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Bootstrap Fusions and Tricritical Potts Model away from Criticality

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Abstract

We present the analysis of the tricritical 3-state Potts model perturbed by the energy density field $\epsilon = \phi_{(\frac{1}{7},\frac{1}{7})}$ and the S-matrices of the (conjectured) field theory. A general scheme for solving the minimal integrable models starting from the possible bootstrap fusions is also discussed.

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The completely integrable 2-D models obey the remarkable property of having infinite set of commuting conserved charges P_s. As a consequence their S-matrices are purely elastic and factorizable into products of two particle S-matrices. Although the problem of classification of all possible sets of conservation laws P_s is far from the solution, the recent observation by Zamolodchikov [1,2,3] allows to classify a subset of integrable models, those having an action $A = A_{conf.} + g_i\phi_i$ (ϕ_i are given fields from the conformal grid), i.e. the models which RG fixed points or certain limits (say $g_i \to 0$) are described by the conformal minimal models. The Zamolodchikov's conjecture is that for each conformal minimal model there exist a few integrable perturbations ϕ_{12}, ϕ_{21}, ϕ_{13}, etc. (?) leading to *"minimal" integrable models* with finite mumber of massive particles. In other words one should find as a consistent solution of eqs.(2), (3) and (5) below a finite set of meromorphic functions $S_{AB}(\theta)$ (in the physical strip $0 \leq \mathrm{Im}\theta \leq \pi$) having finite number of physical poles. The last statement is supported by the explicit solution for the thermal and magnetic perturbations of Ising model [1,3], thermal perturbations for tricritical Ising model [4] and 3-state Potts model [2] and the analysis for the nonunitary Yang-Lee edge singularity model [5]. In this letter we present the solution for the thermal perturbation $(h, \bar{h}) = (\frac{1}{7}, \frac{1}{7})$ of the tricritical 3-state Potts model ($c = \frac{6}{7}$). The main result is that the corresponding "minimal" integrable model describing off-critical behaviour of tricritical 3-state Potts model has the following six particle mass spectrum:

$$\begin{aligned} m_{\bar{a}} &= m_a, & m_b &= m_{\bar{b}} = m_a(\sqrt{3}+1)/\sqrt{2}, \\ m_c &= \sqrt{2} m_a, & m_d &= m_a(\sqrt{3}+1). \end{aligned} \tag{1}$$

The Zamlodchikov's recipe [1,2,3] for describing and solving of the *minimal integrable models* can be summarized as follows:

a) to find the spin s of the *coservation laws* for the perturbed model

$$P_{2s+1} = \int \left(T_{2s+2}\mathrm{d}z + \Theta_{2s}\mathrm{d}\bar{z}\right), \qquad P_{2s} = \int \left(W_{2s+1}\mathrm{d}z + Q_{2s-1}\mathrm{d}\bar{z}\right),$$

by studying the corresponding deformations of the well known conformal conservation laws

$$\partial_{\bar{z}} T^{(i)}_{2S+2} = 0 = \partial_{\bar{z}} W^{(i)}_{2s+1}.$$

Here $T^{(i)}_{2S+2}$ and $W^{(i)}_{2s+1}$ are the quasiprimary descendents of the stress-energy tensor T (of spin $s = 2$) and of the spin $s = \frac{(m-1)(m-2)}{4}, m = 1, 2 \pmod 4$ W currents specific for the Virasoro minimal models*;

b) to analyse the specific *residual symmetries* of the perturbed model, i.e. to describe the fundamental particles a_i, b_j, $i = 1, 2, \cdots, N$ as representations of the corresponding "group" of symmetry. This fixes the number of the independent components of the two by two S-matrices $S_{ij}^{kl}(\theta)$ and the symmetry conditions for them;

c) to satisfy the *crossing symmetry*,

$$S_{a\bar{a}}^{T}(\theta) = S_{aa}(i\pi - \theta), \qquad S_{a\bar{a}}^{R}(\theta) = S_{a\bar{a}}^{R}(i\pi - \theta), \tag{2}$$

unitarity condition,

$$\begin{aligned} S_{a\bar{a}}^{T}(-\theta)S_{a\bar{a}}^{T}(\theta) + S_{a\bar{a}}^{R}(-\theta)S_{a\bar{a}}^{R}(\theta) = 1, \\ S_{a\bar{a}}^{T}(-\theta)S_{a\bar{a}}^{R}(\theta) + S_{a\bar{a}}^{R}(-\theta)S_{a\bar{a}}^{T}(\theta) = 0, \end{aligned} \qquad S_{aa}(-\theta)S_{aa}(\theta) = 1, \tag{3}$$

and the factorization (Yang-Baxter) equations (see, for example [7]). The conditions (2) and (3) above are purposely written only for the very specific case of particle-antiparticle scattering in the $U(1)$- or Z_{2N+1}-invariant models;

d) to assume the *bootstrap*, i.e. that all 2-particle bound states belong to the set of the fundamental particles of the model; to find an ansatz for the bootstrap 2-particle fusions of some "minimal subset" of particles consistent with the conservation laws and the symmetries of the models. For example

$$a\ a \to b + a, \quad b\ b \to c, \quad c\ c \to a. \tag{4}$$

In other words this means to find some of the physical poles $\theta = iU_{AB}^{C}$ of $S_{AB}(\theta)$, $(A, B = a, b, c)$ by using of the conservation laws P_s and the symmetries of the model;

e) to solve the *bootstrap equations* [3]

$$\begin{aligned} S_{CD}(\theta) &= S_{BD}(\theta - i\bar{U}_{BC}^{A})S_{AD}(\theta + i\bar{U}_{AC}^{B}), \\ \bar{U} &= \pi - U, \qquad U_{AB}^{C} + U_{BC}^{A} + U_{CA}^{B} = 2\pi, \end{aligned} \tag{5}$$

following from the factorization of S_{ABD}

$$S_{ABD}(\theta_1, \theta_2, \theta_3) = S_{AB}(\theta_{12})S_{AD}(\theta_{13})S_{BD}(\theta_{23}),$$

* the so called extended or W_n minimal models obey more than one such currents [6].

and the bootstrap fusions discussed above.

We note that points b) and d) in this recipe are not independent. If we know all the symmetries of the perturbed model and the representations consistent with conservation laws P_s then the bootstrap fusions are given by the tensor products rules for these representations. What is know in fact are only few hints about these symmetries:

1) the relation between some of the "minimal" integrable models, the affine Toda field theories (say $\hat{A}_1, \hat{A}_2, \hat{E}_6, \hat{E}_7, \hat{E}_8$ etc.) for specific rational values of the squared coupling constants [8,9] and the different coset realizations of the initial conformal model.

2) Although the exact S-matrices for some of the simplest models trivially satisfy the Y-B equations, the classical and quantum group representation method [10,11] can play a role of the symmetry principles at least for the integrable models based on WZW minimal model.

The lack of a general scheme (analogous to BPZ [12] for the minimal conformal models) forces us to use "heuristic" arguments as the bootstrap minimal fusions consistent with $\{P_s\}$ and the residual symmetries remaining unbroken by the perturbation (for example S_3 symmetry for the 3-state Potts models). In fact one can try to derive the fusions from the conservation laws. Here we shall present our analysis of the reversed problem: given a finite set of massive particles $a_i(p_i)$ (or states $|a_i(p_i)\rangle_{\text{in(out)}}$ $i = 1, 2, \cdots, N$) with the 2-particle fusions:

$$a_i(p_i)a_j(p_j) \to \sum_k C_{ijk} a_k(p_i + p_j), \tag{6a}$$

satisfying the momenta conservation laws (P_1):

$$\begin{aligned} m_k^2 - m_i^2 - m_j^2 &= 2m_i m_j \cos U_{ij}^k, \\ U_{ij}^k + U_{jk}^i + U_{ki}^j &= 2\pi. \end{aligned} \tag{6b}$$

Find an infinite set of conserved charges P_s consistent with the bootstrap fusions (6), such that

$$\begin{aligned} P_s a_i(p_i) &= \gamma_s^i \left(\frac{p_i}{m_i}\right)^s a_i(p_i), \\ P_s \prod_{i=1}^n a_i(p_i) &= \sum_{i=1}^n \gamma_s^i \left(\frac{p_i}{m_i}\right)^s \prod_{j=1}^n a_j(p_j). \end{aligned}$$

For $N = 1$ we have only the fusion:

$$a(p_1)\ a(p_2) \to a(p_1 + p_2).$$

In the rapidity variables $p = me^{\theta}$, $\bar{p} = me^{-\theta}$, the consistency conditions reads [3]

$$2\cos\frac{s\pi}{3} = 1, \qquad s = 1, 5 \pmod 6. \tag{7}$$

For N=2 we can have one reducible fusion $a\ a \to a$, $b\ b \to b$ and two new ones:

$$a\ a \to b, \qquad b\ b \to a,$$

or

$$a\ a \to a + b, \qquad b\ b \to a,$$

In the first case the cosistency conditions are

$$2\gamma_s^a \cos(s\bar{U}_{ab}^a) = \gamma_s^b, \qquad 2\gamma_s^b \cos(s\bar{U}_{ba}^b) = \gamma_s^a,$$

and for $\gamma_s^{a,b} \neq 0$:

$$\cos(s\bar{U}_{ab}^a)\cos(s\bar{U}_{ba}^b) = \frac{1}{4}. \tag{8}$$

Two solutions of eq.(8) are given by

$$\bar{U}_{ab}^a = \frac{\pi}{12}, \qquad \bar{U}_{ba}^b = \frac{5\pi}{12}, \tag{8a}$$

with $s = 1, 4, 5, 7, 8, 11 \pmod{12}$, i.e. the exponents of $E_6^{(1)}$ and

$$\bar{U}_{ab}^a = \frac{\pi}{5}, \quad \bar{U}_{ba}^b = \frac{2\pi}{5}, \quad s = 1, 3, 7, 9 \pmod{10}, \tag{8b}$$

In the second case we have to solve the system of equations (7) and (8) and the corresponding solutions for the spins s are the common solutions of (8a, b) and (7).

For $N = 3$ we bave

$$a\ a \to b, \qquad b\ b \to c \qquad, c\ c \to a, \tag{9a}$$

with consistensy condition

$$\cos(sx_1)\cos(sx_2)\cos(sx_3) = \frac{1}{8}, \tag{9b}$$

and solutions:

$$x_1 = \frac{\pi}{9}, x_2 = \frac{2\pi}{9}, x_3 = \frac{4\pi}{9}, \qquad s = 1, 3, 5, 7, 11, 13, 15, 17 \pmod{18},$$
$$x_1 = \frac{\pi}{7}, x_2 = \frac{2\pi}{7}, x_3 = \frac{3\pi}{7}, \qquad s \neq 0 \pmod 7,$$
$$x_1 = \frac{\pi}{20}, x_2 = \frac{4\pi}{20}, x_3 = \frac{9\pi}{20}, \qquad s = 1, 3, 7, 9, 11, 13, 17, 19 \pmod{20},$$

etc. The other possibility is

$$a\,a \to b + c, \qquad b\,b \to c, \qquad c\,c \to a + b, \tag{10}$$

obeying the system of equations

$$\begin{aligned} \cos(sx_1)\cos(sx_2)\cos(sx_3) &= \frac{1}{8}, \\ \cos(sx_4)\cos(sx_2) &= \frac{1}{4}, \\ \cos(sx_5)\cos(sx_3) &= \frac{1}{4}. \end{aligned}$$

The corresponding solutions are given by the common solutions of (8a,b) and (9). The next fusions

$$a\,a \to b + c, \qquad b\,b \to c, \qquad c\,c \to a,$$

lead to the following system

$$\begin{aligned} \cos(sx_1)\cos(sx_2)\cos(sx_3) &= \frac{1}{8}, \\ \cos(sx_4)\cos(sx_3) &= \frac{1}{4}, \end{aligned}$$

which solutions are obvious combinations of the previous ones. We give here two more nontrivial fusions

$$a\,a \to b + c, \qquad b\,b \to c + a, \qquad c\,c \to a + b,$$

$$\begin{aligned} \cos(sx_1)\cos(sx_2)\cos(sx_3) &= \frac{1}{8}, \qquad & \cos(sx_1)\cos(sx_5) &= \frac{1}{4}, \\ \cos(sx_4)\cos(sx_5)\cos(sx_6) &= \frac{1}{8}, \qquad & \cos(sx_2)\cos(sx_6) &= \frac{1}{4}, \end{aligned}$$

$$\cos(sx_3)\cos(sx_4) = \frac{1}{4},$$

and

$$a\,a \to b + c, \qquad b\,b \to a, \qquad c\,c \to a,$$

$$\cos(sx_1)\cos(sx_3) = \frac{1}{4}, \qquad \cos(sx_2)\cos(sx_4) = \frac{1}{4}.$$

Continuing in this way one can exhaust all possible fusions of 3-particles . But we didn't have a proof of exhausting all possible solutions of the corresponding consistency conditions.

For $N=4$ it is still possible to exhaust all the fusions. One of the simplest fusions is:

$$a\,a\to b,\quad b\,b\to c,\quad c\,c\to d,\quad d\,d\to a,$$
$$\cos(sx_1)\cos(sx_2)\cos(sx_3)\cos(sx_4)=\frac{1}{16},$$

with solutions:

$$x_1=\frac{\pi}{15},x_2=\frac{2\pi}{15},x_3=\frac{4\pi}{15},x_4=\frac{7\pi}{15},\quad s\neq 0 \ (\mathrm{mod}\ 15),$$
$$x_1=\frac{\pi}{20},x_2=\frac{3\pi}{20},x_3=\frac{7\pi}{20},x_4=\frac{9\pi}{20},\quad s=1,3,4,7,8,9,11,12,13,16,17,19 \ (\mathrm{mod}\ 20),$$
$$x_1=\frac{\pi}{30},x_2=\frac{7\pi}{30},x_3=\frac{11\pi}{30},x_4=\frac{13\pi}{30},$$
$$s=1,2,4,6,7,8,10,11,12,13,14,16,17,18,19,20,22,23,24,26,28,29 \ (\mathrm{mod}\ 30),$$

etc. This type of fusions can be analysed for general N:

$$a_i\,a_i\to a_{i+1},\qquad a_N\,a_N\to a_1,$$
$$\prod_{i=1}^{N}\cos(sx_i)=2^{-N}.$$

Some of the solutions of this equations can be found using the formula

$$\prod_{j=1,(j,m)=1}^{m-1}(2\sin\frac{\pi j}{m})=\begin{cases}P \text{ if } m=P^k,\\ 1 \text{ otherwise.}\end{cases}$$

The case when $\gamma_s^i=0$ only for some i and some s requires more detailed analysis.

Let us assume that the initial finite set of particles contains also antiparticles and neutral particles. One can take $U(1)$, Z_N or $SU(N)$ etc. charged particles. We choose S_3 for our discussions. The case of one charged doublet $(a,\bar{a})$ with symmetry operations as follows:

$$\Omega a=e^{\frac{2\pi i}{3}}a,\qquad \Omega\bar{a}=e^{\frac{-2\pi i}{3}}\bar{a},$$
$$Ca=\bar{a},\qquad C^2=1=\Omega^3,$$

was considered in detail in ref. [2]. It describes minimal integrable model with one massive doublet $m_a=m_{\bar{a}}$ which can be identified as thermal perturbation of the conformal Potts model ($c=4/5$). The explicit construction of the lowest conservation laws shows the even spin P_{2s} ($s=1$, 2) charges are C-odd:

$$CP_{2s}=-P_{2s}C,\qquad \text{i.e.}\quad P_{2s}\bar{a}=-\gamma_{2s}\left(\frac{p}{m}\right)^{2s}\bar{a},$$

and the odd spin P_{2s+1} are C-even. The same phenomena also occured in the case of the thermal perturbation of tricritical 3-state Potts model by the energy density field $\epsilon = \phi_{(\frac{1}{7},\frac{1}{7})}$. In fact at the critical point there exists another chain of conservation laws $\partial_{\bar{z}} W_{(2s+1,0)} = 0$ generated by the descendents of spin-5 current $W_{(5,0)}$ which is C-odd. Repeating the analysis of spins and dimensions of ref. [2] for our case

$$\partial_{\bar{z}} W_{(5,0)} = \lambda B_1 + \cdots, \qquad [\lambda] = (\frac{6}{7}, \frac{6}{7}),$$

one can conclude that $B_1 = \rho \partial_z \phi_{(\frac{22}{7},\frac{1}{7})}$ which is the only C-odd (spin 3) field we can construct. Therefore the lowest odd conservation laws is

$$\partial_{\bar{z}} W_{(5,0)} = \lambda \rho \partial_z \phi_{(\frac{22}{7},\frac{1}{7})},$$

i.e.

$$P_4 = \int \left(W_{(5,0)} \mathrm{d}z + \lambda \rho \phi_{(\frac{22}{7},\frac{1}{7})} \mathrm{d}\bar{z} \right).$$

To prove that the next odd conserved charge is P_8, we shall apply later the Zamolodchikov's counting argument [3] comparing the number of quasiprimary descendents of W_5 with the number of descendents of $\phi_{\frac{22}{7}}$ at level s which are not in the form $L_{-1}(\phi_{\frac{22}{7}})_{s-1}$.

To continue the general discussion of the conservation laws corresponding to given fusion procedure for S_3 charged particles, we conjecture that the even spin P_{2s} are C-odd. The first nontrivial case is of two doublets $(a, \bar{a})$, $(b, \bar{b})$:

$$a\ a \to \bar{a} + \bar{b}, \qquad b\ b \to \bar{a}, \tag{12a}$$

with consistency condition

$$\begin{gathered} \gamma_s^a \cos(s x_a) = \pm \gamma_s^b, \qquad \gamma_s^b \cos(s x_b) = \pm \gamma_s^a, \\ 2\cos(\frac{s\pi}{3}) = \pm 1, \end{gathered} \tag{12b}$$

where the $+$ sign is for s-odd and $-$ sign for s-even. The solution of this system is

$$x_a = \frac{\pi}{12}, \qquad x_b = \frac{5\pi}{12}, \quad s = 1,4,5,7,8,11 \ (\mathrm{mod}\ 12). \tag{12c}$$

With this sign modification one can easily repeat previous analysis for chargeless particles for the case of S_3 particles also. At the end of this general discussion the natural question arises: can we find S-matrices satisfying eqs.(2), (3) and the corresponding modified bootstrap equations for any given bootstrap fusions and the corresponding infinite set of conservation laws? If yes, which are the conformal models generating these minimal integrable models? We have a definite answer only for the case given by eqs.(12).

We shall first prove the existence of some nontrivial conservation laws with spin $s = 1, 4, 5, 7, 8, 11$ in the tricritical 3-state Potts model perturbed by the field $\epsilon(z,\bar{z}) = \phi_{(\frac{1}{7},\frac{1}{7})}$. Since the field $\phi_{(\frac{1}{7},\frac{1}{7})}$ have zero Z_3 charges this perturbation does not destroy the S_3 symmetry of the model. Let $\hat{\mathrm{T}}_{s+1}$ be the space of quasiprimary descendent fields of identity field I at level $s+1$: $\hat{\mathrm{T}}_{s+1} = \mathrm{T}_{s+1}/\partial_z \mathrm{T}_s$, and $\hat{\phi}_s^{h,\bar{h}}$ be the space of quasiprimary descendent fields of the field $\phi^{h,\bar{h}}$ at level s. If $\dim(\hat{\mathrm{T}}_{s+1}) > \dim(\hat{\phi}_s^{h,\bar{h}})$, it means that the mapping $\partial_{\bar{z}} : \hat{\mathrm{T}}_{s+1} \longrightarrow \lambda\hat{\phi}_s^{h,\bar{h}}$ should have a nonvanishing kernel. Therefore there exist fields $\mathrm{T}_{s+1}(z,\bar{z}) \in \mathrm{T}_{s+1}$ and $\phi_{s-1}^{h,\bar{h}}(z,\bar{z}) \in \phi_{s-1}^{h,\bar{h}}$ such that

$$\partial_{\bar{z}} \mathrm{T}_{s+1}(z,\bar{z}) = \lambda \partial_z \phi_{s-1}^{h,\bar{h}}(z,\bar{z}). \tag{13}$$

Analogous arguments were already discussed for the case of descendents of spin-5 current and spin-3 field $\phi_{(\frac{22}{7},\frac{1}{7})}$.

The generating function for the dimensions of the spaces are given in terms of the characters [13]:

$$\chi_{h_{r,s}} = \frac{1}{\prod_{n=1}^{\infty}(1-q^n)} \sum_{n=-\infty}^{\infty} \left(q^{\frac{(2m(m+1)n+(m+1)r-ms)^2-1}{4m(m+1)}} - q^{\frac{(2m(m+1)n+(m+1)r+ms)^2-1}{4m(m+1)}} \right), \tag{14}$$

for the highest weight field $\phi_{h_{r,s}}$ in the minimal model with central charge $c = 1 - \frac{6}{m(m+1)}$. The tricritical 3-state Potts model have $c = \frac{6}{7}$ or $m = 6$. We have then (in an obvious notation):

$$\begin{aligned}
&\sum_{s=0}^{\infty} q^s \dim(\hat{\mathrm{T}}_s) = (1-q)\chi_0(q) + q, \\
&\sum_{s=0}^{\infty} q^{s+\frac{1}{7}} \dim(\hat{\phi}_s^{(\frac{1}{7},\frac{1}{7})}) = (1-q)\chi_{\frac{1}{7}}(q), \\
&\sum_{s=0}^{\infty} q^{s+5} \dim(\hat{\mathrm{W}}_{s+5}) = (1-q)\chi_5(q), \\
&\sum_{s=0}^{\infty} q^{s+\frac{22}{7}} \dim(\hat{\phi}_{s+3}^{(\frac{22}{7},\frac{1}{7})}) = (1-q)\chi_{\frac{22}{7}}(q),
\end{aligned} \tag{15}$$

where $\chi_0(q) = \chi_{h_{1,1}}(q, m = 6)$, and $\chi_{\frac{1}{7}}(q) = \chi_{h_{1,2}}(q, m = 6)$, $\chi_5(q) = \chi_{h_{6,1}}(q, m = 6)$, $\chi_{\frac{22}{7}}(q) = \chi_{h_{1,5}}(q, m = 6)$. The results of the calculation for $s \leq 17$ are shown in Table 1 and Table 2 below.

Table 1: Dimensions of the spaces $\hat{T}_{s+1}$ and $\hat{\phi}_s^{(\frac{1}{7},\frac{1}{7})}$

S	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
$\dim(\hat{T}_{s+1})$	1	0	1	0	2	0	3	1	4	2	7	3	10	7	14	11	22
$\dim(\hat{\phi}_s^{(\frac{1}{7},\frac{1}{7})})$	0	0	1	1	1	2	2	3	4	5	6	9	10	13	17	21	25

Table 2: Dimensions of the spaces $\hat{W}_{s+1}$ and $\hat{\phi}_s^{(\frac{22}{7},\frac{1}{7})}$

S	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
$\dim(\hat{W}_{s+1})$				1	0	1	1	2	1	3	3	5	5	8	8	13	14
$\dim(\hat{\phi}_s^{(\frac{22}{7},\frac{1}{7})})$			1	0	1	1	2	1	4	3	6	6	9	10	16	16	24

From these two tables one dicscovers the C-even conservation laws with spin $s = 1, 5, 7, 11$ and C-odd ones with spins $s = 4, 8$. We conjecture that there exist an infinite number of conservation laws with spin $s = 1, 4, 5, 7, 8, 11 \pmod{12}$. These spins are exactly the Coxeter exponents of E_6 which was expected because of the existing of an alternative construction of the conformal tricritical Potts model as the first model of the WE_6-extended algebra.

As we have shown eqs.(12) give the bootstrap fusions consistent with the conservation laws and with the S_3 symmetry of the perturbed model. According to the analysis of ref.[14] when the anti-particle appears as bound state of the 2-particle a a scattering the corresponding reflection amplitude vanishes, i.e. $S^R_{a\bar{a}} = 0$. Then the Y-B equations are trivially satisfied by S and S^T and we can start solving the bootstrap equations by using of the knowledge (12) of the three poles of the S-matrices:

$$\bar{U}^a_{a\bar{b}} = \frac{1}{12}\pi, \qquad \bar{U}^{\bar{b}}_{a\bar{b}} = \frac{5}{12}\pi, \qquad \bar{U}^a_{a\bar{a}} = \frac{1}{3}\pi.$$

The simplest equations are related with the fusion $a\ a \to \bar{a}$:

$$S^T_{a\bar{a}}(\theta) = S_{aa}(\theta - i\frac{\pi}{3})S_{aa}(\theta + i\frac{\pi}{3}),$$
$$S_{aa}(\theta) = S^T_{a\bar{a}}(\theta - i\frac{\pi}{3})S^T_{a\bar{a}}(\theta + i\frac{\pi}{3}),$$

or equivalently

$$S_{aa}(\theta)S_{aa}(\theta - i\frac{2\pi}{3})S_{aa}(\theta + i\frac{2\pi}{3}) = 1. \tag{16}$$

The minimal solution of eq.(16) which satisfies the unitarity condition has the form:

$$S_{aa}(\theta) = \frac{sh(\frac{\theta}{2} + i\frac{\pi}{3})sh(\frac{\theta}{2} + i\frac{\pi}{12})sh(\frac{\theta}{2} + i\frac{\pi}{4})}{sh(\frac{\theta}{2} - i\frac{\pi}{3})sh(\frac{\theta}{2} - i\frac{\pi}{12})sh(\frac{\theta}{2} - i\frac{\pi}{4})} \equiv f_{\frac{1}{3}}(\theta)f_{\frac{1}{12}}(\theta)f_{\frac{1}{4}}(\theta),$$

where $f_x(\theta) = \frac{sh(\frac{\theta}{2}+ix\pi)}{sh(\frac{\theta}{2}-ix\pi)}$. Obviously $S_{aa}(\theta)$ has two simple poles with positive residues at $\theta_1 = i\frac{2\pi}{3}$ and $\theta_2 = i\frac{\pi}{6}$ which correspond to the physical particles $\bar{a}$ and $\bar{b}$ with masses $m_{\bar{a}} = m_a$ and $m_{\bar{b}} = m_a\frac{\sqrt{3}+1}{\sqrt{2}}$ according to eq.(6b). The simple pole $\theta = i\frac{\pi}{2}$ has negative residue and represents a particle in the cross-channel. In fact by using of the crossing symmetry we get

$$S^T_{a\bar{a}}(\theta) = S_{aa}(i\pi - \theta) = -f_{\frac{1}{6}}(\theta)f_{\frac{1}{4}}(\theta)f_{\frac{5}{12}}(\theta),$$

which has only one simple pole with positive residue at $\theta = i\frac{\pi}{2}$. This pole corresponds to the neutral physical particle c, arising as an $a\ \bar{a}$ bound state:

$$a\ \bar{a} \to c,$$

with mass $m_c = \sqrt{2}m_a$.

The scattering amplitude $S_{\bar{a}b}(\theta)$ can be obtained from the bootstrap equation:

$$S_{\bar{a}b}(\theta) = S_{\bar{a}a}(\theta - i\frac{\pi}{12})S_{\bar{a}a}(\theta + i\frac{\pi}{12}),$$

and the result is:

$$S_{\bar{a}b}(\theta) = f_{\frac{3}{8}}(\theta)f_{\frac{1}{8}}(\theta)f_{\frac{1}{24}}(\theta)f_{\frac{5}{24}}(\theta)\left(f_{\frac{7}{24}}(\theta)\right)^2.$$

The pole structure of $S_{\bar{a}b}$ leads to the following fusions:

$$\bar{a}\ b \to c + d,$$

with $m_d = m_a(\sqrt{3}+1)$. Proceeding in this way we obtain the following full solution of the bootstrap equations with only six particles:

$$
\begin{aligned}
&S_{aa} = f_{\frac{1}{12}} f_{\frac{1}{4}} f_{\frac{1}{3}}, && S^T_{a\bar{a}} = -f_{\frac{1}{6}} f_{\frac{1}{4}} f_{\frac{5}{12}}, \\
&S_{ab} = S_{\bar{a}\bar{b}} = f_{\frac{1}{8}} (f_{\frac{5}{24}})^2 f_{\frac{7}{24}} f_{\frac{3}{8}} f_{\frac{11}{24}}, && S_{a\bar{b}} = S_{\bar{a}b} = f_{\frac{1}{24}} f_{\frac{1}{8}} f_{\frac{5}{24}} (f_{\frac{7}{24}})^2 f_{\frac{3}{8}}, \\
&S_{ac} = S_{\bar{a}c} = f_{\frac{1}{8}} f_{\frac{5}{24}} f_{\frac{7}{24}} f_{\frac{3}{8}}, && S_{ad} = S_{\bar{a}d} = f_{\frac{1}{12}} (f_{\frac{1}{6}})^2 (f_{\frac{1}{4}})^2 (f_{\frac{1}{3}})^2 f_{\frac{5}{12}}, \\
&S_{bb} = (f_{\frac{1}{12}})^2 (f_{\frac{1}{6}})^2 (f_{\frac{1}{4}})^3 (f_{\frac{1}{3}})^3 f_{\frac{5}{12}}, && S^T_{b\bar{b}} = -f_{\frac{1}{12}} (f_{\frac{1}{6}})^3 (f_{\frac{1}{4}})^3 (f_{\frac{1}{3}})^2 (f_{\frac{5}{12}})^2, \\
&S_{bc} = S_{\bar{b}c} = f_{\frac{1}{12}} (f_{\frac{1}{6}})^2 (f_{\frac{1}{4}})^2 (f_{\frac{1}{3}})^2 f_{\frac{5}{12}}, && S_{bd} = S_{\bar{b}d} = f_{\frac{1}{24}} (f_{\frac{1}{8}})^3 (f_{\frac{5}{24}})^4 (f_{\frac{7}{24}})^4 (f_{\frac{3}{8}})^3 f_{\frac{11}{24}}, \\
&S_{cc} = -f_{\frac{1}{12}} f_{\frac{1}{6}} (f_{\frac{1}{4}})^2 f_{\frac{1}{3}} f_{\frac{5}{12}}, && S_{dd} = -(f_{\frac{1}{12}})^3 (f_{\frac{1}{6}})^5 (f_{\frac{1}{4}})^6 (f_{\frac{1}{3}})^5 (f_{\frac{5}{12}})^3, \\
&S_{cd} = f_{\frac{1}{24}} (f_{\frac{1}{8}})^2 (f_{\frac{5}{24}})^3 (f_{\frac{7}{24}})^3 (f_{\frac{3}{8}})^2 f_{\frac{11}{24}}.
\end{aligned}
\tag{17}
$$

The full fusion structure of tricritical 3-state Potts model away from criticality is given as follows:

$$
\begin{aligned}
&a\ a \rightarrow \bar{a} + \bar{b}, && a\ \bar{a} \rightarrow c, \\
&a\ b \rightarrow \bar{a} + \bar{b}, && a\ \bar{b} \rightarrow c + d, \\
&a\ c \rightarrow a + b, && a\ d \rightarrow b, \\
&b\ b \rightarrow \bar{a}, && b\ \bar{b} \rightarrow \text{no bound states}, \\
&b\ c \rightarrow a, && b\ d \rightarrow a, \\
&c\ c \rightarrow c + d, && d\ d \rightarrow \text{no bound states}, \\
&c\ d \rightarrow c.
\end{aligned}
\tag{18}
$$

As one can see from (17) the scattering S-matrices we have obtained have a rich multipole structure. The apperance of double poles was observed [15] and explained [16] in the case of scattering amplitudes of Sine-Gordon model. This mechanism was recently generalized by Christe and Mussardo in ref. [4] to multipoles. They also argue that in the case of higher odd poles (of order 3, 5, etc) one can have new bound states if the residue of the corresponding simple pole apperaring in the Laurent expansion of $S(\theta)$ has appropriate value. The direct check for the higher odd poles in (17) shows that no new particles appeared. The only change is in the five of our fusions (18), namely:

$$
\begin{aligned}
&b\ b \rightarrow \bar{a} + (\bar{b}), && b\ \bar{b} \rightarrow (d), \\
&b\ d \rightarrow a + (b), && c\ d \rightarrow c + (d), \\
&d\ d \rightarrow (c) + (d),
\end{aligned}
\tag{20}
$$

where parenthesis denotes the bound states coming through the odd multipoles.

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