}{|x-y|} \equiv i\delta^{ab} \Delta(x-y) \quad , \qquad (3.2)$$

$$\Delta(x) = -\frac{1}{4\pi |x|} = \int \frac{d^3p}{(2\pi)^3} \, e^{ipx} \, \frac{1}{p^2} \quad , \qquad (3.3)$$

whereas the inversion of the differential operator in the bosonic part of S_{quadr} needs some care because it is not diagonal in the $\{A^a_\mu \, , \, B^a\}$ space. One finds

$$< A^a_\mu(x) A^b_\nu(y) > = \frac{i}{k} \delta^{ab} \epsilon_{\mu\nu\rho} \frac{(x-y)^\rho}{|x-y|^3} = i \frac{4\pi}{k} \delta^{ab} \epsilon_{\mu\nu\rho} \partial^\rho_x \Delta(x-y) \quad , \qquad (3.4)$$

$$< A^a_\mu(x) B^b(y) > = -\frac{i}{k} \delta^{ab} \frac{(x-y)_\mu}{|x-y|^3} = -i \frac{4\pi}{k} \delta^{ab} \partial^x_\mu \Delta(x-y) \quad , \qquad (3.5)$$

$$< B^a(x) B^b(y) > = 0 \quad . \qquad (3.6)$$

The action (2.7) is renormalizable by power counting. The relevant proper vertices which are potentially divergent are: the vector self-energy $\Pi^{ab}_{\mu\nu}$, the ghost self-energy Π^{ab}, the three-vector vertex $\Gamma^{abc}_{\mu\nu\rho}$ and the ghost-vector vertex Γ^{abc}_μ. As mentioned before, on general grounds the gauge invariance implies that the coupling constant k must be an integer. For this reason one expects vanishing of the β-function. In order to verify this statement, an explicit computation is necessary. The analysis reported below shows that the β-function vanishes at two loops at least; even more, the one- and two-loop corrections to $\Pi^{ab}_{\mu\nu}$ and $\Gamma^{abc}_{\mu\nu\rho}$ are not only finite, but vanish identically!

The relevant one-loop graphs are shown in Fig.1 and Fig.2. The dashed and the

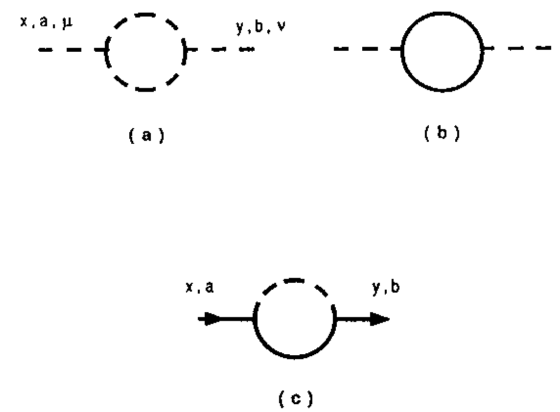

Fig.1 One-loop self-energy graphs.

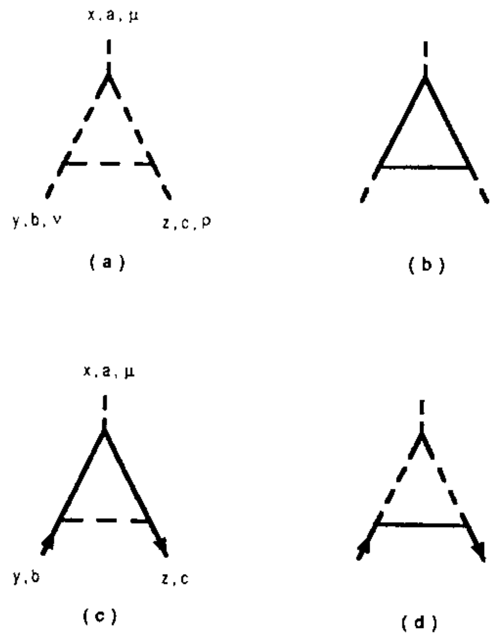

Fig.2 One-loop vertex graphs.

continuous lines represent the gauge and ghost fields respectively. Here and in what follows it is understood that the ghost loop includes both the possible orientations. The individual diagrams in Fig.1 and Fig.2 are divergent and need to be regularized. One possible regularization is the following. Since all the propagators depend on $\Delta(x)$ and there are at most linear (actually logarithmic) divergences, one may simply regularize by substituting

$$\frac{1}{p^2} \mapsto \frac{1}{p^2} - \frac{1}{p^2+M^2} \tag{3.7}$$

in eq.(3.3). Here M is a sort of Pauli-Villars mass. As we shall see in a moment, the theory is at least two-loop finite and then the $M \to \infty$ limit can be trivially performed

(there is no need of counterterms). The values of the graphs displayed in Fig.1 are:

$$(1.a) = c_v \delta^{ab}\, \partial^x_\mu \Delta(x-y)\partial^x_\nu \Delta(x-y) \quad , \tag{3.8}$$

$$(1.b) = -c_v \delta^{ab}\, \partial^x_\mu \Delta(x-y)\partial^x_\nu \Delta(x-y) \quad , \tag{3.9}$$

$$(1.c) = \frac{4\pi}{k} c_v \delta^{ab} \epsilon_{\mu\nu\rho} \partial^x_\mu \partial^x_\nu \Delta(x-y)\partial^x_\rho \Delta(x-y) = 0 \quad . \tag{3.10}$$

Here c_v is the Casimir operator in the adjoint representation

$$\delta^{ab}\, c_v = f^{acd} f^{bcd} \quad , \tag{3.11}$$

$c_v = N$ for $SU(N)$. Equations (3.8-10) obviously imply that at one loop

$$\Pi^{(1)ab}_{\mu\nu}(x-y) = 0 \quad , \qquad \Pi^{(1)ab}(x-y) = 0 \quad . \tag{3.12}$$

Concerning the graphs contributing to $\Gamma^{abc}_{\mu\nu\rho}$ (Fig.2a,b) one has

$$(2.a) = f^{lam} f^{mbn} f^{ncl} \big[\delta^{\mu\alpha}\delta^{\rho\beta}\delta^{\nu\gamma} + \delta^{\nu\alpha}\delta^{\mu\beta}\delta^{\rho\gamma}$$

$$+\delta^{\rho\alpha}\delta^{\nu\beta}\delta^{\mu\gamma} - \delta^{\rho\alpha}\delta^{\mu\beta}\delta^{\nu\gamma}\big] \partial^x_\alpha \Delta(x-y)\partial^y_\beta \Delta(y-z)\partial^z_\gamma \Delta(z-x) \quad , \tag{3.13}$$

$$(2.b) = -f^{lam} f^{mbn} f^{ncl} \left[\delta^{\mu\alpha}\delta^{\nu\beta}\delta^{\rho\gamma} + \delta^{\nu\alpha}\delta^{\rho\beta}\delta^{\mu\gamma}\right] \partial^x_\alpha \Delta(x-y)\partial^y_\beta \Delta(y-z)\partial^z_\gamma \Delta(z-x) \quad . \tag{3.14}$$

The sum of (3.13) and (3.14) is proportional to the completely antisymmetrized product

$$\partial^x_{[\alpha} \Delta(x-y)\partial^y_\beta \Delta(y-z)\partial^z_{\gamma]} \Delta(z-x) \quad , \tag{3.15}$$

which can be easily shown to vanish. Therefore,

$$\Gamma^{(1)abc}_{\mu\nu\rho}(x-y, y-z, z-x) = 0 \quad . \tag{3.16}$$

Finally, for the ghost vertex Γ^{abc}_μ one finds (see Fig.2c,d)

$$(2.c) = \frac{4\pi}{k} f^{lam} f^{mcn} f^{nbl} \epsilon_{\nu\rho\sigma} \partial^x_\mu \partial^x_\nu \Delta(x-y)\partial^y_\rho \Delta(y-z)\partial^z_\sigma \Delta(z-x) \quad , \tag{3.17}$$

$$(2.d) = -\frac{4\pi}{k} f^{lam} f^{mcn} f^{nbl} \epsilon_{\nu\rho\sigma} [\partial^x_\mu \partial^x_\nu \Delta(x-y)\partial^y_\rho \Delta(y-z)\partial^z_\sigma \Delta(z-x)$$
$$+ \partial^y_\mu (\partial^x_\nu \Delta(x-y)\partial^y_\rho \Delta(y-z)\partial^z_\sigma \Delta(z-x))] \quad . \tag{3.18}$$

The sum of (3.17) and (3.18) is also proportional to (3.15) and therefore

$$\Gamma^{(1)abc}_{\mu}(x-y, y-z, z-x) = 0 \quad . \qquad (3.19)$$

The above results imply that the theory is one-loop finite. In addition, the two- and three-point functions remain the *free* ones. This remarkable property reflects the geometrical character of the theory. As we shall show in the next section, the free two-point function (3.4) is related to the Gauss (linking) number. The result (3.12) simply means that there are no "radiative corrections" to the linking number, as it should be.

We turn now to the full expression of the one-loop effective action $\Gamma[A]$. The determinant resulting from the integration of the ghost variables is $\mathrm{Det}(\partial^\mu D_\mu)$, whereas the determinant coming from the bosonic variables is

$$\mathrm{Det}^{-\frac{1}{2}}\left(\mathcal{D}(A)\right) \quad , \qquad (3.20)$$

where

$$\mathcal{D}(A) = \begin{pmatrix} \epsilon^{\mu\nu\rho} D^{ab}_{\nu} & \delta^{ab}\partial^\mu \\ -\delta^{ab}\partial^\rho & 0 \end{pmatrix} \quad . \qquad (3.21)$$

Therefore the one-loop effective action is given by

$$\exp\left(i\Gamma[A]\right) = \mathrm{Det}(\partial^\mu D_\mu)\mathrm{Det}^{-\frac{1}{2}}(\mathcal{D}) \quad . \qquad (3.22)$$

In the Abelian case, the expression (3.22) is related to the Ray-Singer torsion of the three-manifold [1]. Let us now elaborate on the A_μ-dependence of $\Gamma[A]$. Introducing for convenience the matrix $\widehat{A}_\mu$ defined by

$$\widehat{A}^{ab}_{\mu} \equiv f^{acb} A^{c}_{\mu} \quad , \qquad (3.23)$$

one finds for the ghost contribution

$$-i\,\mathrm{Tr}\,\ln\left(1 - \partial^{-2}\partial^\rho \widehat{A}_\rho\right) \quad . \qquad (3.24)$$

Analogously, the contribution coming from $\mathcal{D}$ is

$$\frac{i}{2}\mathrm{Tr}\,\ln\left[1 + \begin{pmatrix} \partial^{-2}\partial_\nu \widehat{A}_\mu - \delta_{\mu\nu}\partial^{-2}\partial_\rho \widehat{A}^\rho & 0 \\ -\epsilon_{\nu\rho\sigma}\partial^{-2}\partial^\rho \widehat{A}^\sigma & 0 \end{pmatrix}\right] =$$

$$= \frac{i}{2}\mathrm{Tr}\,\ln\left(\delta_{\mu\nu} + \partial^{-2}\partial_\nu \widehat{A}_\mu - \delta_{\mu\nu}\partial^{-2}\partial_\rho \widehat{A}^\rho\right) \quad . \qquad (3.25)$$

Finally, in compact form

$$\Gamma^{(1)}[A] = \frac{i}{2}\mathrm{Tr}\,\ln\left(\delta_{\mu\nu} + \partial^{-2}\partial_\nu \widehat{A}_\mu - \delta_{\mu\nu}\partial^{-2}\partial_\rho \widehat{A}^\rho\right) - i\,\mathrm{Tr}\ln\left(1 - \partial^{-2}\partial^\rho \widehat{A}_\rho\right) \quad . \qquad (3.26)$$

In spite of the relative simplicity of expression (3.26), we have not found any direct argument (apart from the explicit computation) for the vanishing of $\Pi^{(1)ab}_{\mu\nu}$ and $\Gamma^{(1)abc}_{\mu\nu\rho}$. One may ask whether a complete cancellation between ghosts and gauge contributions takes place also for the n-point A_μ-proper vertices with $n \geq 4$. This would be very amusing, but unfortunately it does not happen. For instance, the four-point function is of the form

$$\Gamma^{(1)abcd}_{\mu\nu\rho\sigma} = T^{abcd} F_{\mu\nu\rho\sigma}(x, y, z, w) \quad , \qquad (3.27)$$

where

$$T^{abcd} = f^{lam} f^{mbn} f^{ncp} f^{pdl} \qquad (3.28)$$

and $F_{\mu\nu\rho\sigma}$ is non-vanishing. The symmetry properties of T^{abcd} are useful at two-loops.

The diagrams contributing to the two-loop self-energy $\Pi^{(2)ab}_{\mu\nu}$ are shown in Fig.3. The graph (3.a) vanishes because of eq.(3.12). It is clear that the graphs (3.b) and (3.c) cancel. In fact, the sum of the subdiagrams inside the boxes gives precisely $\Pi^{(1)ab}_{\mu\nu}$. Analogously, diagrams (3.d) and (3.e) cancel because of the vanishing of $\Gamma^{(1)abc}_{\mu\nu\rho}$. The same cancellation mechanism works also for the diagrams (3.f) and (3.g) because of equation (3.19). Therefore

$$\Pi^{(2)ab}_{\mu\nu}(x-y) = 0 \quad . \qquad (3.29)$$

In the same way one can show that

$$\Pi^{(2)ab}(x-y) = 0 \quad . \qquad (3.30)$$

Eqs.(3.29,30) simply follow from the Schwinger-Dyson equations and the one-loop vanishing.

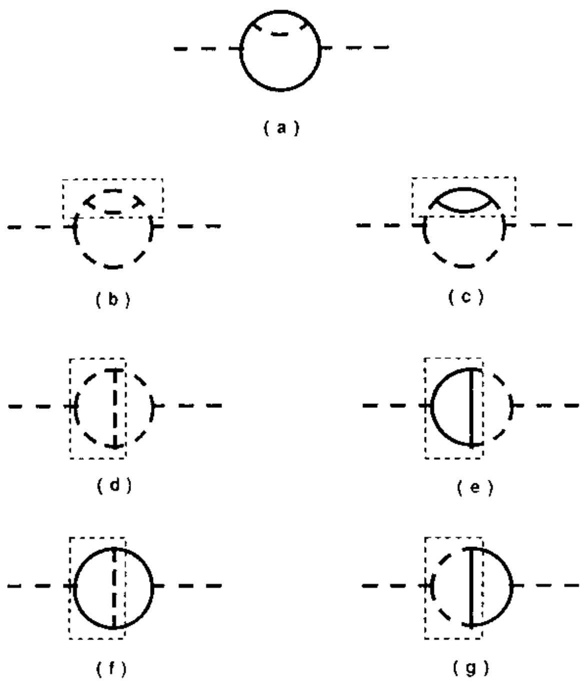

Fig.3 Two-loop self-energy graphs.

Let us consider now the two-loop vertex $\Gamma^{(2)abc}_{\mu\nu\rho}$. The cancellation mechanism explained above immediately applies to the planar graphs contributing to $\Gamma^{(2)abc}_{\mu\nu\rho}$. The remaining graphs are shown in Fig.4. Each of them vanishes because of the peculiar structure in the internal indices. Indeed one can always select a three-vector vertex (as shown in Fig.4) contracted with a diagram contributing to $\Gamma^{(1)abcd}_{\mu\nu\rho\sigma}$. Any such diagram is proportional to the tensor T^{abcd} defined in eq.(3.28) and the contraction is of the form $T^{abcd}f^{ace}$. This tensor structure vanishes because $T^{abcd} = T^{cbad}$. Therefore

$$\Gamma^{(2)abc}_{\mu\nu\rho}(x-y, y-z, z-x) = 0 \quad . \tag{3.31}$$

This concludes the proof of the two-loop finiteness of the C-S model.

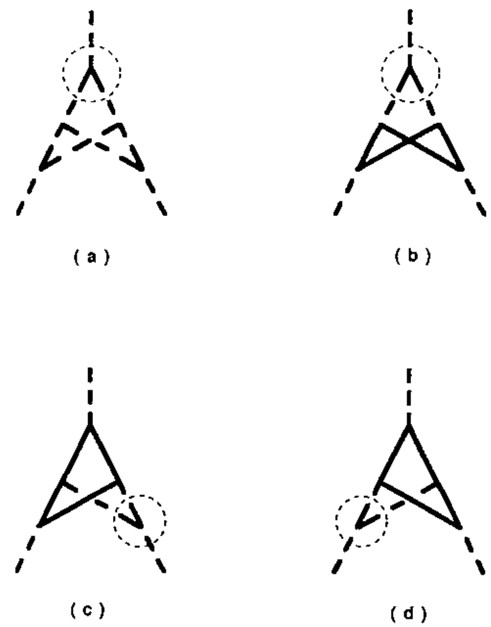

Fig.4 Two-loop non-planar vertex graphs.

Summarizing, we have shown that it is possible to quantize the C-S model in such a way that the symmetries of the classical action (2.1) are realized on the physical subspace defined by the BRS cohomology. Moreover, the quantum C-S theory has a meaningful perturbative expansion. Not only is the theory two-loop finite but, at this order, there are no radiative corrections to $< A^a_\mu(x)A^b_\nu(y) >$ and $< A^a_\mu(x)A^b_\nu(y)A^c_\rho(z) >$ at all. We expect this nice feature to persist at higher orders of the loop expansion.

4. Computing Wilson Line Expectation Values

As is well known, the trace of the holonomy operator along the knot C has the following expansion

$$W_R(C) = \mathrm{Tr}\Big[1 + i\oint_C dx^\mu A_\mu(x) - \oint_C dx^\mu \int^x dy^\nu A_\nu(y)A_\mu(x)$$
$$-i\oint_C dx^\mu \int^x dy^\nu \int^y dz^\rho A_\rho(z)A_\nu(y)A_\mu(x)$$
$$+\oint_C dx^\mu \int^x dy^\nu \int^y dz^\rho \int^z dw^\sigma A_\sigma(w)A_\rho(z)A_\nu(y)A_\mu(x) + \cdots\Big] \quad , \qquad (4.1)$$

where $A_\mu = A^a_\mu R^a$ and R^a are the generators of the R-representation of G. All line integrals are performed on the same contour C. With an explicit parametrization $\{x^\mu(t) : 0 \le t \le 1\}$ of C the multiple integrals take the form

$$\oint_C dx^\mu \int^x dy^\nu \equiv \int_0^1 ds \int_0^s dt\ \dot{x}^\mu(s)\dot{x}^\nu(t) \quad , \qquad (4.2)$$

and so on.

In computing $< W_{R_1}(C_1)\cdots W_{R_n}(C_n) >$ one has to integrate correlation functions of the type $< A_{\mu_1}(x_1)\cdots A_{\mu_j}(x_j) >$ over the paths $C_1, ..., C_n$ in different combinations. As shown before, $< A_{\mu_1}(x_1)\cdots A_{\mu_j}(x_j) >$ present no problems because the theory is renormalizable and, in our case, actually finite. The problems occur when some of the fields appearing in the correlation functions have to be integrated over the same path. Differently from the case of the standard Yang-Mills theory, here the difficulties are not related to the appearance of divergences in the expression of $< W(L) >$, but rather to some kind of ambiguities in finite terms! As we shall see, all these ambiguities can be eliminated on geometrical grounds. It should be noted that the finiteness of $< W(L) >$ is of primary importance both in relating the C-S model with two-dimensional conformal field theory and interpreting $< W(L) >$ as a link invariant. The connection of the C-S theory with quantum groups [4,23,24] also requires $< W(L) >$ to be finite.

In this section we shall illustrate all the above features of the C-S model by computing explicitly $< W(L) >$ at second order in perturbation theory.

4A. Self-Linking Number

Let us consider a single knot C. The expectation value $< W_R(C) >$ has an expansion in powers of $\frac{2\pi}{k}$. Let us denote by $< W_R(C) >^{(n)}$ the $\left(\frac{2\pi}{k}\right)^n$-contribution. Clearly,

$$< W_R(C) >^{(0)} = dim R \quad , \qquad (4.3)$$

whereas the $\frac{2\pi}{k}$-contribution to $< W_R(C) >$ is

$$-\mathrm{Tr}(R^b R^a)\oint_C dx^\mu \int^x dy^\nu < A^b_\nu(y)A^a_\mu(x) > = -i\frac{2\pi}{k} dim R\, c_2(R)\,\varphi(C) \quad , \qquad (4.4)$$

where

$$c_2(R)\mathbf{1} = R^a R^a \qquad (4.5)$$

and $\varphi(C)$ is given by

$$\varphi(C) = \frac{1}{4\pi}\oint_C dx^\mu \oint_C dy^\nu \epsilon_{\mu\nu\rho}\frac{(x-y)^\rho}{|x-y|^3}$$
$$= \frac{1}{4\pi}\int_0^1 ds \int_0^1 dt\, \epsilon_{\mu\nu\rho}\dot{x}^\mu(s)\dot{x}^\nu(t)\frac{(x(s)-x(t))^\rho}{|x(s)-x(t)|^3} \quad . \qquad (4.6)$$

It is well known that the integral (4.6) is *well defined* and *finite*; a detailed analysis of $\varphi(C)$ has been performed for instance by Calugareanu in [25]. It is perhaps instructive to recall the main point of Calugareanu's argument. At first sight one could suspect that the integrand of (4.6) diverges at $t = s$. Anyone who has tried to compute (4.6) knows however that this is not the case. Indeed, expanding $x(t)$ around $t = s$ one finds

$$\epsilon_{\mu\nu\rho}\dot{x}^\mu(s)\dot{x}^\nu(t)\frac{(x(s)-x(t))^\rho}{|x(s)-x(t)|^3} = -\frac{1}{6}|s-t|\epsilon_{\mu\nu\rho}\frac{\dot{x}^\mu(s)\ddot{x}^\nu(s)\dddot{x}^\rho(s)}{|\dot{x}(s)|^3} + O(|s-t|^2) \quad , \qquad (4.7)$$

which shows that the integrand actually vanishes at $t = s$.

The expression (4.6) is known as the cotorsion of C. Unfortunately, the cotorsion is metric dependent and is not invariant under deformations of C, which seems to contradict the claim that $< W_R(C) >$ is a topological invariant. The point is that there exists a certain freedom in defining the composite operator

$$A_\nu(y)dy^\nu\, A_\mu(x)dx^\mu \qquad (4.8)$$

at coincident points and different prescriptions lead in general to different finite values of $< W_R(C) >^{(1)}$. We shall show now that one can use this freedom to attribute a geometrical meaning to $< W_R(C) >^{(1)}$. In the context of knot theory, this problem, known as the self-linking problem, was considered some time ago by mathematicians [11,12,25]. The solution consists of introducing a *framing* contour C_f defined by

$$x^\mu(t) \mapsto y^\mu(t) = x^\mu(t) + \epsilon n^\mu(t) \quad , \quad (\epsilon > 0 \quad , \quad |n(t)| = 1) \quad , \tag{4.9}$$

where n^μ is a vector field orthogonal to C (see Fig.5). Consider then $\varphi_f(C)$ defined by

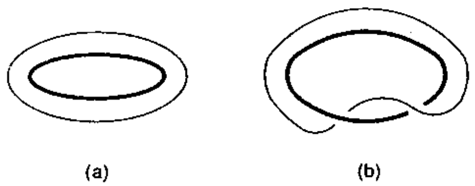

Fig.5 Two examples of framing. The thin line represents the framed contour defined by eq.(4.9) in terms of n^μ.

$$\varphi_f(C) = \lim_{\epsilon\to 0} \frac{1}{4\pi} \int_0^1 ds \int_0^1 dt\, \epsilon_{\mu\nu\rho}\, \dot{x}^\mu(s)(\dot{x}^\nu(t) + \epsilon\dot{n}^\nu(t)) \frac{(x(s) - x(t) - \epsilon n(t))^\rho}{|x(s) - x(t) - \epsilon n(t)|^3}$$

$$= \lim_{\epsilon\to 0} \frac{1}{4\pi} \oint_C dx^\mu \oint_{C_f} dy^\nu \epsilon_{\mu\nu\rho} \frac{(x-y)^\rho}{|x-y|^3} = \lim_{\epsilon\to 0} \chi(C, C_f) \quad . \tag{4.10}$$

In the expression $\chi(C, C_f)$ one recognizes the linking number for two non-intersecting curves C and C_f given by the Gauss integral

$$\chi(C, C_f) = \frac{1}{4\pi} \oint_C dx^\mu \oint_{C_f} dy^\nu \epsilon_{\mu\nu\rho} \frac{(x-y)^\rho}{|x-y|^3} \quad . \tag{4.11}$$

The integral (4.11) is well defined, takes integer values and equals the number of windings of C_f around C. Since $\chi(C, C_f)$ is a topological invariant and, in particular, it is ϵ-independent, the $\epsilon \to 0$ limit in (4.10) is trivially performed and gives

$$\varphi_f(C) = \chi(C, C_f) \quad . \tag{4.12}$$

If the ϵ-limit and the integration in (4.10) were commuting, then $\varphi_f(C)$ would coincide with $\varphi(C)$. This is not the case however; in fact, one has for the self-linking number

$$\varphi_f(C) = \varphi(C) + \tau_f(C) \quad , \tag{4.13}$$

where

$$\tau_f(C) = \frac{1}{2\pi} \int_0^1 ds\, \epsilon_{\mu\nu\rho}\, \dot{n}^\mu(s) \dot{x}^\nu(s) n^\rho(s) |\dot{x}(s)|^{-1} \tag{4.14}$$

is known as the torsion of C. A proof of eq.(4.13) can be found in [11].

Summarizing, the ambiguities that one finds in the computation of $< W_R(C) >$ are finite terms which are related to the behaviour of (4.8) at coincident points. The framing procedure is a way of defining (4.8) at coincident points which permits us to express $< W_R(C) >^{(1)}$ in metric-independent terms:

$$< W_R(C) >_f^{(1)} = -i\frac{2\pi}{k}\, dimR\, c_2(R)\, \varphi_f(C) \quad . \tag{4.15}$$

The expression (4.15) is now a topological invariant but the price one has to pay is a dependence of $< W_R(C) >_f^{(1)}$ on the choice of the framing contour C_f. It turns out that at the quantum level one cannot define a metric-independent $< W_R(C) >$ exclusively in terms of C. Later we shall discuss this point in more detail. For notational simplicity, from now on we will omit the subscript f in $< W_R(C) >_f$.

One may think that the framing dependence of $< W_R(C) >$ leads to a loss of information. On the contrary, the behaviour of $< W_R(C) >$ under a change of framing introduces [3] a new physically meaningful variable related to the conformal weights of the associated two-dimensional conformal field theory.

At this stage one can imagine that the geometrical support of the Wilson line operator behaves like a band; this band is determined by the original path C and the framing vector n^μ entering in (4.9). A twist of the band is equivalent to a change of framing in which $\varphi_f(C)$ gets modified by the unity (Fig.5). As we shall see later, the interpretation of the Wilson line as a band is useful in relating $< W(L) >$ with the link invariants.

Now the question is if the framing procedure consistently extends to higher orders in perturbation theory. In the Abelian case this extension is trivial because $< W(L) >$

can be entirely expressed in terms of linking and self-linking integrals. With a non-Abelian group, the problem becomes more complicated because of the field interactions and the non-commutativity of the group.

4B. The Next Knot Invariant

Let us analyse now the $\left(\frac{2\pi}{k}\right)^2$ part of $< W_R(C) >$. As in the previous subsection, we first write down the expression before the framing has been performed. Since the one-loop radiative correction to $< A_\mu(x)A_\nu(y) >$ vanishes, there are only two contributions, V_1 and V_2. V_1 involves one interaction Lagrangian contracted with the A^3-term of eq.(4.1) and reads

$$V_1 = \left(\frac{2\pi}{k}\right)^2 dim R\, c_v\, c_2(R)\, \varrho_1(C) \quad , \tag{4.16}$$

where

$$\varrho_1(C) = -\frac{1}{32\pi^3} \oint_C dx^\mu \int^x dy^\nu \int^y dz^\rho \epsilon^{\alpha\beta\gamma}\, \epsilon_{\mu\alpha\sigma}\, \epsilon_{\nu\beta\lambda}\, \epsilon_{\rho\gamma\tau}\, I^{\sigma\lambda\tau}(x,y,z) \quad , \tag{4.17}$$

with

$$I^{\sigma\lambda\tau}(x,y,z) = \int d^3w\, \frac{(w-x)^\sigma}{|w-x|^3} \frac{(w-y)^\lambda}{|w-y|^3} \frac{(w-z)^\tau}{|w-z|^3} \quad . \tag{4.18}$$

As shown in Appendix A, the integral (4.18) gives

$$I^{\sigma\lambda\tau}(x,y,z) = \partial_y^\lambda\, \partial_z^\tau\, I^\sigma(y-x, z-x) \quad , \tag{4.19}$$

where

$$I^\sigma(a,b) = \frac{2\pi(|a|+|b|-|a-b|)}{(|a||b|+a\cdot b)} \left[\frac{a^\sigma}{|a|} + \frac{b^\sigma}{|b|}\right] \quad . \tag{4.20}$$

The explicit expression of the integrand of (4.17) is given by eq.(A.8) in Appendix A and one may verify that $\varrho_1(C)$ is finite.

Let us consider now the contribution V_2, which is obtained by contracting the A^4-term of the expansion (4.1); one has

$$V_2 = -\frac{1}{k^2} dim R\, c_2^2(R) \oint_C dx^\mu \int^x dy^\nu \int^y dz^\rho \int^z dw^\sigma \left[\epsilon_{\sigma\rho\alpha}\epsilon_{\nu\mu\beta} \frac{(w-z)^\alpha}{|w-z|^3} \frac{(y-x)^\beta}{|y-x|^3}\right.$$

$$\left. + \epsilon_{\sigma\nu\alpha}\epsilon_{\rho\mu\beta} \frac{(w-y)^\alpha}{|w-y|^3} \frac{(z-x)^\beta}{|z-x|^3} + \epsilon_{\sigma\mu\alpha}\epsilon_{\rho\nu\beta} \frac{(w-x)^\alpha}{|w-x|^3} \frac{(z-y)^\beta}{|z-y|^3}\right]$$
$$+ \frac{1}{2k^2} dim R\, c_v\, c_2(R) \oint_C dx^\mu \int^x dy^\nu \int^y dz^\rho \int^z dw^\sigma \epsilon_{\sigma\nu\alpha}\epsilon_{\rho\mu\beta} \frac{(w-y)^\alpha}{|w-y|^3} \frac{(z-x)^\beta}{|z-x|^3} \quad . \tag{4.21}$$

The integrand of the first integral in (4.21) is a symmetric function in the exchange of any two variables. Consequently, there the integration domain can be taken to be $D = C \times C \times C \times C$, provided one divides by 4!. Afterwards, the three terms of the first integral in (4.21) give the same contribution when integrated over D. Therefore

$$V_2 = -\frac{1}{2}\left(\frac{2\pi}{k}\right)^2 dim R\, c_2^2(R)\, \varphi^2(C) + \left(\frac{2\pi}{k}\right)^2 dim R\, c_v\, c_2(R)\, \varrho_2(C) \quad , \tag{4.22}$$

where $\varphi(C)$ is given by eq.(4.6) and

$$\varrho_2(C) = \frac{1}{8\pi^2} \oint_C dx^\mu \int^x dy^\nu \int^y dz^\rho \int^z dw^\sigma \epsilon_{\sigma\nu\alpha}\epsilon_{\rho\mu\beta} \frac{(w-y)^\alpha}{|w-y|^3} \frac{(z-x)^\beta}{|z-x|^3} \quad . \tag{4.23}$$

The argument following eq.(4.6) implies that $\varrho_2(C)$ is well defined.

Summarizing, $V_1 + V_2$ is finite but unfortunately metric-dependent because of the $\varphi^2(C)$-term. Note that even if the sum

$$\varrho(C) = \varrho_1(C) + \varrho_2(C) \tag{4.24}$$

were metric-dependent, it could never compensate the metric-dependence of $\varphi^2(C)$ because of the different Casimir factors. As discussed below, there is a strong evidence that $\varrho(C)$ is actually a topological invariant.

Let us examine now what happens after framing. First of all, we must extend the framing procedure to operator products of the form

$$A_{\mu_1}(x_1)dx_1^{\mu_1} A_{\mu_2}(x_2)dx_2^{\mu_2} \ldots A_{\mu_n}(x_n)dx_n^{\mu_n} \quad , \tag{4.25}$$

which have to be integrated over the same contour C. Of course, in shifting the operators one must use the same vector field n^μ introduced in (4.9). The simplest recipe for framing consists of shifting the argument of the j-th factor of (4.25) according to

$$A_{\mu_j}(x_j)dx_j^{\mu_j} \mapsto A_{\mu_j}(x_j + (j-1)\epsilon n)d(x_j + (j-1)\epsilon n)^{\mu_j} \quad .$$

Clearly, for $\varepsilon \neq 0$ all factors of the framed version of (4.25) always occur in different points. After performing all the Wick-contractions and integrations, one should take the $\varepsilon \to 0$ limit. Two possibilities may occur: the limit depends on n^μ (see e.g. eqs.(4.13,14)) or the limit is framing-independent. The sum $V_1 + V_2$ illustrates both the situations. First of all, one finds that after framing, the $\varphi^2(C)$-contribution in (4.22) is replaced by $\varphi_f^2(C)$. As expected, the $\varphi_f^2(C)$-term appears with the right coefficient in order to exponentiate the $\frac{1}{k}$-result, precisely like in the Abelian case.

Let us concentrate now on $\varrho(C)$. Our investigation shows that:

i) $\varrho(C)$ is *framing-independent*, that is, the $\varepsilon \to 0$ limit leads precisely to the same expression (4.24) obtained without framing. The details on this point are given in Appendix B;

ii) $\varrho(C)$ represents in *analytic form* the second coefficient of the Alexander-Conway polynomial, which in turn is related to the Arf- and Casson-invariants [14,15]. This remarkable result just follows from the existence of a quantum field theory description of the link invariant polynomials. We shall discuss this point in sect.7.

Concerning the explicit form of $\varrho(C)$, some comments are in order. Let us recall that, apart from the Gauss integral (4.11), the known analytic and explicit expressions of link invariants include the "higher order linking coefficients" (HOLC) [26]. These link invariants $\{\chi_n(L)\}$ are labelled by a positive integer n in such a way that $\chi_n(L)$ makes sense only if $\chi_1(L) = \ldots = \chi_{n-1}(L) = 0$. The HOLC can be expressed [26] in terms of the one-forms

$$u(C_i) = u(C_i;x)_\mu dx^\mu = \frac{1}{4\pi}\left(\oint_{C_i} dy^\nu \, \epsilon_{\mu\nu\rho} \frac{(x-y)^\rho}{|x-y|^3}\right) dx^\mu \quad , \quad C_i \in L \quad . \qquad (4.26)$$

The form $u(C)$ is known as the Alexander dual of the knot C. The first linking coefficient χ_1 can be written in terms of the Gauss integral (4.11) as follows:

$$\chi_1(L) = \max_{i \neq j} |\chi(C_i, C_j)| \quad .$$

A simple example which illustrates the role of HOLC is represented by the Borromean

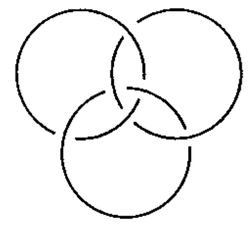

Fig.6 Link B: the Borromean Rings.

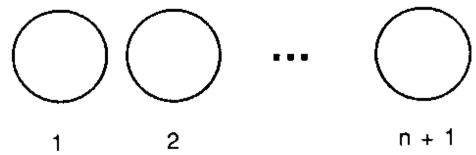

Fig.7 Link U_n: distant union of $(n+1)$ unknots.

rings B, see Fig.6. Exactly like the link U_2 shown in Fig.7, B has vanishing χ_1. It is the second order linking coefficient which distinguishes between B and U_2. Indeed, one has $\chi_2(U_2) = 0$ but $\chi_2(B) = 1$.

A natural question one may pose is whether $\varrho(C)$ is somehow related to the HOLC. Of course, being an invariant of a single knot, $\varrho(C)$ is *not* a HOLC. One could suspect, however, that $\varrho(C)$ can be obtained from the HOLC by performing an appropriate limit in which the different components C_i of a link L coincide. In fact, the expression of $\varrho_1(C)$, eqs.(4.17) and (4.18), looks similar to the integral over the space of the external product of three Alexander duals. However, the integration region of the multiple line integrals *cannot* be reduced to the product $C \times C \times C$; this is also the reason why $\varrho_1(C)$ is actually non-vanishing.

The expression of $\varrho_2(C)$, eq.(4.23), also looks very similar to a "double" Gauss integral. But here again the particular structure of the domain, in which the line integrals have to be performed, plays a crucial role in making $\varrho_2(C)$ substantially different from the product of two Gauss integrals. Note that ϱ_2 vanishes for closed non-intersecting curves on the plane.

Another important issue concerning $\varrho(C)$ is its relation with the coefficients of the HOMFLY polynomials [5]. As is well known, each of these coefficients represents an invariant of ambient isotopy but, apart from a few of them, their topological meaning is completely unknown. One of the reasons for this unpleasant situation it that even if one can compute the coefficients of the HOMFLY polynomials by means of the skein relation, there are no analytic expressions for them. We will show in sect.7 that $\varrho(C)$ is simply related to the second coefficient $a_2(C)$ of the Alexander-Conway polynomial, which in turn is a particular case of the HOMFLY polynomial. It is known [14] that $a_2(C)$ mod 2 is the Arf-invariant associated to C; it is also known [15] that $a_2(C)$ gives the difference between the values of the Casson-invariant associated with two consecutive Dehn surgeries constructed with the knot C. We find it quite remarkable that the C-S field theory provides an analytic expression for $a_2(C)$.

Finally it is worth mentioning that since $\varrho(C)$ represents a $(\frac{1}{k^2})$-contribution to $< W_R(C) >$, the I-symmetry introduced in sect.2 implies $\varrho(C) = \varrho(\widetilde{C})$, that is, $\varrho(C)$ is a knot invariant which does not distinguish a knot C from its mirror image $\widetilde{C}$.

Because of the rather complicated form of $\varrho(C)$, a direct computation for a generic knot is not an easy matter. As an example, we have reported in Appendix C the complete analytic computation of the value of $\varrho(C)$ for the unknot U_0. It turns out that $\varrho(U_0) = -\frac{1}{12}$.

Summarizing, by means of the framing procedure the contribution at second order to $< W_R(C) >$ reads

$$< W_R(C) >^{(2)} = \left(\frac{2\pi}{k}\right)^2 dimR \left[-\frac{1}{2}c_2^2(R)\varphi_f^2(C) + c_v c_2(R)\varrho(C)\right] \quad . \qquad (4.27)$$

In conclusion, by adding (4.3), (4.15) and (4.27), one has for a single knot

$$< W_R(C) > = dimR \Bigg[1 - i\left(\frac{2\pi}{k}\right) c_2(R)\varphi_f(C)$$
$$-\frac{1}{2}\left(\frac{2\pi}{k}\right)^2 c_2^2(R)\varphi_f^2(C) + \left(\frac{2\pi}{k}\right)^2 c_v c_2(R)\varrho(C)\Bigg] + O\left(\frac{1}{k^3}\right) \quad . \qquad (4.28)$$

4C. Interaction between Knots

In computing $< W(L) >$ for a link L with m components $C_1, ..., C_m$, in addition to the "self-energy" contributions (4.28), one has to take into account also the "interactions" between the different components of L. Consider first the case $m = 2$; the first non-vanishing contribution to the interaction between the two knots C_1 and C_2 is of

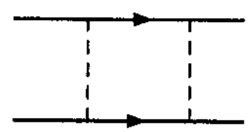

Fig.8 First non-vanishing contribution to the interaction between different knots.

order $(\frac{2\pi}{k})^2$ (see Fig.8) and reads

$$-2\left(\frac{2\pi}{k}\right)^2 \frac{dimR_1\, dimR_2}{dimG} c_2(R_1)\, c_2(R_2)\, \chi^2(C_1, C_2) \quad , \qquad (4.29)$$

where R_1 and R_2 are the representations of G associated with C_1 and C_2. Clearly the expression (4.29) is a topological invariant. In the general case, combining (4.28) with (4.29), one finds

$$< W(L) > = \left(\prod_{i=1}^{m} dimR_i\right) \Bigg\{1 - i\left(\frac{2\pi}{k}\right) \sum_{i=1}^{m} c_2(R_i)\varphi_{f_i}(C_i)$$
$$-\left(\frac{2\pi}{k}\right)^2 \sum_{i=1}^{m} \left[\frac{1}{2}c_2^2(R_i)\varphi_{f_i}^2(C_i) - c_v c_2(R_i)\varrho(C_i)\right]$$
$$-\left(\frac{2\pi}{k}\right)^2 \sum_{i\neq j} c_2(R_i)c_2(R_j)\left[\varphi_{f_i}(C_i)\varphi_{f_j}(C_j) + \frac{\chi^2(C_i, C_j)}{dimG}\right]\Bigg\} + O\left(\frac{1}{k^3}\right) \quad . \qquad (4.30)$$

This equation gives the value of the framed $< W(L) >$ computed in perturbation theory at second order in $(\frac{2\pi}{k})$. Note that, at this stage, the framings of the different components C_i of L are arbitrary and a priori unrelated. We shall see later that, in connecting $< W(L) >$ with the link invariants in R^3, the above arbitrariness is naturally solved.

In the remaining part of the paper, we shall relate $< W(L) >$ with the link invariants and we shall extract the values of the parameters (at order $(\frac{2\pi}{k})^2$) entering the associated link polynomials.

5. Link Invariants

In this section, we recall some basic notions from knot theory [10,16]. Consider two oriented links L_1 and L_2 in R^3 (or S^3). They are called *ambient isotopic* ($L_1 \sim L_2$)

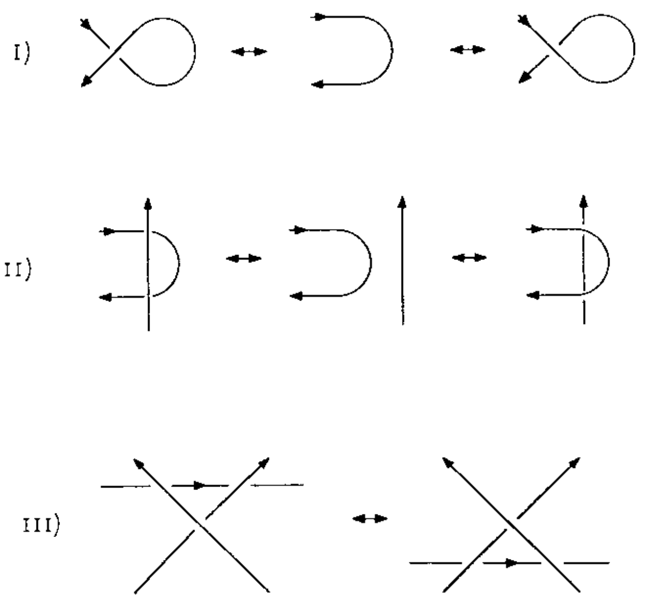

Fig.9 Reidemeister moves of type I, II and III.

if there exists a smooth deformation through embeddings from one to the other. A practical procedure for verifying ambient isotopy is the following one. One associates with any link L a four-valent planar diagram, namely a diagram obtained by projecting

L on a plane in such a way that the projection contains only simple crossing points where the choice of over/under crossing is given. It can be shown that $L_1 \sim L_2$ if and only if there exists a finite sequence of Reidemeister moves, shown in Fig.9, which transforms the diagram of L_1 into the diagram of L_2.

It is useful to introduce also the concept of *regular isotopy for diagrams* [16]. Two diagrams are said to be regular isotopic if one can be obtained from the other by a sequence of Reidemeister moves of type II and type III only. Regular isotopy arises for example when, instead of a link, we consider an oriented twisted band. Clearly, the Reidemeister moves of type I do not relate ambient isotopic bands, see Fig.10. It turns

Fig.10 A twisted band is not invariant under Reidemeister moves of type I.

out that the ambient isotopy of twisted bands is equivalent to regular isotopy for link (or knot) diagrams. In order to clarify this statement, it is useful to introduce the *writhe* (or Tait number) $w(D_L)$ of a *link diagram* D_L. By definition

$$w(D_L) = \sum_p \epsilon(p) \quad , \tag{5.1}$$

where the sum is performed over all crossings of the link diagram and

$$\epsilon(L_\pm) = \pm 1 \quad , \tag{5.2}$$

$L_\pm$ being displayed on Fig.11. Note that the integer $w(D_L)$ is regular isotopy invariant, it is additive and can be computed by analysing locally the diagram D_L associated with L. In order to simplify the notations, instead of D_L, from now on we shall simply write "L", keeping in mind that for regular isotopy invariant quantities L actually indicates a link diagram.

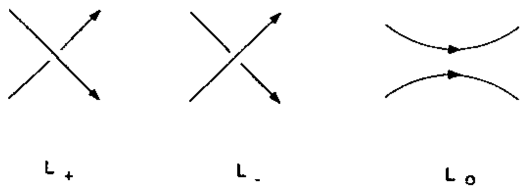

Fig.11 Configurations connected by the skein relation.

The connection between regular isotopy for link diagrams and ambient isotopy of twisted bands can be understood as follows. As we have seen in sect.4, by framing a knot C with an orthonormal vector n^μ, one obtains an oriented band. The Gauss linking number $\varphi_f(C)$ of eq.(4.12) is an ambient isotopy invariant of the band. On the other hand, in passing from a knot C in the space to its projections on a plane, one finds a collection of diagrams. By using Reidemeister moves of type II and III only, these diagrams fall into equivalence classes of regular isotopy. Each class is characterized by an integer number: the writhe w. As shown by Kauffmann [16], the framing defined by

$$\varphi_f(C) = w(C) \tag{5.3}$$

transmutes ambient isotopy of twisted bands into regular isotopy of knot diagrams.

Some examples are illustrated in Fig.12. The *correspondence* (5.3) is called *"vertical framing"* in [4]. Actually, the connection (5.3) was already known to Tait in the early days of knot theory (see [27] p.308). In the context of the self-linking problem, Calugareanu [11] has shown that the vertical framing consists of identifying n^μ with the principal normal in the Frenet basis associated with C. As far as the computation of $< W(L) >$ is concerned, the vertical framing procedure simply means that the framing of each component C_i of any link L is fixed by eq.(5.3). In what follows we shall see that $< W(L) >$, defined by the vertical framing procedure, is directly related to the new link invariants, the S_L-polynomials, introduced in [7].

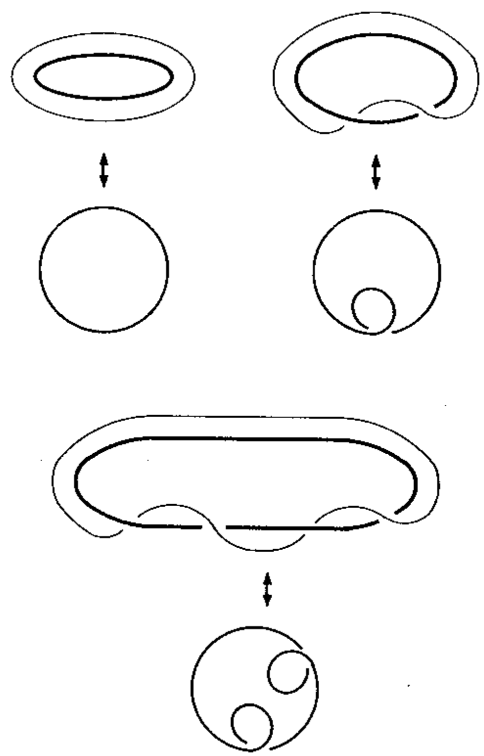

Fig.12 Correspondence between oriented twisted band diagrams and knot diagrams of regular isotopy.

In the particular case of knots in R^3, it is also convenient to introduce an alternative framing procedure, the so-called *"standard framing"*

$$\varphi_f(C) = 0 \quad . \tag{5.4}$$

We shall see that $< W_R(C) >$, computed by standard framing, is simply related to the HOMFLY polynomials.

For discussing the properties of $< W(L) >$, we have introduced in [7] the so-called S_L-polynomials. $S_L(\alpha,\beta,z)$ is a finite Laurent polynomial with integer coefficients, which is regular isotopy invariant. It is defined by the skein relations

$$S_{\widehat{L}_+} = \alpha S_{\widehat{L}_0} \quad , \tag{5.5}$$

$$S_{\widehat{L}_-} = \alpha^{-1} S_{\widehat{L}_0} \quad , \tag{5.6}$$

$$\beta S_{L_+} - \beta^{-1} S_{L_-} = z\, S_{L_0} \quad , \tag{5.7}$$

where the diagrams associated with $\{L_\pm, L_0\}$ and $\{\widehat{L}_\pm, \widehat{L}_0\}$ are shown in Fig.11 and Fig.13 respectively. By means of (4.5-7) and Reidemeister moves of type II and III

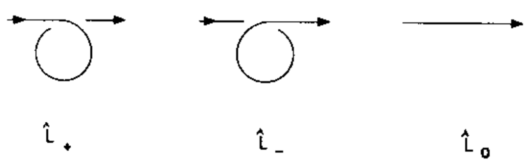

Fig.13 Configurations related by eqs.(5.5) and (5.6).

only, one can construct S_L recursively, starting from the polynomial S_{U_0} of the unknot with zero writhe, which is usually normalized as follows

$$S_{U_0} = 1 \quad . \tag{5.8}$$

As a simple exercise, the reader can derive the S_L polynomials associated with the knot and link diagrams shown in Figs.6,7,14,15. One finds

$$S_{U_n} = u^n \quad , \quad w(U_n) = 0 \quad , \tag{5.9}$$

$$S_T = \alpha^3 \left[z^2 t^{-2} + 2t^{-2} - t^{-4} \right] \quad , \quad w(T) = 3 \quad , \tag{5.10}$$

$$S_K = t^2 - 1 + t^{-2} - z^2 \quad , \quad w(K) = 0 \quad , \tag{5.11}$$

$$S_J = \alpha^{-5} t^4 \left[(1+z^2)(z^2+3-t^2) - t^2 \right] \quad , \quad w(J) = -5 \quad , \tag{5.12}$$

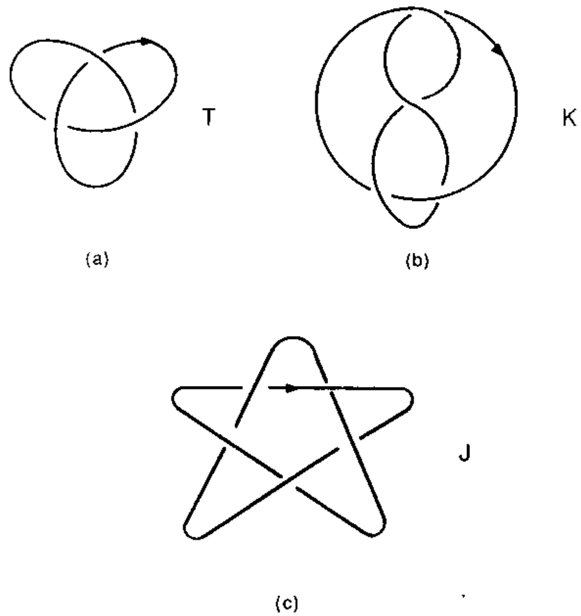

Fig.14 Simple knots: (a) right-handed knot 3_1 (trefoil); (b) knot 4_1; (c) right-handed knot 5_1.

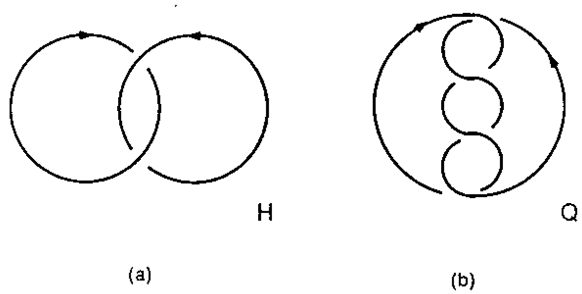

Fig.15 Simple links: (a) link 2_1^2 (Hopf); (b) link 4_1^2.

$$S_H = \alpha^2(ut^{-2} + zt^{-1}) \quad , \quad w(H) = 2 \quad , \tag{5.13}$$

$$S_Q = \alpha^{-4}\left[t^4(1+z^2)u - z^3t^3 - 2zt^3\right] \quad , \quad w(Q) = -4 \quad , \tag{5.14}$$

$$S_B = (1-z^4)u^2 + z^4 \quad , \quad w(B) = 0 \quad , \tag{5.15}$$

where

$$t = \alpha\beta \quad , \quad u = (t - t^{-1})z^{-1} \quad . \tag{5.16}$$

For particular values of the parameters α, β and z in eq.(5.5-7), S_L gives rise to the Kauffman R_L-polynomial [16]

$$R_L(\alpha, z) = S_L(\alpha, \beta = 1, z) \tag{5.17}$$

and to the HOMFLY P_L-polynomial

$$P_L(t, z) = S_L(\alpha = 1, \beta = t, z) \quad . \tag{5.18}$$

Note that R_L is a regular isotopy invariant, while P_L is actually an ambient isotopy invariant because $\alpha = 1$ implies (see eqs.(5.5,6)) invariance under Reidemeister moves of type I also. The S_L- and P_L-polynomials are related each to the other also for arbitrary values of the parameters. One can show [7] that

$$P_L(t = \alpha\beta, z) = \alpha^{-w(L)} S_L(\alpha, \beta, z) \quad . \tag{5.19}$$

As is well known, the link invariant $P_L(1, z)$ is the Alexander-Conway polynomial (or Conway potential function) [13], whereas $P_L(t, z = \sqrt{t} - \frac{1}{\sqrt{t}})$ is the Jones polynomial [6]. For applications to Wilson lines it is useful to recall also that the S_L-polynomials of a link diagram L and its mirror image $\widetilde{L}$ are related according to

$$S_L(\alpha, \beta, z) = S_{\widetilde{L}}(\alpha^{-1}, \beta^{-1}, -z) \quad . \tag{5.20}$$

Eq.(5.20) simply follows from the skein relations (5.5-7). Combining eq.(5.20) with the I-invariance of the C-S action, one can derive the following restriction on the k-dependence of the variables entering the link invariants obtained in the C-S theory:

$$\alpha(k) = \alpha^{-1}(-k) \quad , \quad \beta(k) = \beta^{-1}(-k) \quad , \quad z(k) = -z(-k) \quad . \tag{5.21}$$

We shall see in the next section that eqs.(5.21) are indeed satisfied.

6. Wilson Lines and Link Invariants

As mentioned in sect.2, $< W(L) >=< W_{R_1}(C_1) \cdots W_{R_m}(C_m) >$ computed in the Chern-Simons theory represents a link invariant for any choice of the representations $R_1, ..., R_m$. This has been explicitly verified at second order in perturbation theory, eq.(4.30). Because of the framing dependence, it turns out that $< W(L) >$ can be understood as a *regular* rather than an *ambient* isotopy invariant.

It is of some interest to consider the case when $R_1 = R_2 = ... = R_m$ equal the fundamental representation of $SU(N)$. It was suggested by Witten that in this case $< W(L) >$ is somehow related to the HOMFLY polynomials P_L, (5.18). Even more, assuming that the level ℓ of the associated two-dimensional WZW-model coincides with the coupling constant k of the C-S theory, Witten was able to compute the values of the variables t and z of P_L. The precise relation between $< W(L) >$ and the link invariants has been further clarified in [4,7]. First of all, it is clear that for a generic link L, $< W(L) > \neq P_L$ because $< W(L) >$ is only regular isotopy invariant. By using a variational method, based on the properties of the Wilson line operators in the three-dimensional theory, we have shown [7] that $< W(L) >$ in the fundamental representation satisfies at the first non-trivial order in $\frac{1}{k}$ the skein relations (5.5-7) of a S_L-polynomial with the following values of the parameters:

$$\alpha = 1 - i\frac{2\pi}{k}\left(\frac{N^2-1}{2N}\right) + O\left(\frac{1}{k^2}\right) \quad , \tag{6.1}$$

$$\beta = 1 - i\frac{2\pi}{k}\frac{1}{2N} + O\left(\frac{1}{k^2}\right) \quad , \tag{6.2}$$

$$z = -i\frac{2\pi}{k} + O\left(\frac{1}{k^2}\right) \quad . \tag{6.3}$$

Unfortunately, in the variational approach the $\frac{1}{k^2}$-corrections to α, β and z are not easy to compute. Our purpose in what follows will be to extract from eq (4.30) the values of the parameters (6.1-3) at order $\frac{1}{k^2}$.

For Wilson lines in the fundamental representation of $SU(N)$, eq.(4.30) takes the form

$$< W(L) >= N^m\left\{1 - i\left(\frac{2\pi}{k}\right)\left(\frac{N^2-1}{2N}\right)\sum_{i=1}^{m}\varphi_{f_i}(C_i)\right.$$

$$-\left(\frac{2\pi}{k}\right)^2 \frac{1}{2}\left(\frac{N^2-1}{2N}\right)^2 \sum_{i=1}^{m} \varphi_{f_i}^2(C_i) + \left(\frac{2\pi}{k}\right)^2 N \left(\frac{N^2-1}{2N}\right) \sum_{i=1}^{m} \varrho(C_i)$$

$$-\left(\frac{2\pi}{k}\right)^2 \left(\frac{N^2-1}{2N}\right)^2 \sum_{i\neq j} \varphi_{f_i}(C_i)\varphi_{f_j}(C_j) - \left(\frac{2\pi}{k}\right)^2 \frac{1}{2N} \left(\frac{N^2-1}{2N}\right) \sum_{i\neq j} \chi^2(C_i, C_j)\Big\}$$

$$+O\left(\frac{1}{k^3}\right) \quad . \tag{6.4}$$

In this section we shall use the *vertical framing* procedure, i.e. $\varphi_{f_i}(C_i)$ are fixed by eq.(5.3) in terms of the writhe of the diagrams associated with the single components C_i of L. In this procedure, the skein relations (5.5-6) are associated with a change of the framing given by $\Delta\varphi_{f_i}(C_i) = \Delta w(C_i) = \pm 1$. Consequently, using (6.4) one finds

$$\alpha = 1 - i\left(\frac{2\pi}{k}\right)\left(\frac{N^2-1}{2N}\right) - \frac{1}{2}\left(\frac{2\pi}{k}\right)^2 \left(\frac{N^2-1}{2N}\right)^2 + O\left(\frac{1}{k^3}\right) \quad . \tag{6.5}$$

Since we do not know how $\varrho(C)$ behaves in passing from L_+ and L_- to L_0, we cannot directly verify eq.(5.7) at order $\frac{1}{k^2}$. So, in order to determine β and z at order $\frac{1}{k^2}$, we use an indirect method, which consists of the following steps:

1) We assume that *in vertical framing* the equation

$$\frac{< W(L) >}{< W(U_0) >} = S_L \quad , \tag{6.6}$$

which holds at order $\frac{1}{k}$, extends to higher orders. In order to agree with the convention $S_{U_0} = 1$, we have divided $< W(L) >$ by the value of $< W(U_0) >$.

2) By means of eq.(6.4) and the explicit form of S_L for the link diagrams U_1 and H for example, we determine the $\frac{1}{k^2}$-part of β and z.

3) By using (6.5) and the values of β and z found in point 2), one can then verify that eq.(6.6) holds for whatever link diagram L.

This last point provides a consistency check and shows a posteriori that $< W(L) >$ satisfies the skein relations (5.5-7).

Let us now go through the above steps. From eq.(6.4) one obtains

$$\frac{< W(L) >}{< W(U_0) >} = N^{m-1}\Big\{1 - i\left(\frac{2\pi}{k}\right)\left(\frac{N^2-1}{2N}\right) \sum_{i=1}^{m} \varphi_{f_i}(C_i)$$

$$-\left(\frac{2\pi}{k}\right)^2 \frac{1}{2}\left(\frac{N^2-1}{2N}\right)^2 \sum_{i=1}^{m} \varphi_f^2(C)$$

$$-\left(\frac{2\pi}{k}\right)^2 N \left(\frac{N^2-1}{2N}\right) \left[\varrho(U_0) - \sum_{i=1}^{m} \varrho(C_i)\right]$$

$$-\left(\frac{2\pi}{k}\right)^2 \left(\frac{N^2-1}{2N}\right)^2 \sum_{i\neq j} \varphi_{f_i}(C_i)\varphi_{f_j}(C_j) - \left(\frac{2\pi}{k}\right)^2 \frac{1}{2N} \left(\frac{N^2-1}{2N}\right) \sum_{i\neq j} \chi^2(C_i, C_j)\Big\}$$

$$+O\left(\frac{1}{k^3}\right) \quad , \tag{6.7}$$

and therefore, using that $\varrho(U_0) = -\frac{1}{12}$, one gets

$$\frac{< W(U_1) >}{< W(U_0) >} = N - \frac{1}{24}\left(\frac{2\pi}{k}\right)^2 N(N^2-1) + O\left(\frac{1}{k^3}\right) \quad , \tag{6.8}$$

$$\frac{< W(H) >}{< W(U_0) >} = N - \frac{1}{24}\left(\frac{2\pi}{k}\right)^2 N(N^2-1) - \left(\frac{2\pi}{k}\right)^2 \left(\frac{N^2-1}{2N}\right) + O\left(\frac{1}{k^3}\right) \quad . \tag{6.9}$$

According to eq.(6.6), we now compare the explicit form of S_{U_1} and S_H, see eqs.(5.9) and (5.13), with the values (6.8) and (6.9). In this way one finds for β and z

$$\beta = 1 - i\left(\frac{2\pi}{k}\right)\frac{1}{2N} - \frac{1}{2}\left(\frac{2\pi}{k}\right)^2\left(\frac{1}{2N}\right)^2 + O\left(\frac{1}{k^3}\right) \quad , \tag{6.10}$$

$$z = -i\left(\frac{2\pi}{k}\right) + O\left(\frac{1}{k^3}\right) \quad . \tag{6.11}$$

The values of α, β and z given by (6.5) and (6.10,11) satisfy eqs.(5.21), as they should, because of the invariance of the C-S action under the I-transformations (2.26). In addition to (6.10) and (6.11), one obtains the relation

$$2(\alpha_3 + \beta_3) - N z_3 = \frac{i}{24}\frac{(N^2-1)(N^2-3)}{N} \quad , \tag{6.12}$$

where α_3, β_3 and z_3 are the $\left(\frac{2\pi}{k}\right)^3$-contributions to α, β and z.

Let us concentrate now on point 3) above. We do not have a general proof of eq.(6.6) for *any* link diagram. However, in all the particular cases we have examined, we found complete agreement with eq.(6.6). There are also wide classes of link diagrams, involving connected sums of links, for which it is not difficult to prove eq.(6.6). One of the problems in verifying eq.(6.6) for arbitrary link diagrams is that one has to know $\varrho(C)$ for a generic knot C. Using the explicit analytic expressions (4.17,18) and (4.23), we have performed numerical computations of $\varrho(C)$ for some simple knots. The results fit the predictions

$$\varrho(T) = \frac{23}{12} \quad , \quad \varrho(K) = -\frac{25}{12} \quad , \quad \varrho(J) = \frac{71}{12} \quad , \tag{6.13}$$

following from eq.(6.6).

At order $\frac{1}{k^3}$ the values of the parameters α, β and z, furnished by the C-S theory, are compatible with the expressions

$$\alpha = \exp\left[-i\frac{2\pi}{k}\left(\frac{N^2-1}{2N}\right)\right] \quad , \quad \beta = \exp\left(-i\frac{2\pi}{k}\frac{1}{2N}\right) \quad , \quad z = -2i\sin\frac{\pi}{k} \quad . \tag{6.14}$$

The values (6.14) obey the conditions (5.21) and moreover are consistent with eq.(6.12). Additional highly non-trivial checks on eqs.(6.6) and (6.14) will be performed in the next section.

In the framework of the two-dimensional WZW-model, the parameters α, β and z have been derived in [3]. The expressions (6.14) are in agreement with Witten's results for

$$k = \ell + N \quad , \tag{6.15}$$

where ℓ is the level of the WZW-model. Note that the relation between k and ℓ is not obvious a priori [3,4] and its complete understanding is an open problem [17]. Concerning this point it is useful to recall that:

(a) A finite renormalization of the *bare* coupling constant k_B has nothing to do with the quantum mechanics induced shift $k \mapsto k + N$ (for the *renormalized* coupling constant k) in the values of the skein parameters;

(b) The explicit computation of $< W(L) >$ at second order in perturbation theory, eq.(6.4), confirms eqs.(6.15);

(c) The C-S action has the peculiar perturbatively non-anomalous discrete symmetry I, eq.(2.26). The I-symmetry implies eq.(5.21), which again excludes the shift $k \mapsto k + N$ in $< W(L) >$;

(d) The quantum holonomies of A_μ, computed in an operator approach [24] to the C-S theory, coincide with the monodromies associated with the correlation functions of primary fields in the WZW model (Knizhnik-Zamolodchikov equations [28]), provided that eq.(6.15) holds.

Obviously, any redefinition of k ab initio does not invalidate the above arguments and has nothing to do with a real quantum mechanics induced shift of k.

7. New Relations between the HOMFLY Coefficients

In this section we show that combining the explicit form of the perturbative expansion of $< W(C) >$ with eqs.(6.6) and (6.14), one can extract new information about the coefficients of the HOMFLY polynomials. Let us consider first the relation between the values of $< W(C) >$ computed by *vertical* and *standard* framings. Assuming (6.6), from eqs.(5.3-6) it follows that

$$< W(C) >_{sf} = \alpha^{-w(C)} < W(C) >_{vf} \quad . \tag{7.1}$$

Using eqs.(6.6) and (5.19), by means of (7.1) one finds

$$\frac{< W(C) >_{sf}}{< W(U_0) >} = \alpha^{-w(C)} S_C(\alpha, \beta, z) = P_C(t = \alpha\beta, z) \quad . \tag{7.2}$$

Therefore for knots in R^3 the normalized expectation value of the Wilson operator $W(C)$, computed in the fundamental representation of $SU(N)$ and with standard framing, equals the HOMFLY polynomial. This is in agreement with the results of [4]. Eq.(7.2) is the source of new relations between the HOMFLY coefficients. Indeed, for a single knot C one has [29]

$$P_C(t, z) = \sum_{i \geq 0} \sum_{j \in Z} b_{ij}\, z^{2i}\, t^{2j} \quad , \tag{7.3}$$

where b_{ij} are integers. Inserting in eq.(7.3) the values of the parameters t and z obtained from eq.(6.14), one finds that P_C has an expansion in powers of $\frac{2\pi}{k}$. The first few terms are

$$P_C = \sum_{j \in Z} b_{0j} - ixN \sum_{j \in Z} j\, b_{0j} - x^2 \sum_{j \in Z} \left(\frac{1}{2} N^2 j^2\, b_{0j} + b_{1j}\right)$$

$$+ix^3 \sum_{j\in Z} \left(\frac{N^3}{6} j^3 b_{0j} + N j b_{1j} \right)$$

$$+x^4 \sum_{j\in Z} \left(\frac{N^4}{24} j^4 b_{0j} + \frac{N^2}{2} j^2 b_{1j} + \frac{1}{12} b_{1j} + b_{2j} \right) + \cdots \quad , \tag{7.4}$$

where $x = \frac{2\pi}{k}$. According to (7.2), the expansion (7.4) should coincide (for any N) with the perturbative expansion of $< W(C) >_{sf} < W(U_0) >^{-1}$ in $\frac{2\pi}{k}$. Therefore from eq.(4.28) one finds

$$\sum_{j\in Z} b_{0j} = 1 \quad , \tag{7.5}$$

$$\sum_{j\in Z} j\, b_{0j} = 0 \quad , \tag{7.6}$$

$$\frac{1}{2} N^2 \sum_{j\in Z} j^2 b_{0j} + \sum_{j\in Z} b_{1j} = \frac{1-N^2}{2} \left[\varrho(C) + \frac{1}{12} \right] \quad . \tag{7.7}$$

Let us analyze these equations. As mentioned in sect.5,

$$P_C(t=1,z) = \nabla_C(z) = a_0(C) + a_2(C) z^2 + \cdots \quad , \tag{7.8}$$

where $\nabla_C(z)$ is the Conway potential function. The L.H.S. of eq.(7.5) is precisely the coefficient $a_0(C)$ of ∇_C and eq.(7.5) states that $a_0(C) = 1$, which is indeed known to be the case. The validity of eq.(7.6) has been proven recently by Lickorish and Millet [29]. Since eq.(7.7) holds for any N, one finds

$$\sum_{j\in Z} b_{1j} = \frac{1}{2} \left[\varrho(C) + \frac{1}{12} \right] \quad , \tag{7.9}$$

$$\sum_{j\in Z} j^2 b_{0j} = -\varrho(C) - \frac{1}{12} \quad . \tag{7.10}$$

The L.H.S. of eq.(7.9) coincides with the second coefficient of the Conway potential function and so

$$a_2(C) = \frac{1}{2} \left[\varrho(C) + \frac{1}{12} \right] \quad . \tag{7.11}$$

The result (7.11) is remarkable because it provides an analytic expression for $a_2(C)$. This coefficient in turn plays a distinguished role: it is related to the Arf-invariant [14] by

$$a_2(C) \bmod 2 = \mathrm{Arf}(C) \quad , \tag{7.12}$$

and to the Casson invariant [15] by

$$a_2(C) = \lambda(C_{n+1}) - \lambda(C_n) \quad , \tag{7.13}$$

where $\lambda(C_n)$ is the Casson invariant of the $\frac{1}{n}$-th Dehn surgery constructed by C. It is also known [29] that $a_2(C)$ represents the *total twisting* of the knot C. Finally, eliminating $\varrho(C)$ from eqs.(7.9,10) one gets

$$\sum_{j\in Z} \left(j^2 b_{0j} + 2 b_{1j} \right) = 0 \quad , \tag{7.14}$$

which has been proven in [29].

The existing derivation of eqs.(7.6) and (7.14) [29] is based on the use of the skein relation. Note that in our derivation of eq.(7.14) from eq.(7.7) the particular form of $\varrho(C)$ is irrelevant; it is only the group structure (N^2-1), appearing in the perturbative expansion, which is essential. This fact provides a systematic method for deriving new relations between the HOMFLY coefficients. For example, examining the $\frac{1}{k^3}$- and $\frac{1}{k^4}$-terms in eq.(7.2) one finds [9]

$$\sum_{j\in Z} \left(j^3 b_{0j} + 6 j\, b_{1j} \right) = 0 \quad , \tag{7.15}$$

$$\sum_{j\in Z} \left[j^4 b_{0j} + 2\left(6j^2+1\right) b_{1j} + 24 b_{2j} \right] = 0 \quad . \tag{7.16}$$

Now we are going to derive the general form of the relations of this type. Comparing eqs.(7.5,6,14-16) with eq.(7.4) one realizes that the right-hand sides of eqs.(7.5,6,14-16) are proportional to the coefficients of the x-expansion of P_C for $N = 1$. For a fixed value of N, the parameters t and z are no longer independent. As easily seen from eqs.(6.14), in the case $N = 1$ one has

$$t = \alpha\beta = \exp\left(-\frac{ix}{2} \right) \quad , \qquad z = t - t^{-1} \quad . \tag{7.17}$$

On the other hand, a general argument [29] based on the skein relation implies that for any knot C

$$P_C(t, t - t^{-1}) = 1 \tag{7.18}$$

for any t. A straightforward computation gives for the x-expansion of the L.H.S. of (7.18):

$$P_C\left(\exp -\frac{ix}{2}, -2i\sin\frac{x}{2}\right) = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} \sum_{p=0}^{n} Q_{n,p} \quad , \tag{7.19}$$

where for even q, $0 \leq q \leq n$

$$Q_{n,n-q} = \binom{n}{n-q} \sum_{r\geq 0}^{q/2} \sum_{m=0}^{2r} (-1)^m \binom{2r}{m} (m-r)^q \sum_{k\in Z} k^{n-q} b_{rk}(C) \tag{7.20}$$

and

$$Q_{n,n-q} = 0 \tag{7.21}$$

for odd q. Comparing (7.18) and (7.19), one obtains

$$\sum_{p=0}^{n} Q_{n,p} = \delta_{n0} \quad , \tag{7.22}$$

which is an infinite set ($n = 0, 1, ...$) of relations between the HOMFLY coefficients. For $0 \leq n \leq 4$ one recovers the relations (7.5,6,14-16).

8. Conclusions

The results reported above confirm that the application of quantum field theory for the study of the geometry of three-manifolds and link invariants is a reality. The Chern-Simons theory gives in fact new insights on the subject. The perturbative expansion of $< W(L) >$ provides a set $\mathcal{I}$ of a priori new link and knot invariants. A nice feature is that for each element of $\mathcal{I}$ one can produce in principle an explicit analytic expression. As Lickorish recently stressed [30], the task of finding an analytic interpretation of the link/knot invariants is one of the chief open problems in knot theory. At any given order of perturbation theory there is a finite number of invariants belonging to $\mathcal{I}$. At lowest order one finds the Gauss linking number and the knot invariant $\varrho(C)$. The set $\mathcal{I}$ covers at least all the invariants represented by the coefficients of the HOMFLY polynomial.

The great advantage of the invariants $\mathcal{I}$ is that they have an in principle straightforward generalization to a generic three-manifold M^3. Another application which comes

to mind is the classification problem for knots and links. One may wonder if by means of $\mathcal{I}$ one could distinguish between different links with coinciding HOMFLY polynomials. This fascinating problem, together with the complete understanding of the role of quantum groups emerging as a unifying structure between integrable two-dimensional field theories and link invariants, deserves further investigation.

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Appendix A

We give here the main steps in the computation of the integral $I^{\sigma\lambda\tau}(x,y,z)$ given by eq.(4.18). Obviously one has

$$I^{\sigma\lambda\tau}(x,y,z) = \partial_y^\lambda \partial_z^\tau I^\sigma(y-x, z-x) \quad ,$$

where

$$I^\sigma(a,b) = \int d^3w \, \frac{w^\sigma}{|w|^3 |w-a||w-b|} \quad . \tag{A.1}$$

Introducing the Feynman parameters s and t, one easily performs the integral in w and finds

$$I^\sigma(a,b) = 2\int_0^1 ds \int_0^1 dt \frac{\sqrt{t}}{\sqrt{s(1-s)}} \frac{[sa^\sigma + (1-s)b^\sigma]}{\{s(1-s)(a-b)^2 + t[sa+(1-s)b]^2\}} \quad . \tag{A.2}$$

After the integration in t and the substitution

$$s = \frac{u^2}{1+u^2} \quad ,$$

I^σ takes the form

$$I^\sigma(a,b) = 8\int_0^\infty du \frac{(u^2 a^\sigma + b^\sigma)}{|u^2 a + b|^3} \left[|u^2 a + b| - u|a-b| \arctan\left(\frac{|u^2 a + b|}{u|a-b|}\right)\right] \quad . \tag{A.3}$$

At this stage it is convenient to consider the scalar $a_\sigma I^\sigma(a,b)$ and to perform the integration over u by means of Cauchy's theorem. After some algebra one gets

$$a_\sigma I^\sigma(a,b) = \frac{2\pi}{|b|}(|a| + |b| - |a-b|) \quad . \tag{A.4}$$

Using that $I^\sigma(a,b)$ is symmetric in the exchange $a^\mu \leftrightarrow b^\mu$, one deduces that

$$b_\sigma I^\sigma(a,b) = \frac{2\pi}{|a|}(|a| + |b| - |a-b|) \quad . \tag{A.5}$$

Finally, observing that I^σ has the general structure

$$I^\sigma(a,b) = a^\sigma A(a,b) + b^\sigma A(b,a) \quad , \tag{A.6}$$

one obtains from eqs.(A.4,5) the expression (4.20). Finally,

$$\varrho_1(C) = -\frac{1}{32\pi^3} \oint dx^\mu \int^x dy^\nu \int^y dz^\rho \, H_{\mu\nu\rho}(y-x, z-x) \quad , \tag{A.7}$$

where

$$\begin{aligned} H_{\mu\nu\rho}(a,b) &= C_1 C_2 C_3 \left[\delta_{\nu\rho}(a-b)_\mu + \delta_{\mu\rho} b_\nu - \delta_{\mu\nu} a_\rho\right] \\ &- C_1 C_2^2 C_3 \, \epsilon_{\lambda\sigma\tau} a_\sigma b_\tau \left[\epsilon_{\rho\mu\alpha}\delta_{\lambda\nu}\left(a_\alpha + b_\alpha \frac{|a|}{|b|}\right) + \epsilon_{\nu\mu\alpha}\delta_{\lambda\rho}\left(b_\alpha + a_\alpha\frac{|b|}{|a|}\right)\right] \\ &+ C_1 C_2 \, \epsilon_{\lambda\sigma\tau} a_\sigma b_\tau \left\{\epsilon_{\rho\mu\alpha}\delta_{\lambda\nu}\left[b_\alpha \frac{|a-b|-|a|}{b^2} + (a-b)_\alpha \frac{|a|+|b|}{(a-b)^2}\right]\right. \\ &\left. + \epsilon_{\nu\mu\alpha}\delta_{\lambda\rho}\left[a_\alpha\frac{|a-b|-|b|}{a^2} + (b-a)_\alpha\frac{|a|+|b|}{(a-b)^2}\right]\right\} \end{aligned} \tag{A.8}$$

and

$$\begin{aligned} C_1(a,b) &= \frac{2\pi}{|a||b||a-b|} \quad , \\ C_2(a,b) &= \frac{1}{|a||b| + a_\mu b_\mu} \quad , \\ C_3(a,b) &= |a| + |b| - |a-b| \quad . \end{aligned} \tag{A.9}$$

Appendix B

We show in this appendix the framing-independence of $\varrho(C)$. It turns out that $\varrho_1(C)$ and $\varrho_2(C)$ are separately framing independent. Let us start by analysing $\varrho_2(C)$, whose expression (4.23) is simpler. Using the framing procedure defined after eq.(4.25), one obtains

$$\varrho_2(C)_f = \lim_{\varepsilon\to 0} \frac{1}{8\pi^2} \int_0^1 ds \int_0^s dt \int_0^t du \int_0^u dv \, F(s,t,u,v) \quad , \qquad (B.1)$$

where

$$F(s,t,u,v) = \epsilon_{\sigma\nu\alpha}\epsilon_{\rho\mu\beta}[\dot{x}(s) + 3\varepsilon\dot{n}(s)]^\mu[\dot{x}(t) + 2\varepsilon\dot{n}(t)]^\nu[\dot{x}(u) + \varepsilon\dot{n}(u)]^\rho \dot{x}(v)^\sigma$$
$$\times \frac{[x(t) + 2\varepsilon n(t) - x(v)]^\alpha}{|x(t) + 2\varepsilon n(t) - x(v)|^3} \frac{[x(s) + 3\varepsilon n(s) - x(u) - \varepsilon n(u)]^\beta}{|x(s) + 3\varepsilon n(s) - x(u) - \varepsilon n(u)|^3} \quad . \qquad (B.2)$$

From eq.(B.2) it follows that the potentially dangerous region in the domain of the integral (B.1), which in principle could lead to framing dependence is $u \sim s$ and/or $v \sim t$. In the case when $u \to s$ (or $v \to t$), the specific domain of integration implies that also $t \to s$ (or $u \to t$) and consequently the ε-limit is easily seen to be n^μ-independent. Let us consider the case when $u \to s$ and $v \to t$ simultaneously. Now one has actually $t, u, v \to s$ and following Calugareanu's [11] analysis of $\varphi_f(C)$

$$\varrho_2(C)_f = \varrho_2(C) + \lim_{\delta\to 0}\left[\lim_{\varepsilon\to 0}\int_0^1 ds\, G(s;\varepsilon,\delta)\right] \quad . \qquad (B.3)$$

The integrand of the second term in (B.3), which carries the potential framing dependence, reads

$$G(s;\varepsilon,\delta) = \int_{s-\delta}^s dt \int_{s-\delta}^t du \int_{s-\delta}^u dv \, \frac{N(s,t,u,v)}{D(s,t,u,v)} \qquad (B.4)$$

where

$$N(s,t,u,v) = 4\varepsilon^2 \epsilon_{\sigma\nu\alpha}\epsilon_{\rho\mu\beta}\, n^\alpha(s)\, n^\beta(s)$$
$$\times[(v-s)\ddot{x}^\sigma(s)\dot{x}^\nu(s) + 2\varepsilon\dot{x}^\sigma(s)\dot{n}^\nu(s)][(u-s)\ddot{x}^\rho(s)\dot{x}^\mu(s) + 2\varepsilon\dot{x}^\rho(s)\dot{n}^\mu(s)] \quad , \qquad (B.5)$$

$$D(s,t,u,v) = [(t-v)^2\dot{x}^2(s) + 4\varepsilon^2]^{\frac{3}{2}}[(s-u)^2\dot{x}^2(s) + 4\varepsilon^2]^{\frac{3}{2}} \quad . \qquad (B.6)$$

In studying the limits in (B.3) one can use the following estimates for the integrals in u and v:

$$\left|\int_A^B d\alpha \frac{1}{(\alpha^2 + 4\varepsilon^2)^{\frac{3}{2}}}\right| \le \frac{1}{2\varepsilon^2} \quad , \qquad (B.7)$$

$$\left|\int_A^B d\alpha \frac{\alpha}{(\alpha^2 + 4\varepsilon^2)^{\frac{3}{2}}}\right| \le \frac{\pi}{2\varepsilon} \quad . \qquad (B.8)$$

The $\frac{1}{\varepsilon}$-factors coming from (B.7,8) compensate exactly the ε-factors in the numerator (B.5). Therefore, after performing the t-integration in (B.4) one finds

$$|G(s;\varepsilon,\delta)| \le \delta G_0(s) \quad , \qquad (B.9)$$

where $G_0(s)$ is ε- and δ-independent. Combining (B.3) and (B.9) we see that $\varrho_2(C)$ is indeed framing-independent.

Consider now $\varrho_1(C)$ given by eqs.(A.7-9) and concentrate for example on the term proportional to $C_1C_2C_3$ in (A.8). After some algebra it turns out that the contribution, which is potentially framing-dependent, is proportional to

$$\lim_{\delta\to 0}\left\{\lim_{\varepsilon\to 0}\varepsilon^2\int_0^1 ds\int_{s-\delta}^s dt\int_{s-\delta}^t du \frac{1}{[(s-t)^2+\varepsilon^2]^{\frac{1}{2}}[(s-u)^2+4\varepsilon^2]^{\frac{1}{2}}(s-u)}\right\} \quad . \qquad (B.10)$$

The integral in u can be estimated by

$$\left|\int_A^B d\alpha \frac{1}{\alpha(\alpha^2 + 4\varepsilon^2)^{\frac{1}{2}}}\right| \le \frac{1}{\varepsilon} \quad . \qquad (B.11)$$

Using (B.11) and

$$\lim_{\varepsilon\to 0}\varepsilon\int_A^B d\alpha \frac{1}{(\alpha^2+\varepsilon^2)^{\frac{1}{2}}} = 0 \quad , \qquad (B.12)$$

one finds that the limit (B.10) vanishes. The terms in (A.8), which are proportional to $C_1C_2^2C_3$ and C_1C_2, can be studied analogously and are also n^μ-independent. In conclusion, $\varrho(C)$ is framing-independent.

Appendix C

This appendix is devoted to the computation of $\varrho(U_0)$. As easily seen from eq.(4.23), for a general non-intersecting planar curve C_0 one has $\varrho_2(C_0) = 0$. Therefore, using eqs.(4.17-20) one finds

$$\varrho(C_0) = \varrho_1(C_0) =$$

$$= \frac{1}{16\pi^2}\int_0^1 ds \int_0^s dt \int_0^t du \frac{(|x(s)-x(t)| + |x(u)-x(s)| - |x(t)-x(u)|)}{|x(s)-x(t)||x(u)-x(t)||x(s)-x(u)|}$$

$$\times \frac{\dot{x}_i(s)\,\dot{x}_j(t)\,\dot{x}_k(u)\,\{\delta_{ij}[x(t)-x(s)]_k + \delta_{ik}[x(s)-x(u)]_j + \delta_{jk}[x(u)-x(t)]_i\}}{\{|x(s)-x(t)||x(u)-x(s)| + [x(t)-x(s)]_i[x(u)-x(s)]_i\}} \quad . \qquad (C.1)$$

Consider now the circle

$$U_0 = \{x(s) = (\cos 2\pi s,\ \sin 2\pi s,\ 0) : 0 \le s \le 1\} \quad . \qquad (C.2)$$

Inserting the parametrization (C.2) in (C.1), after some algebra one obtains

$$\varrho(U_0) =$$

$$= -\frac{1}{16\pi^2}\int_0^{2\pi} d\theta \int_0^{\theta} d\chi \int_0^{\chi} d\psi \left[\sin\left(\frac{\theta-\chi}{2}\right) + \sin\left(\frac{\theta-\psi}{2}\right) + \sin\left(\frac{\chi-\psi}{2}\right)\right]^{-1} \quad . \qquad (C.3)$$

The consecutive integrations in (C.3) give

$$\varrho(U_0) = -\frac{1}{16\pi^2}\int_0^{2\pi} d\theta \int_0^{\theta} d\chi \cos^{-2}\left(\frac{\chi-\theta}{4}\right) \ln\left[1 - \coth\left(\frac{\chi-\theta}{4}\right)\tan\frac{\chi}{4}\right] =$$

$$= -\frac{1}{8\pi^2}\int_0^{2\pi} d\theta \left[\theta + 4\coth\frac{\theta}{4}\ln\left(\cos\frac{\theta}{4}\right)\right] = -\frac{1}{12} \quad . \qquad (C.4)$$

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