

A general method for the resummation of event-shape distributions in e^+e^- annihilation

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ABSTRACT: We present a novel method for resummation of event shapes to next-to-next-to-leading-logarithmic (NNLL) accuracy. We discuss the technique and describe its implementation in a numerical program in the case of e^+e^- collisions where the resummed prediction is matched to NNLO. We reproduce all the existing predictions and present new results for oblateness and thrust major.

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1 Introduction

Event-shape variables in e^+e^- annihilation are among the most studied QCD observables. Since they are very sensitive to the pattern of QCD radiation, they have been widely used in the past to measure the QCD coupling constant, and to test non-perturbative hadronization models (see e.g. ref. [1] and references therein). The study of event shapes also led to important advances in the understanding of all-order properties of QCD radiation, for instance through the “discovery” of non-global logarithms [2–4]. Fixed order predictions for observables involving up to three jets in e^+e^- collisions have been available up to next-to-next-to-leading order (NNLO) [5–8] for some years. While fixed order calculations provide a good approximation of hard radiation, which contributes to the region where event shapes have rather large values, resummed calculations are required where the bulk of data lies, i.e. in the region dominated by multiple soft-collinear emissions. Next-to-leading logarithmic (NLL) resummations, that include all terms $\mathcal{O}(\alpha_s^n L^n)$ in the exponent of integrated distributions are available for specific observables [9–22]. In ref. [23] a semi-numerical approach was presented to compute the NLL resummation for all event shapes and jet rates that are recursive infrared and collinear (rIRC) safe and are continuously global. Most NLL resummations have been performed for observables that satisfy these minimal requirements. Some recent works address the problem of resumming ratios of angularities which happen to be not IR safe, but still resumable [24]. Their resummations rely on factorisation theorems for double differential distributions of angularities [25, 26]. The method of ref. [23] was subsequently extended and implemented in the computer program CAESAR [27], that also verifies whether a given observable satisfies these properties. This led to a first systematic study of event shapes in hadronic dijet production at NLL accuracy matched to next-to-leading order (NLO) results at hadron colliders [28, 29].

More recently, some observables have been resummed beyond NLL accuracy. These resummations have been so far obtained through observable-dependent factorisation theorems which lead to a full decomposition of the cross section in the infrared limit in terms of different kinematical subprocesses (i.e. soft, collinear, hard) which are then resummed individually through evolution equations. Despite being systematically extendable to all orders, this approach is strictly observable-dependent and requires that the observable can be factorised in some conjugate space. In particular, full next-to-next-to-leading logarithmic (NNLL) predictions are available for a number of event shapes at lepton colliders like thrust $1-T$ [30, 31], heavy jet mass ρ_H [32], jet broadenings B_T, B_W [33], C -parameter [34] and energy-energy-correlation [35].¹ For $1-T$ and ρ_H all N³LL corrections but the four-loop cusp anomalous dimension are also known. Similar observables have been resummed at the same accuracy also in deep inelastic scattering [37–39]. For hadronic collisions, full NNLL resummations are available for processes where a colour singlet is produced at Born level, specifically for the boson’s transverse momentum [36, 40] and ϕ^* [41], the beam thrust [42, 43] and the leading jet’s transverse momentum [44–47], and for heavy quark

¹Note that the NNLL $A^{(3)}$ coefficient in ref. [35] is incomplete. The correct coefficient has been derived in ref. [36].

pair’s transverse momentum [48, 49]. For an arbitrary number of legs, a NNLL accurate resummation is available for the N -jettiness variable [50, 51].

Currently most of the phenomenological interest is devoted to hadron-hadron collisions. However, in view of a possible future e^+e^- machine (see e.g. [52, 53]), it is desirable to improve our description of *generic* e^+e^- event shapes to next-to-next-to-leading logarithmic (NNLL) level, matched to exact next-to-next-to-leading order (NNLO) results. Furthermore, e^+e^- observables provide a simpler laboratory in which to develop new methods, compared to jet production in hadronic collisions. Therefore, in this work we focus on e^+e^- collisions, with the aim to extend the method suggested here to hadron colliders in a future publication.

In this article we derive a general and systematic method to compute NNLL corrections to event shape distributions in e^+e^- collisions. The method is flexible and can handle any rIRC safe observable which is continuously global, without any additional requirement on factorisability of the observable into kinematic subprocesses. The method relies on a semi-numerical approach in which all real corrections can be expressed in terms of four-dimensional phase space integrals to all orders, and can be efficiently implemented using Monte Carlo techniques. The remaining analytic ingredient is a Sudakov form factor, i.e. the exponential of the so-called “radiator”. In the present paper we do not derive a general expression for the NNLL radiator, but we show that the only unknown contribution is universal for classes of observables which scale in the same fashion for a single soft-collinear emission. The latter property allows us to resum a number of observables by using the radiator of those for which a NNLL resummation was previously known. We derive the method and describe its numerical implementation in the program **ARES** (Automated Resummation for Event Shapes). In the present article we limit ourselves to the resummation of NNLL terms, nevertheless the technique described here can be extended systematically to higher logarithmic orders.

The paper is structured as follows. In section 2 we recall the NLL method of ref. [23] in detail, revisiting all the approximations that lead to the derivation of the master resummation formula. In section 3 we describe all NNLL corrections, showing how to derive them systematically. We then apply the resummation method to the following seven event-shape observables: the thrust $1 - T$, the C -parameter, the heavy-jet mass ρ_H , the total and wide-jet broadenings B_T , B_W , the thrust-major T_M , and the oblateness O , for which data from LEP are available. In section 4 we test the resummation program by expanding the resummed cross section to fixed order in the strong coupling. For observables for which an analytic NNLL resummation was previously available in the literature (i.e. thrust, heavy jet mass and jet broadenings), we check our results against the analytic ones up to (and including) $\mathcal{O}(\alpha_s^3)$. For the remaining observables, for which a NNLL analytic result was not available so far (i.e. C -parameter, thrust-major T_M and oblateness O) we check the expansion of the resummed result against the NLO generator **Event2** [54]. In the second part of section 4 we perform a matching to the NNLO distributions obtained with the event generator **EERAD3** [55]. Our conclusions are reported in section 5. A definition of the observables studied here can be found in appendix A. All analytic ingredients used in this article are reported in appendix B. In appendix C we show that for a class of additive

observables (e.g. $1 - T$, C and ρ_H), all the necessary NNLL corrections can be computed analytically, and we give explicit analytic results. The numerical implementation of our method in a Monte Carlo code is discussed in appendix D.

2 Review of NLL resummation

We consider the resummation of a generic continuously global, recursive infrared and collinear (rIRC) safe event-shape observable V , a function of all final-state momenta, in e^+e^- annihilation. We review here the next-to-leading logarithmic (NLL) resummation for these observables. This section is largely inspired by section 2 of ref. [27], which contains a detailed derivation of the NLL resummation for generic event shapes within the CAESAR approach.

At Born level, the final state consists of a quark \tilde{p}_1 and an antiquark \tilde{p}_2 , which are back-to-back. All event shapes we consider vanish in the Born limit, i.e. $V(\{\tilde{p}_1, \tilde{p}_2\}) = 0$.² Beyond Born level, further radiation (of gluons or gluons splitting into quarks) is present and the final state consists in general of n secondary emissions, k_1, \dots, k_n , and of the primary quark and antiquark which recoil against these additional emissions. We denote the value of an event shape by $V(\{\tilde{p}\}, k_1, \dots, k_n)$, with $\{\tilde{p}\} = \{\tilde{p}_1, \tilde{p}_2\}$.

For any final state event, it is possible to use the thrust axis \vec{n}_T to define two like-light vectors, p_1 and p_2 as

$$p_1 = \frac{Q}{2}(1, \vec{n}_T), \quad p_2 = \frac{Q}{2}(1, -\vec{n}_T), \quad (2.1)$$

where Q denotes the total centre of mass energy of the collision. At Born level clearly \tilde{p}_1 and \tilde{p}_2 coincide with p_1 and p_2 .

In order to compute the resummed distribution for an observable V , it is useful to parametrise each emission k_i and its phase-space in terms of Sudakov variables:

$$k_i = z_i^{(1)} p_1 + z_i^{(2)} p_2 + \kappa_{t,i}, \quad (2.2)$$

where $\kappa_{t,i}$ is a space-like four-vector, orthogonal to p_1 and p_2 . In the reference frame in which p_1 and p_2 are given by eq. (2.1), each $\kappa_{t,i}$ has no time-like component and can be written as $\kappa_{t,i} = (0, \vec{k}_{t,i})$, such that $\kappa_{t,i}^2 = -k_{t,i}^2$. Notice that since k_i is massless

$$k_{t,i}^2 = \frac{2(p_1 k_i)2(p_2 k_i)}{2(p_1 p_2)}.$$

We recall that the thrust axis divides each event in two hemispheres $\mathcal{H}^{(1)}$ and $\mathcal{H}^{(2)}$. If all emissions are soft and/or collinear, \tilde{p}_1 and \tilde{p}_2 belong to different hemispheres. We denote by $\mathcal{H}^{(i)}$ the hemisphere containing \tilde{p}_i . Finally, we introduce the emission's rapidity η_i with respect to the thrust axis, which is given by

$$\eta_i = \frac{1}{2} \ln \frac{z_i^{(1)}}{z_i^{(2)}}, \quad \text{with} \quad |\eta_i| < \ln \frac{Q}{k_{t,i}}, \quad (2.3)$$

where the boundary for η_i is obtained by imposing $z_i^{(\ell)} < 1$ for any leg $\ell = 1, 2$.

²In the case of the thrust, the resummation is actually performed for $\tau \equiv 1 - T$.

We consider observables V that obey the following general parametrisation³ for a single soft emission k collinear to leg ℓ (i.e. parton \tilde{p}_ℓ):

$$V_{\text{sc}}(\{\tilde{p}\}, k) = d_\ell g_\ell(\phi) \left(\frac{k_t}{Q}\right)^a e^{-b_\ell \eta^{(\ell)}}, \quad (2.4)$$

where $\eta^{(1)} = \eta$ and $\eta^{(2)} = -\eta$, and ϕ is the angle that the transverse momentum \vec{k}_t forms with a fixed reference vector \vec{n} orthogonal to the thrust axis. Collinear and infrared safety imposes that $a > 0$ and $b_\ell > -a$.

In order to build the NLL resummed cumulative distribution $\Sigma(v)$

$$\Sigma(v) = \frac{1}{\sigma} \int_0^v dv' \frac{d\sigma(v')}{dv'}, \quad (2.5)$$

it is enough to consider an ensemble of soft-collinear partons, emitted independently off the hard legs, together with the corresponding virtual corrections, as follows:

$$\Sigma(v) = \mathcal{H}(Q^2) \sum_{n=0}^{\infty} \frac{1}{n!} \int \prod_i [dk_i] M^2(k_i) \Theta(v - V(\{\tilde{p}\}, k_1, \dots, k_n)). \quad (2.6)$$

Here $\mathcal{H}(Q^2)$ represents virtual corrections to the Born process, normalised to the total cross section σ , and $[dk]M^2(k)$ is the one-gluon emission probability

$$[dk]M^2(k) = dz^{(1)} dz^{(2)} \frac{d\phi}{2\pi} \frac{dk_t^2}{k_t^2} \delta\left(z^{(1)} z^{(2)} - \frac{k_t^2}{Q^2}\right) \frac{\alpha_s^{\text{CMW}}(k_t) C_F}{4\pi} \frac{z^{(1)} p_{gq}(z^{(1)})}{C_F} \frac{z^{(2)} p_{gq}(z^{(2)})}{C_F}, \quad (2.7)$$

with⁴

$$p_{gq}(z) = C_F \frac{1 + (1-z)^2}{z}. \quad (2.8)$$

Notice that $\alpha_s^{\text{CMW}}(k_t)$ is the QCD coupling in the CMW scheme [57]. In this scheme the QCD coupling is defined as the strength of the soft radiation, inclusive in its branchings, and is related to the coupling in the $\overline{\text{MS}}$ scheme ($\alpha_s = \alpha_s^{\overline{\text{MS}}}$) by

$$\alpha_s^{\text{CMW}}(k_t) = \alpha_s(k_t) \left(1 + \frac{\alpha_s(k_t)}{2\pi} K\right) + \mathcal{O}(\alpha_s^3(k_t)), \quad K = \left(\frac{67}{18} - \frac{\pi^2}{6}\right) C_A - \frac{5}{9} n_f. \quad (2.9)$$

The constant K is a remainder of the cancellation of infrared and collinear singularities between unresolved real emissions⁵ and virtual corrections. This term gives NLL contributions starting at order $\alpha_s^2 L^2$, which are universal for all rIRC safe observables, and proportional to the two-loop cusp anomalous dimension [27]. The CMW scheme is an effective way of incorporating such corrections into a redefinition of the coupling.

³All event shapes for which a NLL resummation is known obey this form.

⁴The azimuthal dependence of the squared amplitude can be ignored in the quark-initiated branching. In hadron-hadron and hadron-lepton collisions the primary branching $g \rightarrow gg$ may occur, and the corresponding azimuth-unaveraged splitting functions must be used for a NNLL resummation [56]. However, in special configurations like colour-singlet production, this azimuthal dependence contributes at most at N³LL.

⁵For a definition of resolved and unresolved emissions see text after eq. (2.11).

The soft-collinear limit of eq. (2.7) is obtained by taking the limit $z^{(1)}, z^{(2)} \rightarrow 0$, giving

$$[dk]M_{\text{sc}}^2(k) = \sum_{\ell=1,2} 2C_\ell \frac{\alpha_s^{\text{CMW}}(k_t)}{\pi} \frac{dk_t}{k_t} d\eta^{(\ell)} \Theta\left(\ln\left(\frac{Q}{k_t}\right) - \eta^{(\ell)}\right) \Theta(\eta^{(\ell)}) \frac{d\phi}{2\pi}, \quad (2.10)$$

where C_ℓ is the Casimir relative to leg ℓ (C_F in the present case) and $\eta^{(\ell)}$ is the rapidity with respect to leg ℓ , as defined after eq. (2.4).

Notice that all integrals in eq. (2.6), as well as the function $\mathcal{H}(Q^2)$, are to be considered as suitably regulated, for instance using dimensional regularisation. At NLL the observable is well approximated by its soft-collinear scaling (2.4). We thus decide to rewrite eq. (2.6) as

$$\begin{aligned} \Sigma(v) = \mathcal{H}(Q^2) \sum_{n=0}^{\infty} \frac{1}{n!} \int \prod_i [dk_i] M^2(k_i) \{ & \Theta(v - V_{\text{sc}}(\{\tilde{p}\}, k_1, \dots, k_n)) \\ & + [\Theta(v - V(\{\tilde{p}\}, k_1, \dots, k_n)) - \Theta(v - V_{\text{sc}}(\{\tilde{p}\}, k_1, \dots, k_n))] \}, \end{aligned} \quad (2.11)$$

where $V_{\text{sc}}(\{\tilde{p}\}, k_1, \dots, k_n)$ denotes the observable with all emissions treated as if they were soft and collinear. We decide to divide the integrals in the real term into a contribution due to emissions with $V_{\text{sc}}(\{\tilde{p}\}, k) > \epsilon v$ (that we refer to as *resolved*), and one due to emissions with $V_{\text{sc}}(\{\tilde{p}\}, k) < \epsilon v$ (that we refer to as *unresolved*). Here ϵ is a small parameter, that can be chosen such that $\epsilon \ll 1$ with $\ln(1/\epsilon) \ll \ln(1/v)$. Because of rIRC safety, the latter can be ignored in computing the observable, up to power-suppressed corrections $\mathcal{O}(v)$. Due to the factorised form of the multi-gluon matrix element in eqs. (2.6) and (2.11), at NLL the contribution of unresolved emissions fully exponentiates, leading to

$$\begin{aligned} \Sigma(v) = \mathcal{H}(Q^2) e^{\int^{\epsilon v} [dk] M^2(k)} \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\epsilon v} \prod_i [dk_i] M^2(k_i) \{ & \Theta(v - V_{\text{sc}}(\{\tilde{p}\}, k_1, \dots, k_n)) \\ & + [\Theta(v - V(\{\tilde{p}\}, k_1, \dots, k_n)) - \Theta(v - V_{\text{sc}}(\{\tilde{p}\}, k_1, \dots, k_n))] \}, \end{aligned} \quad (2.12)$$

where we have used the shorthand notations

$$\begin{aligned} \int^{\epsilon v} [dk] M^2(k) &= \int [dk] M^2(k) \Theta(\epsilon v - V_{\text{sc}}(\{\tilde{p}\}, k)), \\ \int_{\epsilon v} \prod_i [dk_i] M^2(k_i) &= \prod_i \int [dk_i] M^2(k_i) \Theta(V_{\text{sc}}(\{\tilde{p}\}, k_i) - \epsilon v). \end{aligned} \quad (2.13)$$

The combination of the unresolved emissions with the virtual corrections in $\mathcal{H}(Q^2)$ gives rise to a Sudakov exponent, representing the probability of having no emissions with $V_{\text{sc}}(\{\tilde{p}\}, k_i) > \epsilon v$, which at NLL accuracy (i.e. neglecting corrections of relative order α_s) reads

$$\mathcal{H}(Q^2) e^{\int^{\epsilon v} [dk] M^2(k)} \simeq e^{-R(\epsilon v)}, \quad (2.14)$$

where

$$R(\epsilon v) \equiv \int [dk] M^2(k) \Theta(V_{\text{sc}}(\{\tilde{p}\}, k) - \epsilon v) = R(v) + \int_{\epsilon v}^v [dk] M^2(k). \quad (2.15)$$

In eqs. (2.12), (2.13), (2.14) and (2.15) any integral over the single-emission's matrix element $[dk]M^2(k)$ has to be interpreted as follows

$$\begin{aligned}
 [dk]M^2(k)\Theta(V_{\text{sc}}(\{\tilde{p}\}, k) - \bar{v}) &= [dk]M_{\text{sc}}^2(k) \sum_{\ell=1,2} \Theta\left(d_\ell g_\ell(\phi) \left(\frac{k_t}{Q}\right)^a e^{-b_\ell \eta^{(\ell)}} - \bar{v}\right) \Theta(\eta^{(\ell)}) \\
 &+ \sum_{\ell=1,2} \frac{dk_t^2}{k_t^2} \frac{dz^{(\ell)}}{z^{(\ell)}} \left(z^{(\ell)} p_\ell(z^{(\ell)}) - 2C_\ell\right) \frac{\alpha_s(k_t^2)}{2\pi} \Theta\left(\frac{d_\ell g_\ell(\phi)}{(z^{(\ell)})^{b_\ell}} \left(\frac{k_t}{Q}\right)^{a+b_\ell} - \bar{v}\right), \quad (2.16)
 \end{aligned}$$

where in the second line we made the replacement $e^{-\eta^{(\ell)}} = k_t/(Qz^{(\ell)})$. Furthermore, the two step functions in eq. (2.16) have to be expanded in order to avoid power suppressed contributions and undesired subleading logarithmic terms. At NLL, one can perform the following approximations in computing the radiator:

$$\begin{aligned}
 \Theta\left(d_\ell g_\ell(\phi) \left(\frac{k_t}{Q}\right)^a e^{-b_\ell \eta^{(\ell)}} - \bar{v}\right) &\simeq \Theta\left(\ln\left(\frac{k_t}{Q}\right)^a e^{-b_\ell \eta^{(\ell)}} - \ln \bar{v}\right) \\
 &+ \delta\left(\ln\left(\frac{k_t}{Q}\right)^a e^{-b_\ell \eta^{(\ell)}} - \ln \bar{v}\right) \ln d_\ell g_\ell(\phi), \quad (2.17)
 \end{aligned}$$

$$\Theta\left(\frac{d_\ell g_\ell(\phi)}{z^{b_\ell}} \left(\frac{k_t}{Q}\right)^{a+b_\ell} - \bar{v}\right) \simeq \Theta\left(\ln\left(\frac{k_t}{Q}\right)^{a+b_\ell} - \ln \bar{v}\right). \quad (2.18)$$

This gives

$$\begin{aligned}
 R(v) \simeq R_{\text{NLL}}(v) &\equiv \int [dk]M_{\text{sc}}^2(k) \sum_{\ell=1,2} \Theta\left(\ln\left(\frac{k_t}{Q}\right)^a e^{-b_\ell \eta^{(\ell)}} - \ln v\right) \Theta(\eta^{(\ell)}) \\
 &+ \int [dk]M_{\text{sc}}^2(k) \sum_{\ell=1,2} \ln \bar{d}_\ell \delta\left(\ln\left(\frac{k_t}{Q}\right)^a e^{-b_\ell \eta^{(\ell)}} - \ln v\right) \Theta(\eta^{(\ell)}) \quad (2.19) \\
 &+ \sum_{\ell=1,2} C_\ell B_\ell \int \frac{dk_t^2}{k_t^2} \frac{\alpha_s(k_t^2)}{2\pi} \Theta\left(\left(\frac{k_t}{Q}\right)^{a+b_\ell} - v\right),
 \end{aligned}$$

where

$$\ln \bar{d}_\ell = \int_0^{2\pi} \frac{d\phi}{2\pi} \ln(d_\ell g_\ell(\phi)), \quad (2.20)$$

and

$$C_\ell B_\ell = \int_0^1 \frac{dz}{z} (z p_{gq}(z) - 2C_\ell). \quad (2.21)$$

In our case $C_\ell B_\ell = -3/2C_F$. $R_{\text{NLL}}(v)$ can be parametrised as

$$R_{\text{NLL}}(v) = -L g_1(\lambda) - g_2(\lambda), \quad (2.22)$$

where $L = \ln(1/v)$, $\lambda = \alpha_s(Q)\beta_0 L$ and $\beta_0 = (11N_c - 4n_f T_F)/(12\pi)$. The functions g_1 and g_2 can be written in terms of the constants a , b_ℓ , d_ℓ and the functions $g_\ell(\phi)$ and are given in appendix B.⁶ We notice that all integrals over real emissions in eq. (2.12) involve an

⁶In appendix B we use a modified definition of $L = \ln(x_V/v)$, and hence of λ , in order to estimate theoretical uncertainties from higher-order logarithmic corrections by varying x_V .

upper and a lower bound on each $V_{\text{sc}}(\{\tilde{p}\}, k_i)$ such that $\epsilon v < V_{\text{sc}}(\{\tilde{p}\}, k_i) \lesssim v$. We remind the reader that ϵ is a small parameter satisfying $\epsilon \ll 1$ and $\ln(1/\epsilon) \ll \ln(1/v)$. The upper bound comes implicitly from the constraint that the observable is smaller than v . Therefore the real-emission phase space is at most single-logarithmic, unlike the corresponding phase space region considered in the radiator $R(v)$, which is double logarithmic. As a consequence, for real emissions, at NLL accuracy, one can consider only the soft-collinear matrix element (i.e. the first line of eq. (2.16)) and replace the observable with its soft-collinear approximation, i.e. neglect the term in the second line of eq. (2.12). This leads to

$$\Sigma(v) = e^{-R_{\text{NLL}}(v)} e^{-\int_{\epsilon v}^v [dk] M_{\text{sc}}^2(k)} \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\epsilon v} \prod_i [dk_i] M_{\text{sc}}^2(k_i) \Theta(v - V_{\text{sc}}(\{\tilde{p}\}, k_1, \dots, k_n)). \quad (2.23)$$

Here the second exponential factor provides the *unresolved* emissions that cancel the dependence on the cutoff ϵ in the *resolved* real emissions, so that the result is finite and independent of ϵ . This gives

$$\Sigma(v) \simeq e^{-R_{\text{NLL}}(v)} \mathcal{F}(v), \quad (2.24)$$

where the function $\mathcal{F}(v)$ contains NLL corrections due to an ensemble of soft and collinear gluons, widely separated in rapidity [27],⁷ and reads

$$\mathcal{F}(v) = e^{-\int_{\epsilon v}^v [dk] M_{\text{sc}}^2(k)} \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\epsilon v} \prod_{i=1}^n [dk_i] M_{\text{sc}}^2(k_i) \Theta(v - V_{\text{sc}}(\{\tilde{p}\}, k_1, \dots, k_n)). \quad (2.25)$$

Although the above expression has all ingredients necessary to achieve NLL accuracy, it contains also subleading effects. We will first explain how to eliminate them, if one seeks a pure NLL result, and then discuss how they can be computed at NNLL accuracy in the next section.

We parametrise the phase space in terms of $v_i = V_{\text{sc}}(\{\tilde{p}\}, k_i)$, i.e. the value that the event shape has in the presence of each *individual* emission k_i (eq. (2.4)). First, it is convenient to divide the phase space according to whether an emission is collinear to p_1 ($\eta_i > 0$) or collinear to p_2 ($\eta_i < 0$). For each emission k_i we introduce the rapidity fractions $\xi_i^{(\ell)} = \eta_i^{(\ell)} / \eta_{\text{max}}^{(\ell)}$ defined as the emission's rapidity divided by the largest available rapidity for a given value of v_i . $\eta_{\text{max}}^{(\ell)}$ is defined as

$$\eta_{\text{max}}^{(\ell)} = \frac{1}{a + b_\ell} \ln \frac{g_\ell(\phi_i) d_\ell}{v_i}. \quad (2.26)$$

We introduce the two functions

$$\begin{aligned} R'_1 \left(\frac{v}{d_1 g_1(\bar{\phi})} \right) &= \int [dk] M_{\text{sc}}^2(k) (2\pi) \delta(\phi - \bar{\phi}) v \delta(v - V_{\text{sc}}(\{\tilde{p}\}, k)) \theta(\eta), \\ R'_2 \left(\frac{v}{d_2 g_2(\bar{\phi})} \right) &= \int [dk] M_{\text{sc}}^2(k) (2\pi) \delta(\phi - \bar{\phi}) v \delta(v - V_{\text{sc}}(\{\tilde{p}\}, k)) \theta(-\eta). \end{aligned} \quad (2.27)$$

⁷The contribution from a phase space region where two gluons are close in rapidity is suppressed by one power of the logarithm, hence it contributes only to NNLL and will be discussed later.

Finally, we introduce $R'(v, \phi)$, defined as

$$R'(v, \phi) = R'_1 \left(\frac{v}{d_1 g_1(\phi)} \right) + R'_2 \left(\frac{v}{d_2 g_2(\phi)} \right). \quad (2.28)$$

Using this parametrisation, we can recast the matrix element for each emission as follows

$$\begin{aligned} [dk_i] M_{\text{sc}}^2(k_i) &= \frac{dv_i}{v_i} \frac{d\phi_i}{2\pi} \sum_{\ell_i=1,2} d\xi_i^{(\ell_i)} \Theta(1 - \xi_i^{(\ell_i)}) \Theta(\xi_i^{(\ell_i)}) R'_{\ell_i} \left(\frac{v_i}{d_{\ell_i} g_{\ell_i}(\phi_i)} \right) \\ &= \frac{d\zeta_i}{\zeta_i} \frac{d\phi_i}{2\pi} \sum_{\ell_i=1,2} d\xi_i^{(\ell_i)} \Theta(1 - \xi_i^{(\ell_i)}) \Theta(\xi_i^{(\ell_i)}) R'_{\ell_i} \left(\frac{\zeta_i v}{d_{\ell_i} g_{\ell_i}(\phi_i)} \right), \end{aligned} \quad (2.29)$$

where $\zeta_i = v_i/v$ is defined as the ratio of the observable's value corresponding to the i^{th} emission to the actual observable's value v .

We can now exploit a fundamental property of event shapes. Given a set of emissions $\{k_1, \dots, k_n\}$, as long as one keeps v_i , ϕ_i and the leg ℓ_i to which k_i is collinear fixed, the value of an event shape does not depend on $\xi_i^{(\ell_i)}$, which can be then integrated out analytically. This makes it possible to simplify $\mathcal{F}(v)$ as follows

$$\begin{aligned} \mathcal{F}(v) &= e^{-\int \frac{d\phi}{2\pi} \int_{\epsilon}^1 \frac{d\zeta}{\zeta} R'(\zeta v, \phi)} \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{i=1}^n \int_{\epsilon}^{\infty} \frac{d\zeta_i}{\zeta_i} \int_0^{2\pi} \frac{d\phi_i}{2\pi} \times \\ &\quad \times \sum_{\ell_i=1,2} R'_{\ell_i} \left(\frac{\zeta_i v}{d_{\ell_i} g_{\ell_i}(\phi_i)} \right) \Theta(v - V_{\text{sc}}(\{\tilde{p}\}, k_1, \dots, k_n)), \end{aligned} \quad (2.30)$$

where k_1, \dots, k_n are now soft and collinear emissions with an arbitrary rapidity fraction. As a last simplification, we can expand each R'_{ℓ} around v

$$R'_{\ell} \left(\frac{\zeta v}{d_{\ell} g_{\ell}(\phi)} \right) = R'_{\ell}(v) + \mathcal{O}(R''_{\ell}) \quad R''_{\ell} = -v \frac{dR'_{\ell}(v)}{dv}, \quad (2.31)$$

and neglect all contributions of order R''_{ℓ} . These constitute a NNLL leftover that will be specifically addressed in section 3.3.1. Notice that

$$R'_{\ell}(v) = \int [dk] M_{\text{sc}}^2(k) v \delta \left(v - \frac{V_{\text{sc}}(\{\tilde{p}\}, k)}{d_{\ell} g_{\ell}(\phi)} \right) \theta(\eta)$$

does not depend on ϕ and on d_{ℓ} any more. This function can be further split as $R'_{\ell}(v) = R'_{\text{NLL},\ell}(v) + \delta R'_{\text{NNLL},\ell}(v)$, where $R'_{\text{NLL},\ell}$ and $\delta R'_{\text{NNLL},\ell}$ are defined in eqs. (B.10) and (B.11), respectively. The NNLL term $\delta R'_{\text{NNLL},\ell}$ contains running coupling effects as well as the contribution of the cusp anomalous dimension through the CMW scheme. This, as explained earlier, encodes the contribution of an inclusive soft-gluon splitting. At NNLL one has to take into account the non-inclusive nature of the observable in the presence of the branching of a soft gluon. This non-inclusive correction is contained in the full set of NNLL contributions (see section 3.3.4), therefore the choice of the CMW scheme in the *resolved* real emission becomes irrelevant (see section 3.3.4). With this simplification, $\mathcal{F}(v) \simeq \mathcal{F}_{\text{NLL}}(\lambda)$ where

$$\mathcal{F}_{\text{NLL}}(\lambda) = \int d\mathcal{Z}[\{R'_{\text{NLL},\ell_i}, k_i\}] \Theta \left(1 - \lim_{v \rightarrow 0} \frac{V_{\text{sc}}(\{\tilde{p}\}, \{k_i\})}{v} \right), \quad (2.32)$$

and subleading terms have been neglected. In eq. (2.32) we have introduced the average of a function $G(\{\tilde{p}\}, \{k_i\})$ over the measure $d\mathcal{Z}$:

$$\begin{aligned} & \int d\mathcal{Z}[\{R'_{\text{NLL},\ell_i}, k_i\}]G(\{\tilde{p}\}, \{k_i\}) \\ &= \epsilon^{R'_{\text{NLL}}} \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{i=1}^n \int_{\epsilon}^{\infty} \frac{d\zeta_i}{\zeta_i} \int_0^{2\pi} \frac{d\phi_i}{2\pi} \sum_{\ell_i=1,2} R'_{\text{NLL},\ell_i} G(\{\tilde{p}\}, k_1, \dots, k_n), \end{aligned} \tag{2.33}$$

where $R'_{\text{NLL}} = R'_{\text{NLL},1} + R'_{\text{NLL},2}$. Note that the dependence on the regulator ϵ cancels in eq. (2.33). The limit $v \rightarrow 0$ in eq. (2.32) is necessary to remove contributions that are power suppressed in v . The existence of this limit in the step function of eq. (2.32) is guaranteed by the rIRC safety property of event shapes here considered, which implies that the quantity $V_{\text{sc}}(\{\tilde{p}\}, k_1, \dots, k_n)/v$ is independent of v , with corrections that scale as a power of v . To conclude, neglecting all terms beyond NLL accuracy, we can write $\Sigma(v)$ in the form

$$\Sigma(v) = e^{Lg_1(\lambda)+g_2(\lambda)} \mathcal{F}_{\text{NLL}}(\lambda). \tag{2.34}$$

3 NNLL resummation

In this section we extend the above treatment to NNLL, illustrating how the various corrections arise. We will first discuss the general structure of the NNLL resummation and then derive the relevant corrections.

3.1 Logarithmic counting for the resolved real emissions

Before extending the above treatment to NNLL, it is worth recalling how, given rIRC safety of the observable, one can define a logarithmic hierarchy in the resolved real emissions, and hence give a precise definition of the multiple emissions function $\mathcal{F}(v)$ at a given logarithmic order. We start by considering an ensemble of n soft emissions. The squared matrix element can be expressed iteratively as a sum of products of matrix elements with a lower number of emissions (from 1 to $n - 1$) plus an irreducible remainder $\tilde{M}^2(k_1, \dots, k_n)$. The first few steps of this iterative definition read

$$\begin{aligned} M^2(k_1) &= \tilde{M}^2(k_1), \\ M^2(k_1, k_2) &= M^2(k_1)M^2(k_2) + \tilde{M}^2(k_1, k_2), \\ M^2(k_1, k_2, k_3) &= M^2(k_1)M^2(k_2)M^2(k_3) + (\tilde{M}^2(k_1, k_2)M^2(k_3) + \text{perm.}) + \tilde{M}^2(k_1, k_2, k_3), \\ M^2(k_1, \dots, k_n) &= \dots \end{aligned} \tag{3.1}$$

The product of single-emission matrix elements clearly defines the abelian contribution, while non-abelian colour factors are associated with the $\tilde{M}^2(k_1, \dots, k_m)$ squared amplitudes. This makes each single \tilde{M} in the above decomposition invariant under gauge transformations. The $\tilde{M}^2(k_1, \dots, k_m)$ matrix elements for more than one emission describe the probability of emitting m colour-connected soft partons, and they are therefore suppressed if the involved emissions are very far in rapidity from each other. We will refer to $\tilde{M}^2(k_1, k_2)$ as the *double-correlated* contribution to the squared amplitude for multiple emissions. We

will label the correlated squared matrix elements with more than two emissions in an analogous fashion. We now study the logarithmic structure of each of the terms in eqs. (3.1). Each resolved real emission (i.e. an emission that contributes to the observable) is defined by requiring that $V_{\text{sc}}(\{\tilde{p}\}, k_i) > \epsilon v$, where ϵ is independent of v because of rIRC safety. This condition poses a lower bound on the resolved emission's phase space which can potentially only give rise to a single logarithm of v (see for instance eq. (2.29)). When several emissions are considered, the same argument applies, so that each emission can at most contribute with a single logarithm. This is ensured by rIRC safety since this condition implies that the observable will have the same scaling independently of the number of emissions, and therefore the condition $V_{\text{sc}}(\{\tilde{p}\}, k_i) > \epsilon v$ will still impose a lower cutoff for all resolved emissions. The unresolved emissions below this limit (i.e. $V_{\text{sc}}(\{\tilde{p}\}, k_i) < \epsilon v$) can be ignored in the observable evaluation and their role is simply to cancel the virtual IRC singularities. They contribute exclusively to the Sudakov radiator and therefore we do not need to consider them here. With the above property we can immediately see that a product of n independent emission matrix elements in eq. (3.1) gives rise at most to a $\alpha_s^n L^n$ (i.e. a NLL) contribution, where $L = \ln 1/v$.

We now consider the double-correlated $\tilde{M}^2(k_1, k_2)$ term. It involves a soft-gluon splitting into either a $q\bar{q}$ or gg pair and it could potentially give rise to a $\alpha_s^2 L^3$ term ($\alpha_s L$ associated with the emission of the parent gluon, and at most two extra logarithms coming from its splitting). However, again due to rIRC safety (see for instance section 2.2.4 of ref. [27] for the relevant properties), one can see that the splitting of the parent gluon does not give rise to additional logarithms, leaving us with a NNLL term $\alpha_s^2 L$. The same argument can be applied to terms with more than two correlated partons, and can be used to show that they are at most N³LL. Therefore, the sole rIRC safety property of the observable allows one to define a logarithmic hierarchy in the multiple emissions function and to define the relevant configurations that contribute to a given logarithmic order. The very same argument applies to the case of one or more emissions emitted collinearly to the Born leg with high momentum. Therefore, if we want to limit ourselves to, for instance, NLL (i.e. $\alpha_s^n L^n$ terms in the multiple emissions function $\mathcal{F}(v)$) it is sufficient to consider an ensemble of soft-collinear independent emissions, since any configuration beyond this one would just be at most NNLL. For a NNLL treatment, in addition, one has to include the contribution of a single splitting of a soft gluon (following the above argument it is easy to see that configurations with more than one splitting are subleading), and a single hard collinear emission. This treatment can be extended to higher orders in a very systematic way.

In addition to the matrix element approximation, we would like to approximate the resolved emission's phase space in order to neglect any effects in $\mathcal{F}(v)$ which are beyond the logarithmic accuracy that we want to achieve. We stress that this class of approximations is not strictly necessary for the resummation, since their only purpose is to ensure that $\mathcal{F}(v)$ is free of any contamination from subleading effects. For instance, at NLL, we can approximate the rapidities of all soft-collinear emissions with the kinematic limit as done in section 2, and treat the observable in the pure soft-collinear approximation, all corrections being at most NNLL. For a NNLL resummation, these approximations are of course not valid anymore and one has to repeat the calculation without making them. Alternatively,

one can simply compute the NNLL corrections associated with these approximations with respect to the NLL function $\mathcal{F}_{\text{NLL}}(\lambda)$, as it will be explained in detail in the next section.

The last ingredient that one needs to go beyond NLL is the Sudakov radiator. This function has the role of cancelling the infrared and collinear singularities associated with the unresolved emissions (i.e. $V_{\text{sc}}(\{\tilde{p}\}, k_i) < \epsilon v$) against the virtual corrections. At NLL its structure is remarkably simple since the unresolved real emissions fully exponentiate in the observable's space and the cancellation of singularities is explicit. Beyond this order, one needs to work out the exact details of real-virtual cancellations (for instance, through renormalisation group evolution equations). We will not present a general expression for the radiator in this article, but we will limit ourselves to show that it only depends on the scaling of the observable in the presence of a *single soft and collinear dressed* (i.e. inclusive in its branchings) emission. Therefore, we will show that it is universal for all observables which have the same soft-collinear parametrisation in the single emission case, i.e. the same a and b_ℓ coefficients in eq. (2.4).

3.2 Structure of the NNLL resummation

Using the arguments outlined in the previous section, we now derive the general form of NNLL corrections. We start by recalling the procedure which lead to the NLL result. On the one hand, we approximated the matrix element and the phase space in all emissions appearing in the multiple emissions function of eq. (2.25), neglecting subleading corrections due to the exact rapidity bound for each resolved soft and collinear emission (see eq. (2.31)), and the correct description of the hard-collinear region (neglecting the second line of eq. (2.16)). On the other hand, we replaced the observable with its soft-collinear parametrisation V_{sc} , neglecting the second line of eq. (2.23). We remark that, at NNLL accuracy, these approximations have to be relaxed for a single emission at a time, since relaxing each approximation gives rise a correction of relative order α_s . This implies that configurations in which we correct more than one emission lead to contributions beyond NNLL, that can be neglected accordingly.

A set of NNLL corrections arises from the first line of eq. (2.12):

$$e^{-R_{\text{NLL}}(v)} e^{-\int_{\epsilon v}^v [dk] M^2(k)} \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\epsilon v} \prod_i [dk_i] M^2(k_i) \Theta(v - V_{\text{sc}}(\{\tilde{p}\}, k_1, \dots, k_n)), \quad (3.2)$$

where R_{NLL} is defined in eq. (2.22). Besides the NLL multiple emissions function $\mathcal{F}_{\text{NLL}}(\lambda)$ of eq. (2.32) derived in section 2, eq. (3.2) contains corrections due both to the hard-collinear term of the matrix element (given by the second line of eq. (2.16)), and to the correct rapidity bounds, which at NLL are the same for all emissions (see eq. (2.31)). Such corrections result in the two NNLL contributions $\delta\mathcal{F}_{\text{hc}}$ (section 3.3.2) and $\delta\mathcal{F}_{\text{sc}}$ (section 3.3.1), respectively.

Another category of NNLL corrections is contained in the second line of eq. (2.12), namely

$$e^{-R_{\text{NNLL}}(v)} e^{-\int_{\epsilon v}^v [dk] M_{\text{sc}}^2(k)} \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\epsilon v} \prod_i [dk_i] M^2(k_i) [\Theta(v - V(\{\tilde{p}\}, k_1, \dots, k_n)) - \Theta(v - V_{\text{sc}}(\{\tilde{p}\}, k_1, \dots, k_n))] , \quad (3.3)$$

where we need to relax the soft-collinear approximation made for the observable when an arbitrary emission becomes hard-collinear or is emitted at small rapidities (large angles). We stress that, at NNLL accuracy, it is enough to consider an ensemble of soft and collinear emissions, plus a single extra emission which is free to probe both the hard-collinear and the soft-wide-angle region of the phase space. Configurations containing more than one soft-wide-angle or hard-collinear real emission are subleading. We can then expand further the first step function in eq. (3.3) in order to take into account the correct behaviour of the observable in these limits for a single emission of the ensemble. The corresponding NNLL corrections are: a recoil correction $\delta\mathcal{F}_{\text{rec}}$ (computed in section 3.3.2) which is due to the exact kinematics of a hard-collinear emission which recoils against the soft-collinear ensemble; a soft-wide-angle correction $\delta\mathcal{F}_{\text{wa}}$ (computed in section 3.3.3) which is due to a soft emission that spans the whole rapidity range; a correlated correction $\delta\mathcal{F}_{\text{correl}}$ (computed in section 3.3.4) to the inclusive treatment of the soft gluon decay in the matrix element (encoded in the scheme of the running coupling in the radiator $R(v)$). An important point to stress is that the soft-collinear approximation $V_{\text{sc}}(\{\tilde{p}\}, k_1, \dots, k_n)$ guarantees that all NNLL corrections arising from eq. (3.3) are well defined and finite when the corrected emission becomes unresolved.

One last NNLL contribution is due to the correction to the NLL Sudakov radiator of eq. (2.19). At NLL, the radiator encodes the contribution of unresolved real emissions k_i with $V_{\text{sc}}(\{\tilde{p}\}, k_i) < \epsilon v$ and corresponding virtual corrections. Moreover, each emission is considered to be inclusive in its two-parton branchings. Analogously, the NNLL Sudakov radiator has to include the effect of the inclusive soft three-parton correlation, which can be absorbed in a redefinition of the running coupling analogously to what is done at NLL, together with the correct matrix element for an inclusive double collinear emission. Furthermore, it contains exact $\mathcal{O}(\alpha_s)$ corrections surviving the poles cancellation between real and virtual corrections. In formulae, we introduce a NNLL radiator $R_{\text{NNLL}}(v)$ through the replacement

$$\mathcal{H}(Q^2) e^{\int_{\epsilon v}^v [dk] M^2(k)} \rightarrow e^{-R_{\text{NNLL}}(v) + \int_{\epsilon v}^v [dk] M^2(k)}, \quad (3.4)$$

where

$$R_{\text{NNLL}}(v) = \int [dk] M_{\text{sc}}^2(k) \Theta(V_{\text{sc}}(\{\tilde{p}\}, k) - v) + \sum_{\ell=1,2} \int \frac{dk_t^2}{k_t^2} \int_0^1 \frac{dz}{z} (z p_\ell(z) - 2C_\ell) \frac{\alpha_s(k_t^2)}{2\pi} \Theta\left(\frac{d_\ell g_\ell(\phi)}{z^{b_\ell}} \left(\frac{k_t}{Q}\right)^{a+b_\ell} - v\right) + \frac{\alpha_s(Q)}{\pi} h(\lambda). \quad (3.5)$$

The function $\alpha_s(Q)h(\lambda)/\pi$ contains the contribution of the triple-correlated splitting, the double hard-collinear correction and additional $\mathcal{O}(\alpha_s)$ constant terms arising from real-virtual cancellations, and corresponding running coupling effects. Eq. (3.5) contains some

power suppressed terms due to the integration limits of the non-singular phase space variables, i.e. ϕ in the soft limit and ϕ, z in the hard-collinear limit. In order to neglect these terms we have relaxed the lower bound in the z integration relative to the hard-collinear limit, and set it to zero (the physical bound being $z > k_t/Q$). Moreover, in order to neglect power-suppressed and subleading contributions, we can expand the two Θ -functions of eq. (3.5) as follows:⁸

$$\Theta\left(d_\ell g_\ell(\phi)\left(\frac{k_t}{Q}\right)^a e^{-b_\ell \eta^{(\ell)}} - v\right) \simeq \Theta\left(\ln\left(\frac{k_t}{Q}\right)^a e^{-b_\ell \eta^{(\ell)}} - \ln v\right) + \delta\left(\ln\left(\frac{k_t}{Q}\right)^a e^{-b_\ell \eta^{(\ell)}} - \ln v\right) \ln d_\ell g_\ell(\phi) + \frac{1}{2}\delta'\left(\ln\left(\frac{k_t}{Q}\right)^a e^{-b_\ell \eta^{(\ell)}} - \ln v\right) \ln^2 d_\ell g_\ell(\phi), \quad (3.6)$$

$$\Theta\left(\frac{d_\ell g_\ell(\phi)}{z^{b_\ell}}\left(\frac{k_t}{Q}\right)^{a+b_\ell} - v\right) \simeq \Theta\left(\ln\left(\frac{k_t}{Q}\right)^{a+b_\ell} - \ln v\right) + \delta\left(\ln\left(\frac{k_t}{Q}\right)^{a+b_\ell} - \ln v\right) \ln \frac{d_\ell g_\ell(\phi)}{z^{b_\ell}}. \quad (3.7)$$

We observe that the dependence on the normalisation $d_\ell g_\ell(\phi)$ is a local rescaling of the observable. This induces a local shift of the logarithm $\ln 1/v$ and gives rise to subleading contributions at each logarithmic order. This implies that, at NNLL accuracy, the dependence on $d_\ell g_\ell(\phi)$ in the Sudakov radiator is completely encoded in the first two integrals of eq. (3.5), and it corresponds to a shift in the logarithms of the NLL radiator (before azimuthal integration). An important consequence of this is that the function $h(\lambda)$ depends exclusively on the scaling in η (or equivalently z) and k_t through the a and b_ℓ coefficients. By exploiting this property, one can conclude that the resummations of all observables which have the same soft-collinear scaling in k_t and η (i.e. the same a and b_ℓ coefficients) will have the same $h(\lambda)$ function. For example, the function $h(\lambda)$ will be the same for thrust $1 - T$, C parameter, and heavy jet mass ρ_H , and it can be taken from [30, 31]. Analogously, the function $h(\lambda)$ for the jet broadenings B_T, B_W , thrust major T_M and oblateness O is identical to the one relative to the k_t resummation (which we take from ref. [46] after replacing the constant one-loop virtual corrections with the corresponding ones in $e^+e^- \rightarrow$ hadrons). Practically, the function $h(\lambda)$ can be obtained by computing the resummation for the reference observable (e.g. the thrust) leaving $h(\lambda)$ unspecified, and fixing it by equating the resummation obtained here to the known result in the literature. This is similar in spirit to what has been done for the jet-veto in refs. [46, 58].

We parametrise the final NNLL Sudakov radiator as

$$R_{\text{NNLL}}(v) = -Lg_1(\lambda) - g_2(\lambda) - \frac{\alpha_s(Q)}{\pi}g_3(\lambda). \quad (3.8)$$

The relevant expressions for the g_1, g_2 , and g_3 functions are reported in appendix B. Once all these corrections have been computed, the NNLL expression for $\Sigma(v)$ becomes

$$\Sigma(v) = e^{Lg_1(\lambda)+g_2(\lambda)+\frac{\alpha_s(Q)}{\pi}g_3(\lambda)} \left[\mathcal{F}_{\text{NLL}}(\lambda) + \frac{\alpha_s(Q)}{\pi} \delta\mathcal{F}_{\text{NNLL}}(\lambda) \right]. \quad (3.9)$$

The function

$$\delta\mathcal{F}_{\text{NNLL}} = \delta\mathcal{F}_{\text{sc}} + \delta\mathcal{F}_{\text{hc}} + \delta\mathcal{F}_{\text{rec}} + \delta\mathcal{F}_{\text{wa}} + \delta\mathcal{F}_{\text{correl}}, \quad (3.10)$$

⁸For the NLL radiator, it was sufficient to consider the first two terms in the r.h.s. of eq. (3.6), and the first in the r.h.s. of eq. (3.7), respectively.

represents NNLL corrections due to real radiation, and it will be extensively discussed in the rest of this section.

Before deriving the relevant NNLL corrections to the real radiation it is worth making an important remark. The whole resummation procedure defined in the present section depends on a specific choice of the variable on which the cutoff ϵ is applied. This choice is reflected in the exponentiated part of the resummed cross section. Our default choice is to define unresolved emissions as those for which $V_{\text{sc}}(\{\tilde{p}\}, k) < \epsilon v$, where V_{sc} is defined by eq. (2.4). This choice is clearly arbitrary and one could equally derive the same resummed results (that will be anyway independent of the cutoff ϵ) with a different definition for the unresolved contributions. Different choices will simply lead to different NLL terms (and beyond) in the Sudakov exponent and in the real corrections described by the multiple emissions function, but will not affect the final result which does not depend on such a definition. In the present article we decide to work in the soft-collinear prescription in which the cutoff ϵ is applied on the soft-collinear approximation of the observable for a generic emission k_i . This prescription has two advantages. On the one hand it allows one to expand the multiple emissions function around the NLL result, which is simply determined by the soft-collinear approximation (meaning that the V_{sc} approximation of eq. (2.4) is enough to account for all NLL contributions). It also ensures that all NNLL corrections to the multiple emissions function are finite without further regulators since the singularities of any unresolved emission are encoded in the soft-collinear approximation. On the other hand it allows us to define the NNLL function $h(\lambda)$ in such a way that it is independent of the observable's normalisation $d_\ell g_\ell(\phi)$ and it only depends on the a and b_ℓ coefficients. As stated above, this implies that the function $h(\lambda)$ is universal for all observables which have the same a and b_ℓ scaling in the soft-collinear region.

3.3 NNLL contributions due to resolved emissions

In this section we explicitly derive all corrections to the multiple emission function $\mathcal{F}(v)$ necessary to achieve NNLL accuracy for the cumulative distribution $\Sigma(v)$ for a generic event-shape observable v . In order to do this we have to recall the basic assumptions used to obtain eq. (2.34). They are:

- gluon splitting in $R(v)$ is treated inclusively;
- each real emission k_i contributing to $\mathcal{F}(v)$ is soft, collinear, and such that $\epsilon v < V_{\text{sc}}(\{\tilde{p}\}, k_i) < v$;
- the rapidity bound of all emissions contributing to $\mathcal{F}(v)$ is the same.

By relaxing any of these approximations valid at NLL accuracy, we obtain a number of NNLL corrections induced by real radiation, and introduced in the previous section. We will derive them in the following order:

1. exact rapidity bound and running coupling corrections to the soft and collinear function $\mathcal{F}(v)$ ($\delta\mathcal{F}_{\text{sc}}$);

2. one of the emissions k_i is collinear but not soft, generating hard-collinear ($\delta\mathcal{F}_{\text{hc}}$) and recoil ($\delta\mathcal{F}_{\text{rec}}$) corrections;
3. one of the emissions k_i is soft but at wide angle ($\delta\mathcal{F}_{\text{wa}}$);
4. gluon decay is treated non-inclusively, giving rise to a correlated-emission correction ($\delta\mathcal{F}_{\text{correl}}$).

The necessary amplitudes to compute $\delta\mathcal{F}_{\text{NNLL}}$ are given by the independent emission probability of eq. (2.7), and the probability of a soft gluon branching into either two gluons or a quark-antiquark pair (correlated emission). In fact, for the real ensemble the observable's value $V(\{\tilde{p}\}, \{k_i\})$ is bound both from above and from below. This reduces the phase space of the real emissions to a strip which contributes with one fewer logarithm at each order of α_s with respect to the Sudakov radiator. Therefore, to obtain the whole set of NNLL real corrections, it is enough to use the same probability amplitudes which appear in the definition of the Sudakov exponent at NLL, i.e. the independent soft and/or collinear emission probability, and the correlated soft-gluon splitting.

3.3.1 Soft-collinear NNLL contributions

The first NNLL correction we consider arises from $\mathcal{F}(v)$, when we take into account the exact rapidity bounds for a single emission in the generated soft-collinear ensemble. At NLL, the correct rapidity limit for the emission k_i ,

$$\eta_i^{(\ell_i)} < \frac{1}{a + b_{\ell_i}} \ln \frac{g_{\ell}(\phi_i) d_{\ell}}{\zeta_i v}, \quad (3.11)$$

was effectively replaced by $1/(a + b_{\ell_i}) \ln(1/v)$ through the expansion of eq. (2.31). NNLL corrections to this approximation are obtained by considering the next term in the expansion of R'_{ℓ} , both in real and in virtual corrections, as follows

$$R'_{\ell} \left(\frac{\zeta v}{d_{\ell} g_{\ell}(\phi)} \right) \simeq R'_{\text{NLL},\ell}(v) + \delta R'_{\text{NNLL},\ell}(v) + R''_{\ell}(v) \ln \frac{d_{\ell} g_{\ell}(\phi)}{\zeta}. \quad (3.12)$$

This gives

$$\begin{aligned} \mathcal{F}(v) &\simeq \epsilon^{R'_{\text{NLL}}} \left(1 - \sum_{\ell} \left(\delta R'_{\text{NNLL},\ell} + R''_{\ell} \int \frac{d\phi}{2\pi} \ln(d_{\ell} g_{\ell}(\phi)) \right) \ln \frac{1}{\epsilon} - \frac{1}{2} \sum_{\ell} R''_{\ell} \ln^2 \frac{1}{\epsilon} \right) \times \\ &\times \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{i=1}^n \int_{\epsilon}^{\infty} \frac{d\zeta_i}{\zeta_i} \int_0^{2\pi} \frac{d\phi_i}{2\pi} \sum_{\ell_i=1,2} \left(R'_{\text{NLL},\ell_i} + \delta R'_{\text{NNLL},\ell_i} + R''_{\ell_i} \ln \frac{d_{\ell_i} g_{\ell_i}(\phi_i)}{\zeta_i} \right) \times \\ &\times \Theta \left(1 - \lim_{v \rightarrow 0} \frac{V_{\text{sc}}(\{\tilde{p}\}, k_1, \dots, k_n)}{v} \right) \simeq \mathcal{F}_{\text{NLL}}(\lambda) + \frac{\alpha_s(Q)}{\pi} \delta\mathcal{F}_{\text{sc}}(\lambda). \end{aligned} \quad (3.13)$$

We can simplify the above equation by keeping only terms in the sum which are linear in $R'_{\text{NNLL},\ell}$ or R''_{ℓ} , i.e. by correcting one emission at a time. The latter approximation ensures that no contributions beyond NNLL are included. Moreover, we can express the virtual correction in eq. (3.13) as the integral over an extra dummy emission as follows:

$$\ln \frac{1}{\epsilon} = \int_{\epsilon}^1 \frac{d\zeta}{\zeta}, \quad \frac{1}{2} \ln^2 \frac{1}{\epsilon} = \int_{\epsilon}^1 \frac{d\zeta}{\zeta} \ln \frac{1}{\zeta}. \quad (3.14)$$

The final form of the soft-collinear correction then reads

$$\delta\mathcal{F}_{\text{sc}}(\lambda) = \frac{\pi}{\alpha_s(Q)} \int_0^\infty \frac{d\zeta}{\zeta} \int_0^{2\pi} \frac{d\phi}{2\pi} \sum_{\ell=1,2} \left(\delta R'_{\text{NNLL},\ell} + R''_{\ell_i} \ln \frac{d_\ell g_\ell(\phi)}{\zeta} \right) \int d\mathcal{Z} [\{R'_{\text{NLL},\ell_i}, k_i\}] \times$$

$$\times \left[\Theta \left(1 - \lim_{v \rightarrow 0} \frac{V_{\text{sc}}(\{\tilde{p}\}, k, \{k_i\})}{v} \right) - \Theta(1 - \zeta) \Theta \left(1 - \lim_{v \rightarrow 0} \frac{V_{\text{sc}}(\{\tilde{p}\}, \{k_i\})}{v} \right) \right], \quad (3.15)$$

where the average of a function over the measure $d\mathcal{Z}$ is defined in eq. (2.33). In the first term of eq. (3.15), $k = k(\zeta, \phi, \ell)$ represents an additional real emission, and the second term corresponds to virtual corrections. In eq. (3.15) we have set the ζ lower integration limit to zero, because singular contributions for $\zeta \rightarrow 0$ exactly cancel between real and virtual corrections.

3.3.2 Recoil and hard-collinear NNLL contributions

Another source of NNLL contributions arises when one of the emissions is collinear to any of the legs and hard, i.e. it carries a sizable fraction of emitter's longitudinal momentum. The matrix element squared $M_\ell^2(k)$ for the emission of a gluon k collinear to leg ℓ is given by

$$[dk] M_\ell^2(k) = \frac{C_\ell \alpha_s^{\text{CMW}}(\tilde{k}_t^{(\ell)})}{4\pi} \frac{d\phi d(\tilde{k}_t^{(\ell)})^2}{2\pi (\tilde{k}_t^{(\ell)})^2} dz^{(\ell)} p_\ell(z^{(\ell)}), \quad (3.16)$$

where, in our case, $p_\ell(z) = p_{gq}(z)$, given in eq. (2.8) and $C_\ell = C_F$, for $\ell = 1, 2$. In the above equation $\tilde{k}_t^{(\ell)}$ is the relative transverse momentum between the emitted gluon and the final state parton \tilde{p}_ℓ . The vectors $\tilde{k}_t^{(\ell)}$ satisfy

$$(\tilde{k}_t^{(1)})^2 = \frac{2(\tilde{p}_1 k) 2(p_2 k)}{2(\tilde{p}_1 p_2)}, \quad (\tilde{k}_t^{(2)})^2 = \frac{2(p_1 k) 2(\tilde{p}_2 k)}{2(p_1 \tilde{p}_2)}. \quad (3.17)$$

In eq. (3.16), we have identified the energy fraction relative to the splitting with the Sudakov variable $z^{(\ell)}$ defined in eq. (2.2). This is justified by the fact that all remaining emissions are soft and hence do not change the energy fraction in an appreciable way.

Due to recoil, the generated transverse momentum $\tilde{k}_t^{(\ell)}$ is different from the Sudakov transverse momentum k_t of eq. (2.2), which is relative to the thrust axis. In order to compute reliably $V(\{\tilde{p}\}, k, k_1, \dots, k_n)$ we need to relate $\tilde{k}_t^{(\ell)}$ and k_t . For simplicity we consider the case $\ell = 1$ and rename $\tilde{k}_t^{(1)} \rightarrow \tilde{k}_t$. We start from the Sudakov parametrisation of k with respect to p_1 and \tilde{p}_1 , respectively

$$k = z^{(1)} p_1 + z^{(2)} p_2 + \kappa_t = \tilde{z}^{(1)} \tilde{p}_1 + \tilde{z}^{(2)} p_2 + \tilde{\kappa}_t, \quad (3.18)$$

where κ_t and $\tilde{\kappa}_t$ are space-like vectors with $\kappa_t^2 = -k_t^2$ and $\tilde{\kappa}_t^2 = -\tilde{k}_t^2$. They can be related to the Sudakov parametrisation in the thrust axis reference frame (2.2) by plugging in the parametrisation of the recoiled momentum \tilde{p}_1 in terms of the Born momenta p_1 and p_2

$$\tilde{p}_1 = z_p^{(1)} p_1 + z_p^{(2)} p_2 + \pi_{t,1}, \quad \pi_{t,1}^2 = -p_{t,1}^2, \quad z_p^{(2)} = \frac{p_{t,1}^2}{z_p^{(1)} Q^2}, \quad (3.19)$$

and requiring the resulting decomposition to be equal to the initial parametrisation eq. (2.2), obtaining

$$\vec{k}_t = \vec{k}_t - z^{(1)} \frac{\vec{p}_{t,1}}{z_p^{(1)}}. \quad (3.20)$$

From energy-momentum conservation and the fundamental property of the thrust axis, i.e. that transverse momentum is conserved separately in each hemisphere, one has

$$z_p^{(1)} \simeq 1 - \sum_{i \in \mathcal{H}^{(1)}} z_i^{(1)} - z^{(1)} \simeq 1 - z^{(1)}, \quad \vec{p}_{t,1} = - \sum_{i \in \mathcal{H}^{(1)}} \vec{k}_{t,i} - \vec{k}_t. \quad (3.21)$$

Substituting the expressions of $z_p^{(1)}$ and $\vec{p}_{t,1}$ in eq. (3.20) we obtain

$$\vec{k}_t \simeq \vec{k}_t - z^{(1)} \frac{\vec{p}_{t,1}}{1 - z^{(1)}} = \vec{k}_t + \frac{z^{(1)}}{1 - z^{(1)}} \left(\sum_{i \in \mathcal{H}^{(1)}} \vec{k}_{t,i} + \vec{k}_t \right) = \frac{\vec{k}_t - z^{(1)} \vec{p}'_{t,1}}{1 - z^{(1)}}, \quad (3.22)$$

where

$$\vec{p}'_{t,1} = - \sum_{i \in \mathcal{H}^{(1)}} \vec{k}_{t,i} \quad (3.23)$$

is the recoil due to all soft and collinear emissions. Defining also $\vec{k}'_t \equiv \vec{k}_t - z^{(1)} \vec{p}'_{t,1}$ we have that $\vec{k}_t = \vec{k}'_t / (1 - z^{(1)})$. Since \vec{k}_t and \vec{k}'_t are related by a simple rescaling, in the collinear matrix element squared of eq. (3.16) we can replace $d\vec{k}_t^2 / \vec{k}_t^2$ with $dk_t'^2 / k_t'^2$. We then obtain the relation between the transverse momentum with respect to the thrust axis \vec{k}_t and the transverse momentum \vec{k}'_t which enters the collinear emission phase space:

$$\vec{k}_t = \vec{k}'_t + z^{(1)} \vec{p}'_{t,1}. \quad (3.24)$$

This implies that the input momentum k becomes a function of $\vec{k}'_t, \vec{p}'_{t,1}, z^{(1)}$. For the sake of simplicity, we drop the vector superscript from now on.

We have two NNLL contributions coming from hard-collinear radiation. The first comes from eq. (3.3), in which we have to take into account the exact expression of the observable when a single emission is hard and collinear:

$$\begin{aligned} \mathcal{F}_{\text{rec}}(v) = & e^{-\int_{\epsilon v}^v [dk] M_{\text{sc}}^2(k)} \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\epsilon v}^v \prod_{i=1}^n [dk_i] M_{\text{sc}}^2(k_i) \sum_{\ell=1,2} \int_0^1 dz p_{\ell}(z) \int_0^{2\pi} \frac{d\phi}{2\pi} \int \frac{dk_t'^2}{k_t'^2} \frac{\alpha_s(k_t')}{2\pi} \times \\ & \times \left[\Theta\left(v - V_{\text{hc}}^{(k)}(\{\tilde{p}\}, k[k'_t, p'_{t,\ell}, z], k_1, \dots, k_n)\right) - \Theta\left(v - V_{\text{sc}}(\{\tilde{p}\}, k[k'_t, p'_{t,\ell}, 0], k_1, \dots, k_n)\right) \right]. \quad (3.25) \end{aligned}$$

In the above expression, $V_{\text{hc}}^{(k)}(\{\tilde{p}\}, k, k_1, \dots, k_n)$ denotes the expression of the observable V where all emissions but k are treated in the soft-collinear approximation. In the second term, the one containing $V_{\text{sc}}(\{\tilde{p}\}, k, k_1, \dots, k_n)$, also emission k has been treated as if it were soft and collinear, so that its transverse momentum with respect to the emitting leg k'_t is equal to k_t . Notice that, in eq. (3.25) we can replace k'_t with k_t in the integration since this variable is integrated over, and use the short-hand notation

$$k' = k[k_t, p'_{t,1}, z], \quad k = k[k_t, p'_{t,1}, 0].$$

To NNLL accuracy it is possible to further simplify the phase-space for k . Introducing

$$\zeta = \frac{1}{v} \frac{d_\ell g_\ell(\phi)}{z^{b_\ell}} \left(\frac{k_t}{Q} \right)^{a+b_\ell}, \quad (3.26)$$

we have, at NNLL accuracy

$$\frac{dk_t^2}{k_t^2} \frac{\alpha_s(k_t)}{2\pi} = \frac{\alpha_s((z^{b_\ell} \zeta v / (d_\ell g_\ell(\phi)))^{1/(a+b_\ell)} Q)}{\pi(a+b_\ell)} \frac{d\zeta}{\zeta} \simeq \frac{\alpha_s(v^{1/(a+b_\ell)} Q)}{\pi(a+b_\ell)} \frac{d\zeta}{\zeta}. \quad (3.27)$$

In fact, rIRC safety constrains the variable ζ to be of order one, so that further terms arising from the expansion of the QCD coupling around $v^{1/(a+b_\ell)} Q$ are of relative order α_s^2 , hence at most N³LL.

Following what we did in sections 2 and 3.3.1, we eliminate all subleading contributions and obtain $\mathcal{F}_{\text{rec}}(v) \simeq (\alpha_s(Q)/\pi) \delta\mathcal{F}_{\text{rec}}(\lambda)$, where

$$\begin{aligned} \delta\mathcal{F}_{\text{rec}}(\lambda) &= \sum_{\ell=1,2} \frac{\alpha_s(v^{1/(a+b_\ell)} Q)}{\alpha_s(Q)(a+b_\ell)} \int_0^\infty \frac{d\zeta}{\zeta} \int_0^{2\pi} \frac{d\phi}{2\pi} \int d\mathcal{Z}[\{R'_{\text{NLL},\ell_i}, k_i\}] \times \\ &\times \int_0^1 dz p_\ell(z) \left[\Theta \left(1 - \lim_{v \rightarrow 0} \frac{V_{\text{hc}}^{(k')}(\{\tilde{p}\}, k', \{k_i\})}{v} \right) - \Theta \left(1 - \lim_{v \rightarrow 0} \frac{V_{\text{sc}}(\{\tilde{p}\}, k, \{k_i\})}{v} \right) \right]. \end{aligned} \quad (3.28)$$

The second NNLL contribution coming from hard collinear radiation arises from eq. (3.2):

$$\begin{aligned} \mathcal{F}_{\text{collinear}}(v) &= e^{-\int_{\epsilon v}^v [dk] M_{\text{sc}}^2(k)} \sum_{n=0}^\infty \frac{1}{n!} \int_{\epsilon v}^v \prod_{i=1}^n [dk_i] M_{\text{sc}}^2(k_i) \sum_{\ell=1,2} \int_0^1 dz p_\ell(z) \int_0^{2\pi} \frac{d\phi}{2\pi} \int \frac{dk_t^2}{k_t^2} \frac{\alpha_s(k_t)}{2\pi} \times \\ &\times [\Theta(v - V_{\text{sc}}(\{\tilde{p}\}, k, k_1, \dots, k_n)) - \Theta(v - V_{\text{sc}}(\{\tilde{p}\}, k_1, \dots, k_n)) \Theta(v - V_{\text{sc}}(\{\tilde{p}\}, k))] , \end{aligned} \quad (3.29)$$

where the second term in the square brackets represents virtual corrections. From the above equation we see that, if k is also soft, i.e. $z \rightarrow 0$, the function $\mathcal{F}_{\text{collinear}}(v)$ contains configurations that have been already taken into account in the function $\mathcal{F}(v)$ of eq. (2.25). We eliminate this double counting by subtracting the NLL contribution

$$\begin{aligned} \mathcal{F}_{\text{collinear}}^{\text{sub.}}(v) &= e^{-\int_{\epsilon v}^v [dk] M_{\text{sc}}^2(k)} \sum_{n=0}^\infty \frac{1}{n!} \int_{\epsilon v}^v \prod_{i=1}^n [dk_i] M_{\text{sc}}^2(k_i) \int_0^{2\pi} \frac{d\phi}{2\pi} \int \frac{dk_t^2}{k_t^2} \frac{\alpha_s(k_t)}{2\pi} \sum_{\ell=1,2} 2C_\ell \int_0^1 \frac{dz}{z} \\ &\times [\Theta(v - V_{\text{sc}}(\{\tilde{p}\}, k, k_1, \dots, k_n)) - \Theta(v - V_{\text{sc}}(\{\tilde{p}\}, k_1, \dots, k_n)) \Theta(v - V_{\text{sc}}(\{\tilde{p}\}, k))] . \end{aligned} \quad (3.30)$$

Performing the same manipulations as for \mathcal{F}_{rec} we arrive at:

$$\mathcal{F}_{\text{collinear}}(v) - \mathcal{F}_{\text{collinear}}^{\text{sub.}}(v) \simeq \frac{\alpha_s(Q)}{\pi} \delta\mathcal{F}_{\text{hc}}(\lambda), \quad (3.31)$$

where

$$\begin{aligned} \delta\mathcal{F}_{\text{hc}}(\lambda) &= \sum_{\ell=1,2} \frac{\alpha_s(v^{1/(a+b_\ell)} Q)}{\alpha_s(Q)(a+b_\ell)} \int_0^\infty \frac{d\zeta}{\zeta} \int_0^{2\pi} \frac{d\phi}{2\pi} \int d\mathcal{Z}[\{R'_{\text{NLL},\ell_i}, k_i\}] \times \\ &\times \int_0^1 \frac{dz}{z} (z p_\ell(z) - 2C_\ell) \left[\Theta \left(1 - \lim_{v \rightarrow 0} \frac{V_{\text{sc}}(\{\tilde{p}\}, k, \{k_i\})}{v} \right) - \Theta \left(1 - \lim_{v \rightarrow 0} \frac{V_{\text{sc}}(\{\tilde{p}\}, \{k_i\})}{v} \right) \Theta(1 - \zeta) \right]. \end{aligned} \quad (3.32)$$

3.3.3 Soft wide-angle NNLL contributions

This contribution arises when one of the soft gluons is emitted at wide angles. We can parametrise the observable dependence on the momentum of this extra gluon k as

$$V_{\text{wa}}^{(k)}(\{\tilde{p}\}, k) = \left(\frac{k_t}{Q}\right)^a f_{\text{wa}}(\eta, \phi). \quad (3.33)$$

In general, when η is close to zero (wide angles), the above expression might differ from the expression of the observable after a soft and collinear emission k

$$V_{\text{sc}}(\{\tilde{p}\}, k) = \left(\frac{k_t}{Q}\right)^a f_{\text{sc}}(\eta, \phi), \quad f_{\text{sc}}(\eta, \phi) = d_1 e^{-b_1 \eta} g_1(\phi) \Theta(\eta) + d_2 e^{b_2 \eta} g_2(\phi) \Theta(-\eta). \quad (3.34)$$

For a fixed value of k_t, η, ϕ for an extra emission k , we denote with $V_{\text{wa}}^{(k)}(\{\tilde{p}\}, k, k_1, \dots, k_n)$ the observable computed by keeping the full η, ϕ dependence of emission k , and using the soft-collinear approximation for all other emissions.

This gives rise to the following correction

$$\begin{aligned} \mathcal{F}_{\text{wa}}(v) &= e^{-\int_{\epsilon v}^v [dk] M_{\text{sc}}^2(k)} \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\epsilon v}^v \prod_{i=1}^n [dk_i] M_{\text{sc}}^2(k_i) 2C_F \int_0^{\infty} \frac{dk_t}{k_t} \frac{\alpha_s(k_t)}{\pi} \int_{-\infty}^{\infty} d\eta \int_0^{2\pi} \frac{d\phi}{2\pi} \times \\ &\times \left[\Theta \left(1 - \lim_{v \rightarrow 0} \frac{V_{\text{wa}}^{(k)}(\{\tilde{p}\}, k, k_1, \dots, k_n)}{v} \right) - \Theta \left(1 - \lim_{v \rightarrow 0} \frac{V_{\text{sc}}(\{\tilde{p}\}, k, k_1, \dots, k_n)}{v} \right) \right]. \end{aligned} \quad (3.35)$$

We can modify the phase space integration for the extra soft gluon as follows:

$$\frac{dk_t}{k_t} \frac{\alpha_s(k_t)}{\pi} = \frac{d\zeta}{\zeta} \frac{\alpha_s((\zeta v)^{1/a} Q)}{a\pi} \simeq \frac{d\zeta}{\zeta} \frac{\alpha_s(v^{1/a} Q)}{a\pi}, \quad (3.36)$$

where

$$\zeta = \frac{1}{v} \left(\frac{k_t}{Q}\right)^a \quad (3.37)$$

is constrained to be of order one for rIRC safe observables. This ensures that the approximation in eq. (3.36) is valid, up to corrections beyond NNLL accuracy. This gives $\mathcal{F}_{\text{wa}}(v) \simeq (\alpha_s(Q)/\pi) \delta \mathcal{F}_{\text{wa}}(\lambda)$, where

$$\begin{aligned} \delta \mathcal{F}_{\text{wa}}(\lambda) &= \frac{2C_F}{a} \frac{\alpha_s(v^{1/a} Q)}{\alpha_s(Q)} \int_0^{\infty} \frac{d\zeta}{\zeta} \int_{-\infty}^{\infty} d\eta \int_0^{2\pi} \frac{d\phi}{2\pi} \int d\mathcal{Z}[\{R'_{\text{NLL}, \ell_i}, k_i\}] \\ &\times \left[\Theta \left(1 - \lim_{v \rightarrow 0} \frac{V_{\text{wa}}^{(k)}(\{\tilde{p}\}, k, \{k_i\})}{v} \right) - \Theta \left(1 - \lim_{v \rightarrow 0} \frac{V_{\text{sc}}(\{\tilde{p}\}, k, \{k_i\})}{v} \right) \right]. \end{aligned} \quad (3.38)$$

3.3.4 Soft correlated emission

Unlike the hard-collinear and soft large-angle emissions, an arbitrary amount of soft and collinear emissions contribute to $\Sigma(v)$. Primary gluons emitted off the hard Born legs can give rise to subsequent branchings which need to be taken into account already at NLL accuracy [27]. However, at this accuracy any rIRC observable can be treated inclusively with respect to subsequent branchings of the soft gluons. This results just in a redefinition

of the scheme for the QCD running coupling, which is now defined as the strength of the inclusive soft radiation [57]. Each soft and collinear emission contributing to NLL accuracy is thus to be interpreted as fully inclusive in its branchings.

A generic event-shape variable is commonly non-inclusive for such splittings. However, for rIRC observables, non-inclusiveness only matters starting from NNLL accuracy [27]. At NNLL, the observable is sensitive to the details of the secondary soft splitting, so we need to undo the inclusive branching in order to compute the corresponding NNLL correction. Once again, in order to achieve NNLL accuracy, only a single non-inclusive splitting can be considered. The NNLL correlated correction has been already written in eq. (D.5) of ref. [27], and reads

$$\delta\mathcal{F}_{\text{correl}}(v) = e^{-\int_{\epsilon v}^v [dk] M_{\text{sc}}^2(k)} \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\epsilon v}^v \prod_{i=1}^n [dk_i] M_{\text{sc}}^2(k_i) \frac{1}{2!} \int [dk_a][dk_b] \tilde{M}^2(k_a, k_b) \times \quad (3.39)$$

$$\times [\Theta(v - V_{\text{sc}}(\{\tilde{p}\}, k_a, k_b, k_1, \dots, k_n)) - \Theta(v - V_{\text{sc}}(\{\tilde{p}\}, k_a + k_b, k_1, \dots, k_n))],$$

where $\tilde{M}^2(k_a, k_b)$ is a two-parton correlated matrix element, defined by

$$\tilde{M}^2(k_a, k_b) = M^2(k_a, k_b) - M^2(k_a)M^2(k_b). \quad (3.40)$$

To show explicitly that this contribution starts from NNLL accuracy, we first express the two-parton correlated emission matrix element and phase space as

$$[dk_a][dk_b] \tilde{M}^2(k_a, k_b) = [dk_a][dk_b] M_{\text{sc}}^2(k_a) M_{\text{sc}}^2(k_b) \frac{\tilde{M}^2(k_a, k_b)}{M_{\text{sc}}^2(k_a) M_{\text{sc}}^2(k_b)}. \quad (3.41)$$

Neglecting terms beyond NNLL accuracy, we rewrite the k_a integration as follows:

$$[dk_a] M_{\text{sc}}^2(k_a) = \frac{dv_a}{v_a} \frac{d\phi_a}{2\pi} \int [dk_a] M_{\text{sc}}^2(k_a) \sum_{\ell_a} v_a \delta\left(v_a - \left(\frac{k_{ta}}{Q}\right)^a e^{-b_{\ell_a} \eta_a^{(\ell_a)}}\right) \Theta\left(\eta_a^{(\ell_a)}\right) \quad (3.42)$$

$$\simeq \frac{d\zeta_a}{\zeta_a} \frac{d\phi_a}{2\pi} \int [dk_a] M_{\text{sc}}^2(k_a) \sum_{\ell_a} v \delta\left(v - \left(\frac{k_{ta}}{Q}\right)^a e^{-b_{\ell_a} \eta_a^{(\ell_a)}}\right) \Theta\left(\eta_a^{(\ell_a)}\right),$$

where, in the last line, we have defined $\zeta_a = v_a/v$, and neglected terms beyond NNLL accuracy, using the fact that rIRC safety constrains ζ_a to be of order one.

We then parametrise the phase space of the emission k_b in terms of the variables $\kappa = k_{t,b}/k_{t,a}$, $\eta = \eta_b - \eta_a$ and $\phi = \phi_b - \phi_a$. Notice that this is a convenient choice since the correlated matrix element $\tilde{M}^2(k_a, k_b)/(M_{\text{sc}}^2(k_a)M_{\text{sc}}^2(k_b))$ explicitly depends on the correlated momenta through these variables. This leads to

$$[dk_b] M_{\text{sc}}^2(k_b) = \left(\frac{2C_F \alpha_s(k_{t,b})}{\pi}\right) \frac{d\kappa}{\kappa} \Theta(\kappa) d\eta \frac{d\phi}{2\pi} \simeq \left(\frac{2C_F \alpha_s(k_{t,a})}{\pi}\right) \frac{d\kappa}{\kappa} \Theta(\kappa) d\eta \frac{d\phi}{2\pi}, \quad (3.43)$$

where in the last step we have set $k_{t,b} \simeq k_{t,a}$. The latter approximation is valid for rIRC safe observables only, with corrections beyond NNLL accuracy.

Therefore, eq. (3.41) can be rewritten as

$$[dk_a][dk_b] \tilde{M}^2(k_a, k_b) = \frac{d\zeta_a}{\zeta_a} \frac{d\phi_a}{2\pi} \sum_{\ell_a=1,2} \left(\frac{2C_{\ell_a} \lambda}{a\pi\beta_0} R''_{\ell_a}(v)\right) \frac{d\kappa}{\kappa} \Theta(\kappa) d\eta \frac{d\phi}{2\pi} C_{ab}(\kappa, \eta, \phi), \quad (3.44)$$

where

$$C_{ab}(\kappa, \eta, \phi) = \frac{\tilde{M}^2(k_a, k_b)}{M_{\text{sc}}^2(k_a)M_{\text{sc}}^2(k_b)}, \quad (3.45)$$

and

$$\begin{aligned} \int [dk] M_{\text{sc}}^2(k) \Theta(\eta) \left(\frac{2C_F \alpha_s(k_t)}{\pi} \right) v \delta \left(v - \left(\frac{k_t}{Q} \right)^a e^{-b_1 \eta^{(1)}} \right) &= \frac{2C_F \lambda}{a\pi\beta_0} R_1''(v), \\ \int [dk] M_{\text{sc}}^2(k) \Theta(-\eta) \left(\frac{2C_F \alpha_s(k_t)}{\pi} \right) v \delta \left(v - \left(\frac{k_t}{Q} \right)^a e^{-b_2 \eta^{(2)}} \right) &= \frac{2C_F \lambda}{a\pi\beta_0} R_2''(v). \end{aligned} \quad (3.46)$$

We are now in a position to write the final expression for the NNLL correlated correction as $\delta\mathcal{F}_{\text{correl}}(v) \simeq \alpha_s(Q)/\pi \delta\mathcal{F}_{\text{correl}}(\lambda)$, where

$$\begin{aligned} \delta\mathcal{F}_{\text{correl}}(\lambda) &= \int_0^\infty \frac{d\zeta_a}{\zeta_a} \int_0^{2\pi} \frac{d\phi_a}{2\pi} \sum_{\ell_a=1,2} \left(\frac{2C_{\ell_a} \lambda}{a\beta_0} \frac{R_{\ell_a}''(v)}{\alpha_s(Q)} \right) \int_0^\infty \frac{d\kappa}{\kappa} \int_{-\infty}^\infty d\eta \int_0^{2\pi} \frac{d\phi}{2\pi} \frac{1}{2!} C_{ab}(\kappa, \eta, \phi) \times \\ &\times \int d\mathcal{Z} \{ [R'_{\text{NLL}, \ell_i}, k_i] \} [\Theta(v - V_{\text{sc}}(\{\bar{p}\}, k_a, k_b, \{k_i\})) - \Theta(v - V_{\text{sc}}(\{\bar{p}\}, k_a + k_b, \{k_i\}))], \end{aligned} \quad (3.47)$$

where, as usual, the observable's value does not depend on emissions' rapidities, with the only exception of k_b , given by

$$k_b = \kappa k_{t,a}^{(\ell_a)} (\cosh(\eta_a + \eta), \cos(\phi_a + \phi), \sin(\phi_a + \phi), \sinh(\eta_a + \eta)), \quad k_{t,a}^{(\ell_a)} = Q v_a^{\frac{1}{a} - \frac{b_{\ell_a}}{a+b_{\ell_a}}} \xi_a^{(\ell_a)}. \quad (3.48)$$

Furthermore, in order to eliminate subleading effects, in the calculation of the observable we assume that k_b belongs to the same hemisphere as k_a , neglecting *de facto* the contribution of two emissions falling into two different hemispheres.

It is worth commenting on the connection between eq. (3.47) and the CMW scheme for the running coupling defined in eq. (2.9). As already explained in section 2, the term K in eq. (2.9) encodes the contribution of the splitting of a soft gluon into either a $q\bar{q}$ or a gg pair. This gives rise to NLL terms in the Sudakov radiator which are universal for all rIRC safe observables. In the multiple emissions function $\mathcal{F}(v)$, the CMW scheme gives rise to NNLL contributions which are contained in soft-collinear corrections (3.15). In the latter contribution, the branching of a soft gluon is in fact treated inclusively. This approximation is subsequently subtracted in the second theta function in the correlated correction (3.47), which takes into account the correct non-inclusive nature of the observable. Therefore, the choice of the CMW scheme in the multiple-emission function $\mathcal{F}(v)$ is irrelevant at all logarithmic orders, since the appropriate non-inclusive treatment of the observable is guaranteed once one adds up all resolved real emission corrections.

4 Validation and matched results

In this section we apply the algorithm described in section 3 to the following set of seven event-shape variables: thrust $1 - T$, heavy jet mass ρ_H , total and wide broadening B_T , B_W , C -parameter, thrust-major T_M , and oblateness O . For the two observables T_M , O an NNLL resummation was not previously available. For C , a numerical result was presented in [34]. On the other hand, for the remaining four event shapes, analytic results can be

found in the literature ($1 - T$ [30], ρ_H [32], B_T, B_W [33]). As described in the previous section, we use the $h(\lambda)$ function of thrust $1 - T$ also for the resummation of both the C -parameter and heavy jet mass ρ_H . For $1 - T$, we compare our resummation formulae to the analytic result of ref. [31], and extract the corresponding $h(\lambda)$ function, reported in eq. (B.14). For ρ_H we then obtained the same resummed result of [32]. Analogously, for B_T and B_W we have compared our numerical expansion to the relative analytic expressions of ref. [33] and found full agreement up to (and including) terms of order $\alpha_s^3 L^2$. To check the resummation for the observables for which we provide new results (i.e. T_M, O) and for C , we subtract the numerical expansion for the differential distributions from the predictions obtained by generating three-jet NLO distributions with `Event2` [54]. In order to get more stable distributions, we compute differences of observables, and plot the following quantity:

$$\Delta(v_1, v_2) = \left(\frac{1}{\sigma_0} \frac{d\sigma^{\text{NLO}}}{d \ln \frac{1}{v_1}} - \frac{1}{\sigma_0} \frac{d\sigma^{\text{NNLL}}|_{\text{expanded}}}{d \ln \frac{1}{v_1}} \right) - \{v_1 \rightarrow v_2\}. \quad (4.1)$$

The results are shown in figure 1. There we see that $\Delta(v_1, v_2)$ tends to zero for $v \rightarrow 0$, providing a check of the validity of the NNLL resummation up to $\mathcal{O}(\alpha_s^2)$. In order to check the expansion to $\mathcal{O}(\alpha_s^3)$ one would have to produce either NNLO 3-jet or NLO 4-jet distributions which are sufficiently stable in the deep infrared region. This can be achieved through long runs e.g. with the generators `EERAD3` [55], however, we have not been able to obtain distributions that were stable enough. Alternatively, one could use `NLOJET++` [59] to generate four-jet distributions at NLO and consider differences of observables. On the other hand, the checks against the analytic results for ρ_H, B_T and B_W at $\mathcal{O}(\alpha_s^3)$ provide us with a proof of the validity of our NNLL resummation at this order.

As a last step, we match the resummed NNLL distributions to NNLO fixed-order differential cross sections obtained with `EERAD3` [55]. The matching is performed according to the log-R scheme [10, 31]. As it is customary in resummed calculations, to probe the size of subleading logarithmic terms we introduce a rescaling constant x_V as

$$\ln \frac{1}{v} = \ln \frac{x_V}{v} - \ln x_V, \quad (4.2)$$

and expand the cross section around $\ln x_V/v$ neglecting subleading terms.⁹ Eventually we modify the resummed logarithm $\ln x_V/v$ in order to impose that the total cross section is reproduced at the kinematical endpoint v_{max}

$$\ln \frac{x_V}{v} \rightarrow \frac{1}{p} \ln \left(1 + \left(\frac{x_V}{v} \right)^p - \left(\frac{x_V}{v_{\text{max}}} \right)^p \right). \quad (4.3)$$

Here, p denotes a positive number which controls how quickly the logarithms are switched off close to the endpoint. In the following we use $p = 1$.

To obtain our central predictions we set $\mu_R = Q = M_Z$, corresponding to $\alpha_s(\mu_R) = 0.118$, and [28, 60]

$$\ln x_V = \frac{1}{2} \sum_{\ell=1,2} \left(\ln d_\ell + \int_0^{2\pi} \frac{d\phi}{2\pi} \ln g_\ell(\phi) \right). \quad (4.4)$$

⁹For details about how the resummed formula and the expansion coefficients change see e.g. ref. [31] where one has to replace $\ln x_L \rightarrow -\ln x_V$.

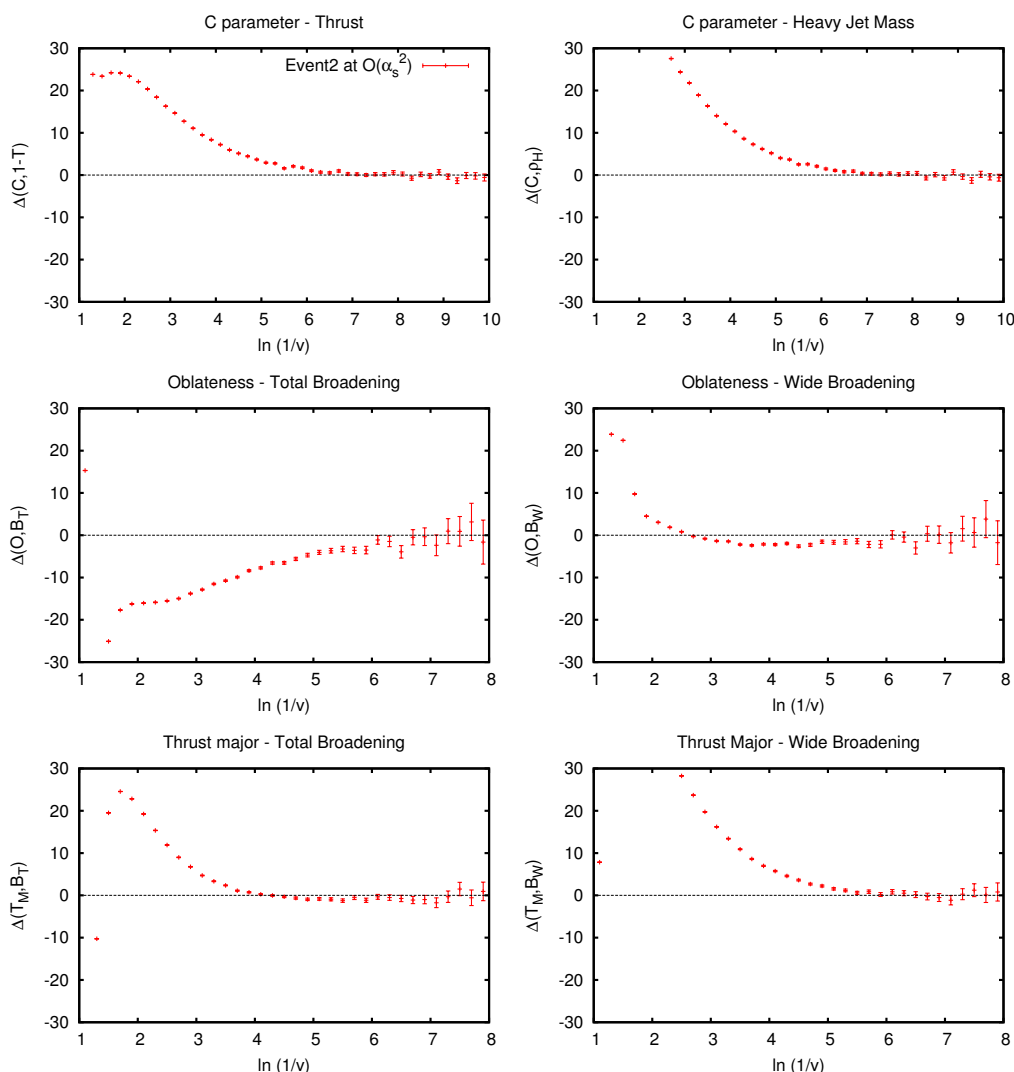


Figure 1. Difference between the NLO differential distributions of pairs of observables after subtracting the expansion of the NNLL resummation formula up to (and including) $\mathcal{O}(\alpha_s^2 L^0)$ (see eq. (4.1)). To obtain these distributions we used about 10^{11} events.

We then construct the uncertainty bands by varying μ_R and x_V individually by a factor of two in either direction. Figure 2 shows the comparison of the NNLL+NNLO prediction (red bands) to the pure fixed order at NNLO accuracy (light blue bands) for the thrust ($1 - T$), C -parameter, heavy-jet mass (ρ_H), wide- and total-broadening (B_W , B_T) and thrust major (T_M). As expected, at large values of the observables the matched results approach smoothly the fixed order distributions. On the other hand we observe large corrections at small values of the observables, where the NNLO distributions tend to diverge, while the NNLL+NNLO results have a smooth Sudakov behaviour. We also notice that at small/intermediate values of the observables the fixed-order uncertainties are artificially small, and that the matched results are not within the fixed-order uncertainty bands. In figure 3 we compare the NNLL+NNLO distributions to NLL+NNLO distributions for

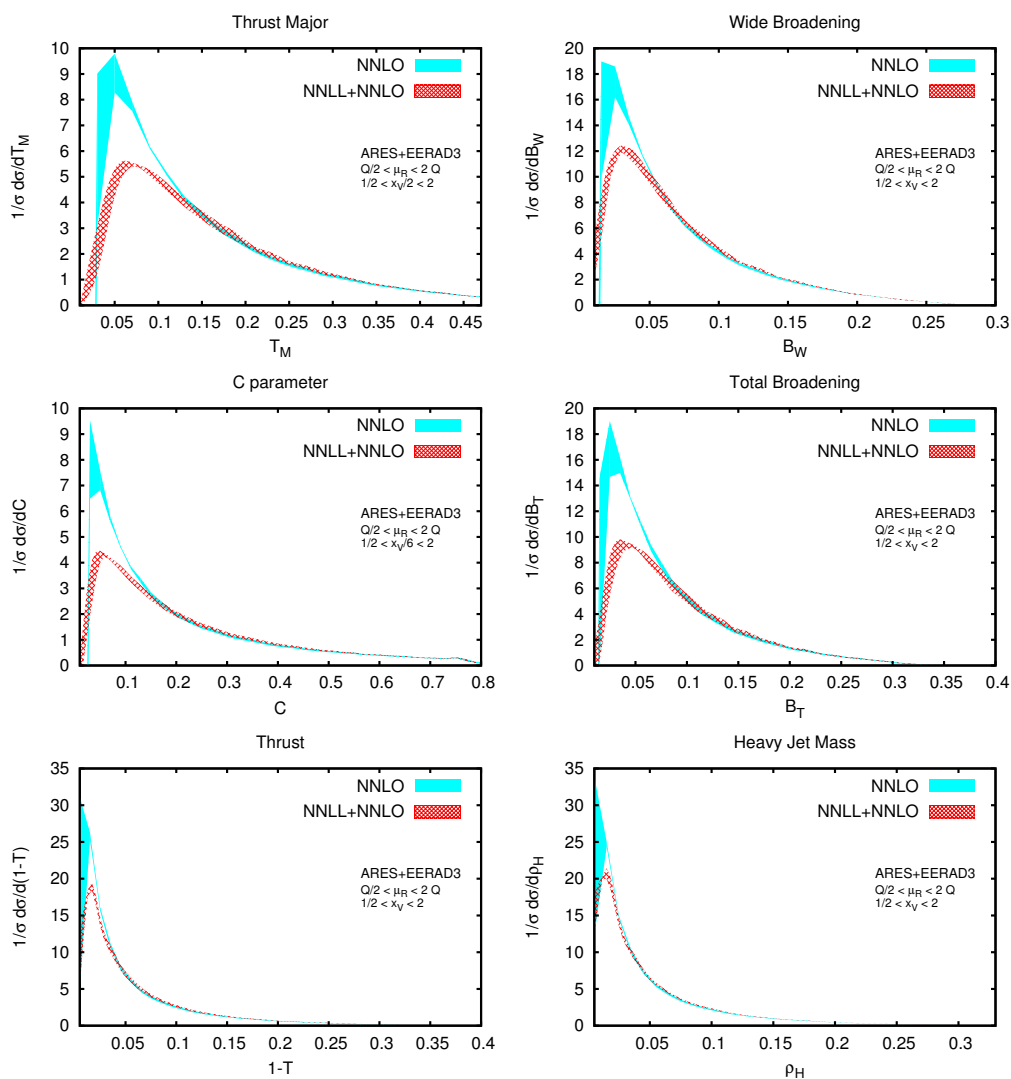


Figure 2. Differential distributions for six of the event-shape observables considered in the article at NNLL+NNLO (red band) and NNLO (light blue band).

the same set of observables. In the hard region, NNLL effects are small, NNLL+NNLO and NLL+NNLO bands overlap, and the uncertainties shown are those of the NNLO distribution. On the other hand, close to the peak of the distributions NNLL effects are important. In general NNLL corrections tend to make the spectrum harder, and we find that uncertainties are reduced when going from NLL+NNLO to NNLL+NNLO. This can be appreciated by looking at the lower panel of each plot of figure 3, representing the ratio of the NLL+NNLO and NNLL+NNLO bands to the corresponding central values. The oblateness distribution, not shown in figures 2 and 3, has the particular feature that it is defined as a difference of two observables. This implies that for sufficiently small values of O the cross section is dominated by cancellations between soft-collinear real emissions (corresponding to single logarithmic contributions) rather than by the double

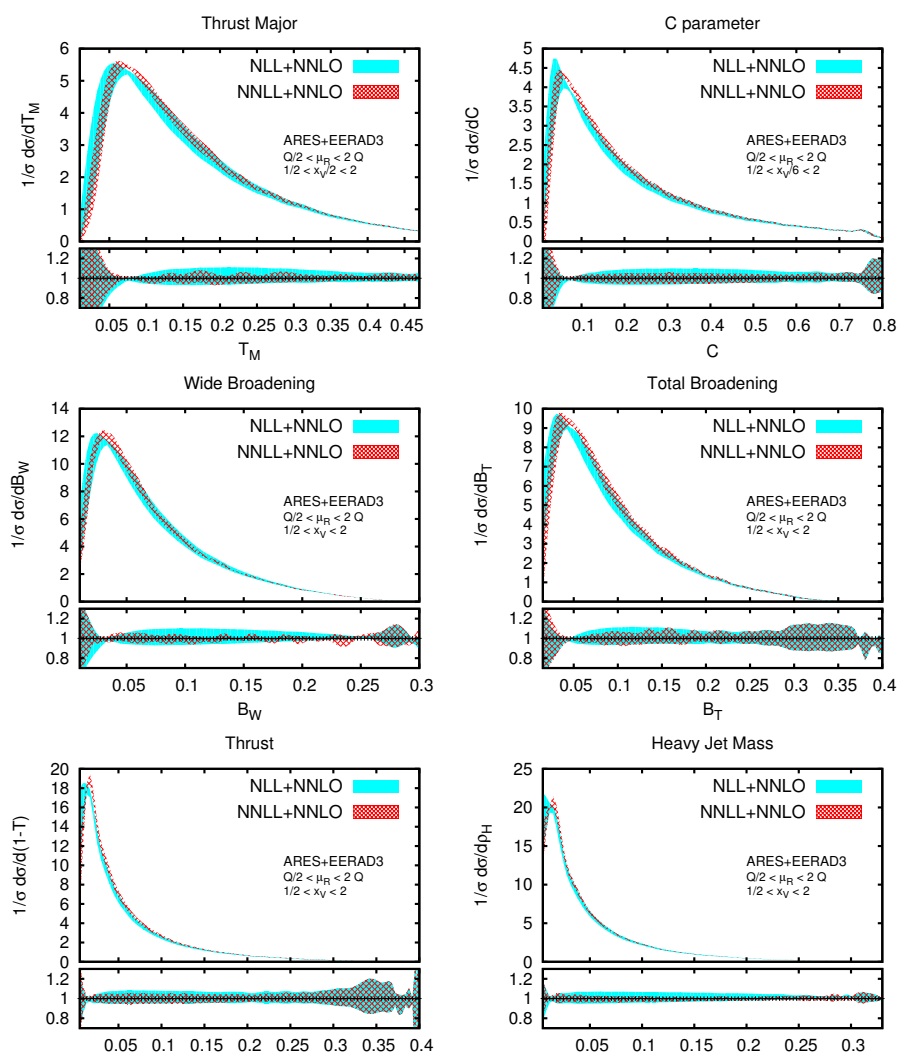


Figure 3. Matched distributions for six of the event-shape observables considered in the article at NNLL+NNLO (red band) and NLL+NNLO (light blue band). The lower panel of each plot shows the ratio of the NNLL+NNLO and NLL+NNLO bands to the corresponding central values.

logarithms present in the Sudakov radiator. As a consequence, the real corrections in the multiple emission function grow faster than their virtual counterpart, resulting in a divergence at $R'_{\text{NLL}} = R'_c \simeq 2$ [27]. In correspondence of this value of R'_{NLL} , the normal hierarchy of logarithms is reversed, and one ends up neglecting subleading logarithmic terms which are actually numerically dominant. A correct treatment of this region would require a resummation of such contributions to all logarithmic orders [61]. The divergence at $R'_{\text{NLL}} = R'_c$ is close to the peak of the distribution, where the bulk of the cross section is. Therefore, one can rely on the resummation only in the tail region, sufficiently away from the singularity. Moreover, the singularity is pushed towards higher values of the oblateness for higher values of x_V , thus one finds a large theory uncertainty due to the x_V variation. Despite spoiling the resummation, the above singularity does not affect the expansion in

powers of the strong coupling, so that our method can still be used to compute correctly the coefficients of the expansion to all orders in α_s .¹⁰

5 Conclusions

We presented a novel method for automated resummation of event-shape distributions to NNLL accuracy. The method is fully general and it can be applied to any global and recursive infrared and collinear safe event shape. Neither the factorisation of the observable into kinematical subprocesses nor an analytic definition is required for the method to be applied. We implemented the algorithm in the fast and stable numerical code **ARES** which will be publicly released soon. For the time being, the method relies on the fact that the NNLL Sudakov radiator is universal for all observables which have the same scaling properties for a single soft-collinear emission. Therefore, for the observables analysed in this article, we could extract the relevant unknown terms of the NNLL radiator from known resummations. Specifically, using the known result for thrust $1 - T$ we can reproduce the known resummation for the heavy jet mass ρ_H and the C parameter. Analogously, we extract the missing term from the k_t resummations in colour singlet production in hadronic collisions, re-derive the known results for jet broadenings B_T , B_W , and obtain new predictions for both thrust-major T_M and oblateness O . The calculation of the NNLL radiator for a generic observable will be addressed in future work.

We performed checks of our results by expanding known resummed distributions up to $\mathcal{O}(\alpha_s^3)$ and comparing them to the results in the literature. For observables for which a NNLL resummation was not previously known, we compare their $\mathcal{O}(\alpha_s^2)$ expansion to the fixed order generator **Event2**. We then presented NNLL+NNLO matched results, where the NNLO results were obtained using the code **EERAD3**. We observed that NNLL corrections are in general sizable and hence play a role in precise determinations of the strong coupling constant using e^+e^- data. A simultaneous fit using distributions of several observables will help disentangle perturbative effects from non-perturbative ones. In fact while perturbative effects are now well-understood and described at NNLL+NNLO or beyond, some deeper understanding is still required to model non-perturbative corrections, which are quite sizable at LEP energies. These corrections will be much more moderate at future lepton colliders. In these conditions, the potential of NNLL resummations can be fully exploited and even data close or at the peak of the distributions can be included in the fit region.

The work presented here represents a first step towards a fully automated resummation of generic global and rIRC observables. Future work includes the calculation of the NNLL radiator in the generic case, the treatment of jet resolution parameters, as well as the extension to observables at hadron colliders. Furthermore, the method could also be applied to derive corrections beyond NNLL.

Note added. When this work was being finalised ref. [62] appeared, where an analytic resummation for the C parameter is presented. We compared their formulae to our analytic result for the C parameter, and found full agreement at NNLL.

¹⁰Analogously to what is observed for both T_M and C , the oblateness, which has $d_\ell = 2$, receives sizable NNLL corrections.

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A Observables definition

In this appendix we recall the definition of the event shapes that we considered in this work.

- Thrust:

$$T \equiv \max_{\vec{n}} \frac{\sum_i |\vec{p}_i \cdot \vec{n}|}{Q}, \quad \tau \equiv 1 - T, \quad (\text{A.1})$$

where Q is the centre-of-mass energy and the vector \vec{n} that maximizes the sum defines the direction of the thrust axis, \vec{n}_T . The thrust axis divides each event into two hemispheres, $\mathcal{H}^{(1)}$ and $\mathcal{H}^{(2)}$.

- Heavy-jet mass:

$$\rho_H \equiv \max_{i=1,2} \frac{M_i^2}{Q^2}, \quad M_i^2 \equiv \left(\sum_{j \in \mathcal{H}^{(i)}} p_j \right)^2. \quad (\text{A.2})$$

- C-parameter:

$$C \equiv 3 \left(1 - \frac{1}{2} \sum_{i,j} \frac{(p_i \cdot p_j)^2}{(p_i \cdot Q)(p_j \cdot Q)} \right), \quad (\text{A.3})$$

where Q^μ is the total four-momentum.

- Total broadening:

$$B_T \equiv B_L + B_R, \quad (\text{A.4})$$

where

$$B_L \equiv \sum_{i \in \mathcal{H}^{(1)}} \frac{|\vec{p}_i \times \vec{n}_T|}{2Q}, \quad B_R \equiv \sum_{i \in \mathcal{H}^{(2)}} \frac{|\vec{p}_i \times \vec{n}_T|}{2Q}. \quad (\text{A.5})$$

- Wide broadening:

$$B_W \equiv \max\{B_L, B_R\}. \tag{A.6}$$

- Thrust-major:

$$T_M \equiv \max_{\vec{n} \cdot \vec{n}_T = 0} \frac{\sum_i |\vec{p}_i \cdot \vec{n}|}{Q}, \tag{A.7}$$

where the vector \vec{n} for which the sum is maximised defines the thrust-major axis.

- Oblateness:

$$O \equiv T_M - T_m, \tag{A.8}$$

where

$$T_m \equiv \frac{\sum_i |p_{i,x}|}{Q}, \tag{A.9}$$

and where x is the direction perpendicular to both the thrust and the thrust-major axes.

B Sudakov radiator

The Sudakov radiator can be parametrised as

$$R(v) = -Lg_1(\lambda) - g_2(\lambda) - \frac{\alpha_s}{\pi} g_3(\lambda) + \dots \tag{B.1}$$

We introduce the resummation scale x_V such that

$$\ln \frac{1}{v} = \ln \frac{x_V}{v} - \ln x_V = \frac{\lambda}{\alpha_s \beta_0} - \ln x_V, \tag{B.2}$$

where $\lambda = \alpha_s \beta_0 \ln x_V / v$. The functions g_1 , g_2 and g_3 can be parametrised as

$$g_i(\lambda) = \sum_{\ell=1,2} g_i^{(\ell)}(\lambda) \tag{B.3}$$

where $g_i^{(\ell)}$ can be expressed in terms of scaling parameters a and b_ℓ as follows:

$$g_1^{(\ell)}(\lambda) = \frac{A_1 \left((a + b_\ell - 2\lambda) \ln \left(1 - \frac{2\lambda}{a+b_\ell} \right) - (a - 2\lambda) \ln \left(1 - \frac{2\lambda}{a} \right) \right)}{4\pi b_\ell \beta_0 \lambda}, \tag{B.4}$$

$$\begin{aligned} g_2^{(\ell)}(\lambda) = & \frac{A_2 \left(a \ln \left(1 - \frac{2\lambda}{a} \right) - (a + b_\ell) \ln \left(1 - \frac{2\lambda}{a+b_\ell} \right) \right)}{8\pi^2 b_\ell \beta_0^2} + \frac{B_1 \ln \left(1 - \frac{2\lambda}{a+b_\ell} \right)}{4\pi \beta_0} \\ & + \frac{A_1 \left(\beta_1 (a + b_\ell) \ln^2 \left(1 - \frac{2\lambda}{a+b_\ell} \right) + 2\beta_1 (a + b_\ell) \ln \left(1 - \frac{2\lambda}{a+b_\ell} \right) \right)}{8\pi b_\ell \beta_0^3} \\ & - A_1 \frac{\ln \left(1 - \frac{2\lambda}{a} \right) \left(a\beta_1 \ln \left(1 - \frac{2\lambda}{a} \right) + 2a\beta_1 \right)}{8\pi b_\ell \beta_0^3} \\ & + \frac{A_1 \left(\ln \left(1 - \frac{2\lambda}{a+b_\ell} \right) - \log \left(1 - \frac{2\lambda}{a} \right) \right)}{4\pi b_\ell \beta_0} \ln x_V^2 \\ & - \frac{A_1 \left(\ln \left(1 - \frac{2\lambda}{a+b_\ell} \right) - \ln \left(1 - \frac{2\lambda}{a} \right) \right)}{2\pi b_\ell \beta_0} \ln \bar{d}_\ell, \end{aligned} \tag{B.5}$$

$$\begin{aligned}
 g_3^{(\ell)}(\lambda) = & \frac{\beta_1 B_1 \left((a+b_\ell) \ln \left(1 - \frac{2\lambda}{a+b_\ell} \right) + 2\lambda \right)}{2\beta_0^2 (a+b_\ell - 2\lambda)} \\
 & + \frac{A_2 \beta_1 \left(a^2 (a+b_\ell - 2\lambda) \ln \left(1 - \frac{2\lambda}{a} \right) - (a+b_\ell)^2 (a-2\lambda) \ln \left(1 - \frac{2\lambda}{a+b_\ell} \right) + 6b_\ell \lambda^2 \right)}{8\pi b_\ell \beta_0^3 (a-2\lambda)(a+b_\ell - 2\lambda)} \\
 & + \frac{A_1 \left(\beta_1^2 (a+b_\ell)^2 (a-2\lambda) \ln^2 \left(1 - \frac{2\lambda}{a+b_\ell} \right) - 4b_\ell \lambda^2 (\beta_0 \beta_2 + \beta_1^2) \right)}{8b_\ell \beta_0^4 (a-2\lambda)(a+b_\ell - 2\lambda)} \\
 & - \frac{a A_1 \ln \left(1 - \frac{2\lambda}{a} \right) (2\beta_0 \beta_2 (a-2\lambda) + a \beta_1^2 \ln \left(1 - \frac{2\lambda}{a} \right) + 4\beta_1^2 \lambda)}{8b_\ell \beta_0^4 (a-2\lambda)} \\
 & + \frac{A_1 (a+b_\ell) \ln \left(1 - \frac{2\lambda}{a+b_\ell} \right) (\beta_0 \beta_2 (a+b_\ell - 2\lambda) + 2\beta_1^2 \lambda)}{4b_\ell \beta_0^4 (a+b_\ell - 2\lambda)} \\
 & - \frac{A_1}{8(a-2\lambda)(a+b_\ell - 2\lambda)} \ln^2 x_V^2 + \left[\frac{\pi a \beta_0^2 B_1 + \lambda (A_2 \beta_0 - 2\pi (A_1 \beta_1 + \beta_0^2 B_1))}{4\pi \beta_0^2 (a-2\lambda)(a+b_\ell - 2\lambda)} \right. \\
 & \left. + \frac{A_1 \beta_1 \left((a+b_\ell)(a-2\lambda) \ln \left(1 - \frac{2\lambda}{a+b_\ell} \right) - a(a+b_\ell - 2\lambda) \ln \left(1 - \frac{2\lambda}{a} \right) \right)}{4b_\ell \beta_0^2 (a-2\lambda)(a+b_\ell - 2\lambda)} \right] \ln x_V^2 \\
 & - \frac{A_1}{2(a-2\lambda)(a+b_\ell - 2\lambda)} \ln \bar{d}_\ell^2 + \frac{A_1}{2(a-2\lambda)(a+b_\ell - 2\lambda)} \ln \bar{d}_\ell \ln x_V^2 \\
 & - \frac{\pi a \beta_0^2 B_1 + \lambda (A_2 \beta_0 - 2\pi (A_1 \beta_1 + \beta_0^2 B_1))}{2\pi \beta_0^2 (a-2\lambda)(a+b_\ell - 2\lambda)} \ln \bar{d}_\ell \\
 & + \frac{A_1 \beta_1 \left(a(a+b_\ell - 2\lambda) \ln \left(1 - \frac{2\lambda}{a} \right) - (a+b_\ell)(a-2\lambda) \ln \left(1 - \frac{2\lambda}{a+b_\ell} \right) \right)}{2b_\ell \beta_0^2 (a-2\lambda)(a+b_\ell - 2\lambda)} \ln \bar{d}_\ell \\
 & + A_1 \frac{7}{16b_\ell} \frac{1}{1 - \frac{2}{a+b_\ell} \lambda} \Theta(b_\ell) + h(\lambda),
 \end{aligned} \tag{B.6}$$

where $\ln \bar{d}_\ell^n = \int_0^{2\pi} \frac{d\phi}{2\pi} \ln^n(d_\ell g_\ell(\phi))$ and $\Theta(b_\ell) = 1(0)$ for $b_\ell > 0$ ($b_\ell = 0$). The renormalisation scale dependence can be restored using the following replacements in eq. (B.1)

$$\begin{aligned}
 g_1(\lambda) & \rightarrow g_1(\lambda), \\
 g_2(\lambda) & \rightarrow g_2(\lambda) + \lambda^2 g_1'(\lambda) \ln \frac{\mu_R^2}{Q^2}, \\
 g_3(\lambda) & \rightarrow g_3(\lambda) + \pi \left(\beta_0 \lambda g_2'(\lambda) + \frac{\beta_1}{\beta_0} \lambda^2 g_1'(\lambda) \right) \ln \frac{\mu_R^2}{Q^2} + \pi \left(\beta_0 \lambda^2 g_1'(\lambda) + \frac{\beta_0}{2} \lambda^3 g_1''(\lambda) \right) \ln^2 \frac{\mu_R^2}{Q^2}.
 \end{aligned} \tag{B.7}$$

The coefficients of the QCD β function used above are defined as

$$\beta_0 = \frac{11C_A - 2n_f}{12\pi}, \quad \beta_1 = \frac{17C_A^2 - 5C_A n_f - 3C_F n_f}{24\pi^2}, \tag{B.8}$$

$$\beta_2 = \frac{2857C_A^3 + (54C_F^2 - 615C_F C_A - 1415C_A^2)n_f + (66C_F + 79C_A)n_f^2}{3456\pi^3}. \tag{B.9}$$

The following functions are also used in the text

$$R'_{\text{NLL},\ell}(v) = \frac{A_1 \left(\ln \left(1 - \frac{2\lambda}{a+b_\ell} \right) - \ln \left(1 - \frac{2\lambda}{a} \right) \right)}{2\pi b_\ell \beta_0}, \tag{B.10}$$

$$\delta R'_{\text{NNLL},\ell}(v) = \frac{\alpha_s(Q)}{\pi} \left[-\frac{A_1\beta_1 \left(a(a+b_\ell-2\lambda) \ln\left(1-\frac{2\lambda}{a}\right) - (a+b_\ell)(a-2\lambda) \ln\left(1-\frac{2\lambda}{a+b_\ell}\right) + 2b_\ell\lambda \right)}{2b_\ell\beta_0^2(a-2\lambda)(a+b_\ell-2\lambda)} \right. \\ \left. + \frac{A_1\lambda}{(a-2\lambda)(a+b_\ell-2\lambda)} \ln \frac{\mu_R^2}{Q^2} - \frac{A_1}{2(a-2\lambda)(a+b_\ell-2\lambda)} \ln x_V^2 \right. \\ \left. + \frac{A_2\lambda}{2\pi\beta_0(a-2\lambda)(a+b_\ell-2\lambda)} \right], \quad (\text{B.11})$$

$$R''_\ell(v) = \frac{\alpha_s(Q)}{\pi} \frac{A_1}{(a-2\lambda)(a+b_\ell-2\lambda)}. \quad (\text{B.12})$$

The limit $b_\ell \rightarrow 0$ (relevant for jet broadenings, thrust major and oblateness) is finite and well defined for all the above expressions. The function $h(\lambda)$, implicitly defined in eq. (3.5), is extracted from the resummed expression of

$$\Sigma(v) = \frac{1}{\sigma} \int_0^v dv' \frac{d\sigma(v')}{dv'}, \quad (\text{B.13})$$

(where σ is the total cross section for $e^+e^- \rightarrow \text{hadrons}$) for the two reference observables (i.e. k_t and thrust $1-T$) leading to

$$h^{(1-T)}(\lambda) = -A_3^{(1-T)} \frac{\lambda^2}{8\pi^2\beta_0^2(1-2\lambda)(2-2\lambda)} - B_2^{(1-T)} \frac{\lambda}{8\pi\beta_0(1-\lambda)} \\ + C_F \frac{\pi^2}{24} \frac{1}{1-2\lambda} + C_F \left(\frac{1}{4} - \frac{\pi^2}{12} \right) \frac{1}{1-\lambda} + C_F \left(-\frac{19}{8} + \frac{7}{24}\pi^2 \right), \quad (\text{B.14})$$

and

$$h^{(k_t)}(\lambda) = -A_3^{(k_t)} \frac{\lambda^2}{8\pi^2\beta_0^2(1-2\lambda)^2} - B_2^{(k_t)} \frac{\lambda}{4\pi\beta_0(1-2\lambda)} \\ + C_F \left(\frac{1}{4} - \frac{\pi^2}{24} \right) \frac{1}{1-2\lambda} + C_F \left(-\frac{19}{8} + \frac{7}{24}\pi^2 \right). \quad (\text{B.15})$$

Note that the above expressions imply that $g_3(0) \neq 0$. This means that constant terms appear in the exponent, which can be expanded to $\mathcal{O}(\alpha_s)$ neglecting subleading terms.

The anomalous dimensions A_i and B_i used in the above expressions are:

$$A_1 = 2C_F, \quad (\text{B.16})$$

$$B_1 = -3C_F, \quad (\text{B.17})$$

$$A_2 = C_F \left(C_A \left(\frac{67}{9} - \frac{\pi^2}{3} \right) - \frac{10}{9}n_f \right), \quad (\text{B.18})$$

the coefficient A_2 defines the running coupling in the CMW scheme used in the definition of the soft emission probability. The coefficients B_2 and A_3 are observable dependent in our study. The expressions for the two reference observables read (with $T_F = 1/2$)

$$B_2^{(1-T)} = -2 \left(C_F^2 \left(-\frac{\pi^2}{2} + \frac{3}{8} + 6\zeta_3 \right) + C_F C_A \left(\frac{11\pi^2}{18} + \frac{17}{24} - 3\zeta_3 \right) + C_F T_F n_f \left(-\frac{1}{6} - \frac{2}{9}\pi^2 \right) \right), \quad (\text{B.19})$$

$$B_2^{(k_t)} = B_2^{(1-T)} + 2\pi\beta_0\zeta_2 C_F, \quad (\text{B.20})$$

$$A_3^{(1-T)} = C_F C_A^2 \left(\frac{245}{12} - \frac{67}{27}\pi^2 + \frac{11}{3}\zeta_3 + \frac{22}{5}\zeta_2^2 \right) + C_F T_F n_f \left(-\frac{55}{6} + 8\zeta_3 \right) - \frac{8}{27} T_F^2 n_f^2$$

$$+C_A T_F n_f \left(-\frac{209}{27} + \frac{20}{27} \pi^2 - \frac{28}{3} \zeta_3 \right) + \pi \beta_0 C_F \left(C_A \left(\frac{808}{27} - 28 \zeta_3 \right) - \frac{224}{27} T_F n_f \right), \quad (\text{B.21})$$

$$A_3^{(k_t)} = A_3^{(1-T)} - 8\pi^2 \beta_0^2 \zeta_2 C_F. \quad (\text{B.22})$$

C Analytic NNLL results for additive observables

Some event shapes have the property that they are additive, meaning that for soft emissions

$$V(\{\tilde{p}\}, k_1, \dots, k_n) = \sum_{i=1}^n V(\{\tilde{p}\}, k_i) + \mathcal{O}(V^2), \quad (\text{C.1})$$

while for a hard emission k collinear to leg ℓ , the corresponding $V(\{\tilde{p}\}, k)$ has to be replaced by $V^{(k)}(\{\tilde{p}\}, k[k'_t, p'_{t,\ell}, z^{(\ell)}])$, as defined in section 3.3.2. This is the case for instance for the thrust, the C -parameter and the heavy-jet masses. For this simpler class of observables, the NNLL corrections can be simplified significantly. In this appendix we work out the NNLL corrections of section 3 for these additive observables analytically. This provides a check of the numerical implementation of our method. We also show that for these observables the corresponding NNLL correction $\delta\mathcal{F}_{\text{NNLL}}$ factorises in a coefficient that multiplies the NLL function $\mathcal{F}_{\text{NLL}}(\lambda)$, that for an additive observable reads

$$\mathcal{F}_{\text{NLL}}(\lambda) = \frac{e^{-\gamma_E R'_{\text{NLL}}}}{\Gamma(1 + R'_{\text{NLL}})}. \quad (\text{C.2})$$

C.1 Soft-collinear correction

We consider first the soft-collinear contribution $\delta\mathcal{F}_{\text{sc}}$ of eq. (3.15), and use the fact that for additive observables

$$V_{\text{sc}}(\{\tilde{p}\}, k, \{k_i\}) = \zeta v + V_{\text{sc}}(\{\tilde{p}\}, \{k_i\}). \quad (\text{C.3})$$

$$\begin{aligned} \delta\mathcal{F}_{\text{sc}}(\lambda) &= \frac{\pi}{\alpha_s(Q)} \int_0^\infty \frac{d\zeta}{\zeta} \sum_{\ell=1,2} \left(\delta R'_{\text{NNLL},\ell} + R''_{\ell_i} \ln \bar{d}_\ell + R''_{\ell_i} \ln \frac{1}{\zeta} \right) \int d\mathcal{Z}[\{R'_{\text{NLL},\ell_i}, k_i\}] \times \\ &\times \left[\Theta \left(1 - \zeta - \lim_{v \rightarrow 0} \frac{V_{\text{sc}}(\{\tilde{p}\}, \{k_i\})}{v} \right) - \Theta(1 - \zeta) \Theta \left(1 - \lim_{v \rightarrow 0} \frac{V_{\text{sc}}(\{\tilde{p}\}, \{k_i\})}{v} \right) \right], \quad (\text{C.4}) \end{aligned}$$

where we used the fact that for additive observables the integral over ϕ can be performed analytically.

We can define rescaled momenta $\tilde{k}_1, \dots, \tilde{k}_n$ in the second theta function such that $V_{\text{sc}}(\{\tilde{p}\}, \tilde{k}_i) = V_{\text{sc}}(\{\tilde{p}\}, k_i)/(1 - \zeta)$. Recursive IRC safety of V guarantees that

$$V_{\text{sc}}(\{\tilde{p}\}, \{k_i\}) = (1 - \zeta) V_{\text{sc}}(\{\tilde{p}\}, \{\tilde{k}_i\}). \quad (\text{C.5})$$

Using the explicit expression for $d\mathcal{Z}$, and defining $\tilde{\zeta}_i = V_{\text{sc}}(\{\tilde{p}\}, \tilde{k}_i)/v$, one gets

$$\begin{aligned} \delta\mathcal{F}_{\text{sc}}(\lambda) &= \frac{\pi}{\alpha_s(Q)} \int_0^\infty \frac{d\zeta}{\zeta} \sum_{\ell=1,2} \left(\delta R'_{\text{NNLL},\ell} + R''_{\ell_i} \ln \bar{d}_\ell + R''_{\ell_i} \ln \frac{1}{\zeta} \right) \epsilon^{R'_{\text{NLL}}} \sum_{n=0}^\infty \frac{1}{n!} \prod_{i=1}^n \sum_{\ell_i=1,2} R'_{\text{NLL},\ell_i} \\ &\times \int_0^{2\pi} \frac{d\phi_i}{2\pi} \Theta(1 - \zeta) \left[\int_{\frac{\epsilon}{1-\zeta}}^\infty \frac{d\tilde{\zeta}_i}{\tilde{\zeta}_i} \Theta \left(1 - \lim_{v \rightarrow 0} \frac{V_{\text{sc}}(\{\tilde{p}\}, \{\tilde{k}_i\})}{v} \right) - \int_\epsilon^\infty \frac{d\zeta_i}{\zeta_i} \Theta \left(1 - \lim_{v \rightarrow 0} \frac{V_{\text{sc}}(\{\tilde{p}\}, \{k_i\})}{v} \right) \right]. \quad (\text{C.6}) \end{aligned}$$

We can then rearrange the above equation to reconstruct the function $\mathcal{F}_{\text{NLL}}(\lambda)$. This gives

$$\begin{aligned} \delta\mathcal{F}_{\text{sc}}(\lambda) &= \mathcal{F}_{\text{NLL}}(\lambda) \frac{\pi}{\alpha_s(Q)} \int_0^1 \frac{d\zeta}{\zeta} \sum_{\ell=1,2} \left(\delta R'_{\text{NLL},\ell} + R''_{\ell_i} \ln \bar{d}_\ell + R''_{\ell_i} \ln \frac{1}{\zeta} \right) \left((1-\zeta)^{R'_{\text{NLL}}} - 1 \right) \\ &= -\mathcal{F}_{\text{NLL}}(\lambda) \frac{\pi}{\alpha_s(Q)} \sum_{\ell=1,2} \left((\delta R'_{\text{NLL},\ell} + R''_{\ell_i} \ln \bar{d}_\ell) \left(\psi^{(0)}(1 + R'_{\text{NLL}}) + \gamma_E \right) \right. \\ &\quad \left. + \frac{R''_{\ell_i}}{2} \left(\left(\psi^{(0)}(1 + R'_{\text{NLL}}) + \gamma_E \right)^2 - \psi^{(1)}(1 + R'_{\text{NLL}}) + \frac{\pi^2}{6} \right) \right). \end{aligned} \quad (\text{C.7})$$

C.2 Recoil correction

Let us now consider the recoil contribution $\delta\mathcal{F}_{\text{rec}}$ of eq. (3.28). Considering a hard emission collinear to leg ℓ , for an additive observable one has

$$V_{\text{hc}}^{(k')}(\{\tilde{p}\}, k', \{k_i\}) = \left(\frac{k'_t}{Q} \right)^{a+b_\ell} f^{(\ell)}(z^{(\ell)}, \phi) + V_{\text{sc}}(\{\tilde{p}\}, \{k_i\}), \quad (\text{C.8})$$

and

$$V_{\text{sc}}(\{\tilde{p}\}, k, \{k_i\}) = \left(\frac{k_t}{Q} \right)^{a+b_\ell} f_{\text{sc}}^{(\ell)}(z^{(\ell)}, \phi) + V_{\text{sc}}(\{\tilde{p}\}, \{k_i\}), \quad (\text{C.9})$$

where the presence of k' , rather than k , denotes that the full recoil has been taken into account in the calculation of the observable.

Using the above equations in eq. (3.28) we get

$$\begin{aligned} \delta\mathcal{F}_{\text{rec}}(\lambda) &= \sum_{\ell=1,2} \frac{\alpha_s(v^{1/(a+b_\ell)}Q)}{\alpha_s(Q)(a+b_\ell)} \int_0^\infty \frac{d\zeta}{\zeta} \int_0^{2\pi} \frac{d\phi}{2\pi} \int d\mathcal{Z}[\{R'_{\text{NLL},\ell_i}, k_i\}] \int_0^1 dz p_\ell(z) \times \\ &\quad \times \left[\Theta \left(1 - \zeta f^{(\ell)}(z, \phi) - \lim_{v \rightarrow 0} \frac{V_{\text{sc}}(\{\tilde{p}\}, \{k_i\})}{v} \right) - \Theta \left(1 - \zeta f_{\text{sc}}^{(\ell)}(z, \phi) - \lim_{v \rightarrow 0} \frac{V_{\text{sc}}(\{\tilde{p}\}, \{k_i\})}{v} \right) \right], \end{aligned} \quad (\text{C.10})$$

where ζ is defined in eq. (3.26). We can define rescaled momenta $\tilde{k}_1, \dots, \tilde{k}_n$ in the second theta function such that $V_{\text{sc}}(\{\tilde{p}\}, \tilde{k}_i) = V_{\text{sc}}(\{\tilde{p}\}, k_i) / (1 - \zeta f_{\text{sc}}^{(\ell)}(z, \phi))$. Recursive IRC safety of V guarantees that

$$V_{\text{sc}}(\{\tilde{p}\}, k_1, \dots, k_n) = (1 - \zeta f_{\text{sc}}^{(\ell)}(z, \phi)) V_{\text{sc}}(\{\tilde{p}\}, \tilde{k}_1, \dots, \tilde{k}_n). \quad (\text{C.11})$$

Analogously, we define soft and collinear momenta $\tilde{k}'_1, \dots, \tilde{k}'_n$ in the theta function containing $f^{(\ell)}(z, \phi)$ such that

$$V_{\text{sc}}(\{\tilde{p}\}, k_1, \dots, k_n) = (1 - \zeta f^{(\ell)}(z, \phi)) V_{\text{sc}}(\{\tilde{p}\}, \tilde{k}'_1, \dots, \tilde{k}'_n). \quad (\text{C.12})$$

Using the explicit expression for $d\mathcal{Z}$, one gets

$$\begin{aligned} \delta\mathcal{F}_{\text{rec}}(\lambda) &= \sum_{\ell=1,2} \frac{\alpha_s(v^{1/(a+b_\ell)}Q)}{\alpha_s(Q)(a+b_\ell)} \int_0^\infty \frac{d\zeta}{\zeta} \int_0^{2\pi} \frac{d\phi}{2\pi} \int_0^1 dz p_\ell(z) \epsilon^{R'_{\text{NLL}}} \sum_{n=0}^\infty \frac{1}{n!} \prod_{i=1}^n \sum_{\ell_i=1,2} R'_{\text{NLL},\ell_i} \int_0^{2\pi} \frac{d\phi_i}{2\pi} \times \\ &\quad \times \left[\Theta \left(1 - \zeta f^{(\ell)}(z) \int_0^\infty \frac{d\tilde{\zeta}'_i}{\tilde{\zeta}'_i} \Theta \left(\tilde{\zeta}'_i - \frac{\epsilon}{1 - \zeta f^{(\ell)}(z)} \right) \Theta \left(1 - \lim_{v \rightarrow 0} \frac{V_{\text{sc}}(\{\tilde{p}\}, \tilde{k}'_1, \dots, \tilde{k}'_n)}{v} \right) \right) \right. \\ &\quad \left. - \Theta \left(1 - \zeta f_{\text{sc}}^{(\ell)}(z) \int_0^\infty \frac{d\tilde{\zeta}_i}{\tilde{\zeta}_i} \Theta \left(\tilde{\zeta}_i - \frac{\epsilon}{1 - \zeta f_{\text{sc}}^{(\ell)}(z)} \right) \Theta \left(1 - \lim_{v \rightarrow 0} \frac{V_{\text{sc}}(\{\tilde{p}\}, \tilde{k}_1, \dots, \tilde{k}_n)}{v} \right) \right) \right]. \end{aligned} \quad (\text{C.13})$$

We can then rearrange the above equation to reconstruct the function $\mathcal{F}_{\text{NLL}}(\lambda)$. This gives

$$\begin{aligned} \delta\mathcal{F}_{\text{rec}}(\lambda) &= \mathcal{F}_{\text{NLL}}(\lambda) \sum_{\ell=1,2} \frac{\alpha_s(v^{1/(a+b_\ell)}Q)}{\alpha_s(Q)(a+b_\ell)} \int_0^{2\pi} \frac{d\phi}{2\pi} \int_0^1 dz p_\ell(z) \times \\ &\quad \times \int_0^\infty \frac{d\zeta}{\zeta} \left[(1 - \zeta f^{(\ell)}(z, \phi))^{R'_{\text{NLL}}} \Theta(1 - \zeta f^{(\ell)}(z, \phi)) - (1 - \zeta f_{\text{sc}}^{(\ell)}(z, \phi))^{R'_{\text{NLL}}} \Theta(1 - \zeta f_{\text{sc}}^{(\ell)}(z, \phi)) \right] \\ &= \mathcal{F}_{\text{NLL}}(\lambda) \sum_{\ell=1,2} \frac{\alpha_s(v^{1/(a+b_\ell)}Q)}{\alpha_s(Q)(a+b_\ell)} \int_0^{2\pi} \frac{d\phi}{2\pi} \int_0^1 dz p_\ell(z) \ln \frac{f_{\text{sc}}^{(\ell)}(z, \phi)}{f^{(\ell)}(z, \phi)}. \end{aligned} \quad (\text{C.14})$$

As an example, we consider the thrust. One can show that its expression in terms of Sudakov variables is

$$1 - T = \sum_{i=1}^n \frac{k_{ti}}{Q} e^{-|\eta_i|} + \frac{1}{Q^2} \sum_{\ell=1,2} \frac{\left(\sum_{i \in \mathcal{H}^{(\ell)}} \vec{k}_{ti}^{(\ell)} \right)^2}{1 - \sum_{i \in \mathcal{H}^{(\ell)}} z_i^{(\ell)}}. \quad (\text{C.15})$$

Suppose k is collinear to leg \tilde{p}_1 . Using the Sudakov parametrisation of eq. (3.18) we then have

$$1 - T \simeq \sum_{i=1}^n \frac{k_{ti}}{Q} e^{-|\eta_i|} + \frac{k_t^2}{z^{(1)}Q^2} + \frac{k_t^2}{(1 - z^{(1)})Q^2} = \sum_{i=1}^n \frac{k_{ti}}{Q} e^{-|\eta_i|} + \frac{k_t^2}{z^{(1)}(1 - z^{(1)})Q^2}, \quad (\text{C.16})$$

where we have used the fact that the hard-collinear k_t is larger than all soft-collinear k_{ti} , and therefore $k_t \simeq k'_t$. A hard collinear emission gives an additive contribution to the observable, so that we can apply eq. (C.14) with

$$f^{(\ell)}(z^{(\ell)}, \phi) = \frac{1}{z^{(\ell)}(1 - z^{(\ell)})}, \quad f_{\text{sc}}^{(\ell)}(z^{(\ell)}, \phi) = \frac{1}{z^{(\ell)}}.$$

This gives

$$\begin{aligned} \delta\mathcal{F}_{\text{rec}}(\lambda) &= \mathcal{F}_{\text{NLL}}(\lambda) 2C_F \frac{\alpha_s(\sqrt{\tau}Q)}{2\alpha_s(Q)} \int_0^1 dz \frac{(1 + (1 - z)^2)}{z} \ln(1 - z) \\ &= \mathcal{F}_{\text{NLL}}(\lambda) \frac{C_F \alpha_s(\sqrt{\tau}Q)}{\alpha_s(Q)} \left(\frac{5}{4} - \frac{\pi^2}{3} \right). \end{aligned} \quad (\text{C.17})$$

This result holds also for the C -parameter and the heavy-jet mass, which behave as $1 - T$ in the collinear region.

C.3 Hard-collinear correction

In a similar way, we compute here the hard-collinear function $\delta\mathcal{F}_{\text{hc}}(\lambda)$ of eq. (3.32). Using eq. (C.9) we obtain

$$\begin{aligned} \delta\mathcal{F}_{\text{hc}}(\lambda) &= \sum_{\ell=1,2} \frac{\alpha_s(v^{1/(a+b_\ell)}Q)}{\alpha_s(Q)(a+b_\ell)} \int_0^\infty \frac{d\zeta}{\zeta} \int_0^\pi \frac{d\phi}{2\pi} \int_0^1 \frac{dz}{z} (z p_\ell(z) - 2C_\ell) \int d\mathcal{Z}[\{R'_{\text{NLL},\ell_i}, k_i\}] \times \\ &\quad \times \left[\Theta \left(1 - \zeta - \lim_{v \rightarrow 0} \frac{V_{\text{sc}}(\{\tilde{p}\}, \{k_i\})}{v} \right) - \Theta(1 - \zeta) \Theta \left(1 - \lim_{v \rightarrow 0} \frac{V_{\text{sc}}(\{\tilde{p}\}, \{k_i\})}{v} \right) \right]. \end{aligned} \quad (\text{C.18})$$

Rescaling the momenta in a similar way as we have done in the previous section we get

$$\begin{aligned} \delta\mathcal{F}_{\text{hc}}(\lambda) &= \mathcal{F}_{\text{NLL}}(\lambda) \sum_{\ell=1,2} \frac{\alpha_s(v^{1/(a+b_\ell)}Q)}{\alpha_s(Q)(a+b_\ell)} \int_0^\infty \frac{d\zeta}{\zeta} \int_0^{2\pi} \frac{d\phi}{2\pi} \int_0^1 \frac{dz}{z} (z p_\ell(z) - 2C_\ell) \times \\ &\times \left[(1-\zeta)^{R'_{\text{NLL}}} \Theta(1-\zeta) - \Theta(1-\zeta) \right] = \mathcal{F}_{\text{NLL}}(\lambda) \sum_{\ell=1,2} \frac{\alpha_s(v^{1/(a+b_\ell)}Q)}{\alpha_s(Q)(a+b_\ell)} \times \quad (\text{C.19}) \\ &\times C_\ell B_\ell \int_0^1 \frac{d\zeta}{\zeta} \left[(1-\zeta)^{R'_{\text{NLL}}} - 1 \right]. \end{aligned}$$

For thrust, using the explicit expression for \mathcal{F}_{NLL} (C.2) we obtain

$$\delta\mathcal{F}_{\text{hc}}(\lambda) = \frac{\alpha_s(\sqrt{\tau}Q)}{\alpha_s(Q)} C_F \frac{3}{2} \left(\psi^{(0)}(1 + R'_{\text{NLL}}) + \gamma_E \right) \mathcal{F}_{\text{NLL}}. \quad (\text{C.20})$$

C.4 Soft wide-angle correction

We consider now the case of a NNLL correction induced by a soft large-angle emission, eq. (3.38). Then we have

$$V_{\text{wa}}^{(k)}(\{\tilde{p}\}, k, \{k_i\}) = \left(\frac{k_t}{Q} \right)^a f_{\text{wa}}(\eta, \phi) + V_{\text{sc}}(\{\tilde{p}\}, \{k_i\}), \quad (\text{C.21})$$

$$V_{\text{sc}}(\{\tilde{p}\}, k, \{k_i\}) = \left(\frac{k_t}{Q} \right)^a f_{\text{sc}}(\eta, \phi) + V_{\text{sc}}(\{\tilde{p}\}, \{k_i\}), \quad (\text{C.22})$$

where $f_{\text{sc}}(\eta, \phi)$ and $f_{\text{wa}}(\eta, \phi)$ are defined in eqs. (3.33) and (3.34). Performing a similar rescaling as for the recoil correction one finds

$$\delta\mathcal{F}_{\text{wa}}(\lambda) = \mathcal{F}_{\text{NLL}}(\lambda) \frac{2C_F}{a} \frac{\alpha_s(v^{\frac{1}{a}}Q)}{\alpha_s(Q)} \int_0^{2\pi} \frac{d\phi}{2\pi} \int_{-\infty}^\infty d\eta \ln \frac{f_{\text{sc}}(\eta, \phi)}{f_{\text{wa}}(\eta, \phi)}. \quad (\text{C.23})$$

For the thrust and the heavy-jet mass:

$$f_{\text{wa}}(\eta, \phi) = f_{\text{sc}}(\eta, \phi) = e^{-|\eta|}, \quad (\text{C.24})$$

so that $\delta\mathcal{F}_{\text{wa}}(\lambda) = 0$. In the case of the C -parameter instead we have

$$f_{\text{wa}}(\eta, \phi) = \frac{3}{\cosh \eta} \quad \text{and} \quad f_{\text{sc}}(\eta, \phi) = 6 e^{-|\eta|}. \quad (\text{C.25})$$

This gives

$$\delta\mathcal{F}_{\text{wa}}(\lambda) = \mathcal{F}_{\text{NLL}}(\lambda) 2C_F \frac{\alpha_s(CQ)}{\alpha_s(Q)} \int_{-\infty}^\infty d\eta \ln(2 \cosh \eta e^{-|\eta|}) = \mathcal{F}_{\text{NLL}}(\lambda) C_F \frac{\alpha_s(CQ)}{\alpha_s(Q)} \frac{\pi^2}{6}, \quad (\text{C.26})$$

where C is the value of the C -parameter.

C.5 Soft correlated correction

The correlated correction presented in eq. (3.47) depends on the difference

$$\Theta(v - V_{\text{sc}}(\{\tilde{p}\}, k_a, k_b, k_1, \dots, k_n)) - \Theta(v - V_{\text{sc}}(\{\tilde{p}\}, k_a + k_b, k_1, \dots, k_n)) , \quad (\text{C.27})$$

which is in general non-zero for additive observables. However, the above correction vanishes if the observable V_{sc} is inclusive, i.e. $V_{\text{sc}}(k_a, k_b) = V_{\text{sc}}(k_a + k_b)$. There are observables which are inclusive in particular regions of the phase space. As an example, the thrust T happens to be inclusive only for emissions that propagate into the same hemisphere (defined by the thrust axis itself). In this case, the difference (C.27) is non-zero if the two correlated soft partons k_a, k_b move into opposite hemispheres. However, this configuration requires the parent gluon to be emitted at small rapidities, which in the limit $T \rightarrow 1$ gives rise to a correction which is at most N³LL, and can be neglected accordingly. The other additive observables treated in this article are also inclusive in the relevant phase space regions, so we can conclude that for $T, C,$ and ρ_H , at NNLL

$$\delta\mathcal{F}_{\text{correl}}(\lambda) = 0. \quad (\text{C.28})$$

D Monte Carlo determination of real emission corrections

Both the NLL function $\mathcal{F}_{\text{NLL}}(\lambda)$ and the NNLL correction $\delta\mathcal{F}_{\text{NNLL}}(\lambda)$ can be computed efficiently with a Monte Carlo procedure. In this appendix we recall the procedure devised in ref. [23], simplifying the notation so that it can be easily adapted to the NNLL case. We then discuss the MC determination of all NNLL corrections.

D.1 The function \mathcal{F}_{NLL}

We now recall the procedure of ref. [27] to efficiently compute the function $\mathcal{F}_{\text{NLL}}(\lambda)$ via a Monte Carlo procedure. The first observation is that in the sum in eqs. (2.32) and (2.33) the term with zero emissions is negligibly small due to the factor $\epsilon^{R'_{\text{NLL}}}$. Second, in all other terms we can pick up the hardest emission k_1 (the one for which $V_{\text{sc}}(\{p\}, k_1)$ is the largest of all $V_{\text{sc}}(\{p\}, k_i)$) and neglect all emissions \tilde{k}_i with $v_i < \epsilon v_1$, with corrections suppressed by powers of $v_1 \sim v$. This gives

$$\begin{aligned} \mathcal{F}_{\text{NLL}}(\lambda) &= \epsilon^{R'_{\text{NLL}}} \sum_{\ell_1=1,2} R'_{\text{NLL},\ell_1} \int_0^\infty \frac{d\zeta_1}{\zeta_1} \zeta_1^{R'_{\text{NLL}}} \int_0^{2\pi} \frac{d\phi_1}{2\pi} \times \\ &\times \sum_{n=0}^\infty \frac{1}{n!} \prod_{i=2}^{n+1} \sum_{\ell_i=1,2} R'_{\text{NLL},\ell_i} \int_{\epsilon\zeta_1}^{\zeta_1} \frac{d\zeta_i}{\zeta_i} \int_0^{2\pi} \frac{d\phi_i}{2\pi} \Theta\left(1 - \lim_{v \rightarrow 0} \frac{V_{\text{sc}}(\{\tilde{p}\}, k_1, \dots, k_{n+1})}{v}\right). \end{aligned} \quad (\text{D.1})$$

We now introduce $\tilde{\zeta}_i = \zeta_i/\zeta_1$, with corresponding momenta \tilde{k}_i such that $V_{\text{sc}}(\{\tilde{p}\}, \tilde{k}_i) = v_i/\zeta_1$. Since V is rIRC safe we have

$$V_{\text{sc}}(\{\tilde{p}\}, k_1, \dots, k_{n+1}) = \zeta_1 V_{\text{sc}}(\{\tilde{p}\}, \tilde{k}_1, \dots, \tilde{k}_{n+1}). \quad (\text{D.2})$$

Substituting into eq. (D.1) we have

$$\begin{aligned} \mathcal{F}_{\text{NLL}}(\lambda) &= \epsilon^{R'_{\text{NLL}}} \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{i=2}^{n+1} \sum_{\ell_i=1,2} R'_{\text{NLL},\ell_i} \int_{\epsilon}^1 \frac{d\tilde{\zeta}_i}{\tilde{\zeta}_i} \int_0^{2\pi} \frac{d\phi_i}{2\pi} \times \\ &\times \sum_{\ell_1=1,2} R'_{\text{NLL},\ell_1} \int_0^{\infty} \frac{d\zeta_1}{\zeta_1} \zeta_1^{R'_{\text{NLL}}} \int_0^{2\pi} \frac{d\phi_1}{2\pi} \Theta \left(1 - \zeta_1 \lim_{v \rightarrow 0} \frac{V_{\text{sc}}(\{\tilde{p}\}, \tilde{k}_1, \dots, \tilde{k}_{n+1})}{v} \right). \end{aligned} \quad (\text{D.3})$$

The ζ_1 integration can be trivially performed to get

$$\begin{aligned} \mathcal{F}_{\text{NLL}}(\lambda) &= \sum_{\ell_1=1,2} \frac{R'_{\text{NLL},\ell_1}}{R'_{\text{NLL}}} \int_0^{2\pi} \frac{d\phi_1}{2\pi} \times \\ &\times \epsilon^{R'_{\text{NLL}}} \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{i=2}^{n+1} \sum_{\ell_i=1,2} R'_{\text{NLL},\ell_i} \int_{\epsilon}^1 \frac{d\tilde{\zeta}_i}{\tilde{\zeta}_i} \int_0^{2\pi} \frac{d\phi_i}{2\pi} \exp \left(-R'_{\text{NLL}} \ln \lim_{v \rightarrow 0} \frac{V_{\text{sc}}(\{\tilde{p}\}, \tilde{k}_1, \dots, \tilde{k}_{n+1})}{v} \right). \end{aligned} \quad (\text{D.4})$$

D.2 The function $\delta\mathcal{F}_{\text{sc}}$

We now extend the procedure devised in section D.1 so as to be able to efficiently compute $\delta\mathcal{F}_{\text{sc}}$ with a Monte Carlo procedure. First we observe that without any secondary emission there is no contribution to $\delta\mathcal{F}_{\text{sc}}$. We isolate the hardest emission k_1 among k_1, \dots, k_n .

We first consider the case in which the special emission is not the hardest of all, i.e. $\zeta < \zeta_1$. This gives

$$\begin{aligned} \delta\mathcal{F}_{\text{sc}}^{\leq} &= \frac{\pi}{\alpha_s(Q)} \sum_{\ell_1=1,2} R'_{\text{NLL},\ell_1} \int_0^{\infty} \frac{d\zeta_1}{\zeta_1} \zeta_1^{R'_{\text{NLL}}} \int_0^{2\pi} \frac{d\phi_1}{2\pi} \int_0^{\zeta_1} \frac{d\zeta}{\zeta} \int_0^{2\pi} \frac{d\phi}{2\pi} \sum_{\ell=1,2} \left(\delta R'_{\text{NNLL},\ell} + R''_{\ell} \ln \left(\frac{d_{\ell} g_{\ell}(\phi)}{\zeta} \right) \right) \times \\ &\times \left[\epsilon^{R'_{\text{NLL}}} \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{i=2}^{n+1} \sum_{\ell_i=1,2} R'_{\text{NLL},\ell_i} \int_{\epsilon\zeta_1}^{\zeta_1} \frac{d\zeta_i}{\zeta_i} \int_0^{2\pi} \frac{d\phi_i}{2\pi} \right] \times \\ &\times \left[\Theta \left(1 - \lim_{v \rightarrow 0} \frac{V_{\text{sc}}(\{\tilde{p}\}, k, k_1, \dots, k_{n+1})}{v} \right) - \Theta(1 - \zeta) \Theta \left(1 - \lim_{v \rightarrow 0} \frac{V_{\text{sc}}(\{\tilde{p}\}, k_1, \dots, k_{n+1})}{v} \right) \right]. \end{aligned} \quad (\text{D.5})$$

We now rescale all momenta as in section D.1, and obtain

$$\begin{aligned} \delta\mathcal{F}_{\text{sc}}^{\leq} &= \frac{\pi}{\alpha_s(Q)} \sum_{\ell_1=1,2} R'_{\text{NLL},\ell_1} \int_{\epsilon}^{\infty} \frac{d\zeta_1}{\zeta_1} \zeta_1^{R'_{\text{NLL}}} \int_0^{2\pi} \frac{d\phi_1}{2\pi} \int_0^1 \frac{d\tilde{\zeta}}{\tilde{\zeta}} \int_0^{2\pi} \frac{d\phi}{2\pi} \sum_{\ell=1,2} \left(\delta R'_{\text{NNLL},\ell} + R''_{\ell} \ln \left(\frac{d_{\ell} g_{\ell}(\phi)}{\tilde{\zeta} \zeta_1} \right) \right) \times \\ &\times \left[\epsilon^{R'_{\text{NLL}}} \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{i=2}^{n+1} \sum_{\ell_i=1,2} R'_{\text{NLL},\ell_i} \int_{\epsilon}^1 \frac{d\tilde{\zeta}_i}{\tilde{\zeta}_i} \int_0^{2\pi} \frac{d\phi_i}{2\pi} \right] \times \\ &\times \left[\Theta \left(1 - \zeta_1 \lim_{v \rightarrow 0} \frac{V_{\text{sc}}(\{\tilde{p}\}, \tilde{k}, \tilde{k}_1, \dots, \tilde{k}_{n+1})}{v} \right) - \Theta(1 - \zeta_1 \tilde{\zeta}) \Theta \left(1 - \zeta_1 \lim_{v \rightarrow 0} \frac{V_{\text{sc}}(\{\tilde{p}\}, \tilde{k}_1, \dots, \tilde{k}_{n+1})}{v} \right) \right]. \end{aligned} \quad (\text{D.6})$$

Performing the ζ_1 integration we get

$$\begin{aligned} \delta\mathcal{F}_{\text{sc}}^{\leq} &= \frac{\pi}{\alpha_s(Q)} \sum_{\ell_1=1,2} \frac{R'_{\text{NLL},\ell_1}}{R'_{\text{NLL}}} \int_0^{2\pi} \frac{d\phi_1}{2\pi} \int_0^1 \frac{d\tilde{\zeta}}{\tilde{\zeta}} \int_0^{2\pi} \frac{d\phi}{2\pi} \times \left[\epsilon^{R'_{\text{NLL}}} \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{i=2}^{n+1} \sum_{\ell_i=1,2} R'_{\text{NLL},\ell_i} \int_{\epsilon}^1 \frac{d\tilde{\zeta}_i}{\tilde{\zeta}_i} \int_0^{2\pi} \frac{d\phi_i}{2\pi} \right] \times \\ &\times \left[\sum_{\ell=1,2} \left(\left(\delta R'_{\text{NNLL},\ell} + R''_{\ell} \left(\frac{1}{R'_{\text{NLL}}} + \ln \left(\frac{d_{\ell} g_{\ell}(\phi)}{\tilde{\zeta}} \right) \right) + \ln \lim_{v \rightarrow 0} \frac{V_{\text{sc}}(\{\tilde{p}\}, \tilde{k}, \tilde{k}_1, \dots, \tilde{k}_{n+1})}{v} \right) \right) \times \right. \\ &\times \exp \left(-R'_{\text{NLL}} \ln \lim_{v \rightarrow 0} \frac{V_{\text{sc}}(\{\tilde{p}\}, \tilde{k}, \tilde{k}_1, \dots, \tilde{k}_{n+1})}{v} \right) \\ &\left. - \left(\delta R'_{\text{NNLL},\ell} + R''_{\ell} \left(\frac{1}{R'_{\text{NLL}}} + \ln \left(\frac{d_{\ell} g_{\ell}(\phi)}{\tilde{\zeta}} \right) \right) + \ln \max \left[\tilde{\zeta}, \lim_{v \rightarrow 0} \frac{V_{\text{sc}}(\{\tilde{p}\}, \tilde{k}_1, \dots, \tilde{k}_{n+1})}{v} \right] \right) \right) \times \end{aligned}$$

$$\times \exp \left(-R'_{\text{NLL}} \ln \max \left[\tilde{\zeta}, \lim_{v \rightarrow 0} \frac{V_{\text{sc}}(\{\tilde{p}\}, \tilde{k}_1, \dots, \tilde{k}_{n+1})}{v} \right] \right) \Bigg] \Bigg] . \quad (\text{D.7})$$

The second contribution arises when the special emission is the hardest of all, i.e. $\zeta > \zeta_1$. This gives

$$\begin{aligned} \delta \mathcal{F}_{\text{sc}}^> &= \frac{\pi}{\alpha_s(Q)} \int_0^\infty \frac{d\zeta}{\zeta} \zeta^{R'_{\text{NLL}}} \int_0^{2\pi} \frac{d\phi}{2\pi} \sum_{\ell=1,2} \left(\delta R'_{\text{NNLL},\ell} + R''_\ell \ln \left(\frac{d_\ell g_\ell(\phi)}{\zeta} \right) \right) \times \\ &\times \left[\epsilon^{R'_{\text{NLL}}} \sum_{n=0}^\infty \frac{1}{n!} \prod_{i=1}^n \sum_{\ell_i=1,2} R'_{\text{NLL},\ell_i} \int_{\epsilon\zeta} \frac{d\zeta_i}{\zeta_i} \int_0^{2\pi} \frac{d\phi_i}{2\pi} \right] \times \\ &\times \left[\Theta \left(1 - \lim_{v \rightarrow 0} \frac{V_{\text{sc}}(\{\tilde{p}\}, k, k_1, \dots, k_n)}{v} \right) - \Theta(1 - \zeta) \Theta \left(1 - \lim_{v \rightarrow 0} \frac{V_{\text{sc}}(\{\tilde{p}\}, k_1, \dots, k_n)}{v} \right) \right] . \end{aligned} \quad (\text{D.8})$$

Defining now $\tilde{\zeta}_i = \zeta_i/\zeta$ we obtain

$$\begin{aligned} \delta \mathcal{F}_{\text{sc}}^> &= \frac{\pi}{\alpha_s(Q)} \int_0^\infty \frac{d\zeta}{\zeta} \zeta^{R'_{\text{NLL}}} \int_0^{2\pi} \frac{d\phi}{2\pi} \sum_{\ell=1,2} \left(\delta R'_{\text{NNLL},\ell} + R''_\ell \ln \left(\frac{d_\ell g_\ell(\phi)}{\zeta} \right) \right) \times \\ &\times \left[\epsilon^{R'_{\text{NLL}}} \sum_{n=0}^\infty \frac{1}{n!} \prod_{i=1}^n \sum_{\ell_i=1,2} R'_{\text{NLL},\ell_i} \int_\epsilon^1 \frac{d\tilde{\zeta}_i}{\tilde{\zeta}_i} \int_0^{2\pi} \frac{d\phi_i}{2\pi} \right] \times \\ &\times \left[\Theta \left(1 - \zeta \lim_{v \rightarrow 0} \frac{V_{\text{sc}}(\{\tilde{p}\}, \tilde{k}, \tilde{k}_1, \dots, \tilde{k}_n)}{v} \right) - \Theta(1 - \zeta) \Theta \left(1 - \zeta \lim_{v \rightarrow 0} \frac{V_{\text{sc}}(\{\tilde{p}\}, \tilde{k}_1, \dots, \tilde{k}_n)}{v} \right) \right] . \end{aligned} \quad (\text{D.9})$$

We can now perform the integration over ζ to obtain

$$\begin{aligned} \delta \mathcal{F}_{\text{sc}}^> &= \frac{\pi}{\alpha_s(Q)} \frac{1}{R'_{\text{NLL}}} \int_0^{2\pi} \frac{d\phi}{2\pi} \times \left[\epsilon^{R'_{\text{NLL}}} \sum_{n=0}^\infty \frac{1}{n!} \prod_{i=1}^n \sum_{\ell_i=1,2} R'_{\text{NLL},\ell_i} \int_\epsilon^1 \frac{d\tilde{\zeta}_i}{\tilde{\zeta}_i} \int_0^{2\pi} \frac{d\phi_i}{2\pi} \right] \times \\ &\times \left[\sum_{\ell=1,2} \left(\left(\delta R'_{\text{NNLL},\ell} + R''_\ell \left(\frac{1}{R'_{\text{NLL}}} + \ln(d_\ell g_\ell(\phi)) + \ln \lim_{v \rightarrow 0} \frac{V_{\text{sc}}(\{\tilde{p}\}, \tilde{k}, \tilde{k}_1, \dots, \tilde{k}_{n+1})}{v} \right) \right) \right) \times \right. \\ &\times \exp \left(-R'_{\text{NLL}} \ln \lim_{v \rightarrow 0} \frac{V_{\text{sc}}(\{\tilde{p}\}, \tilde{k}, \tilde{k}_1, \dots, \tilde{k}_{n+1})}{v} \right) \\ &- \left. \left(\delta R'_{\text{NNLL},\ell} + R''_\ell \left(\frac{1}{R'_{\text{NLL}}} + \ln(d_\ell g_\ell(\phi)) + \ln \max \left[1, \lim_{v \rightarrow 0} \frac{V_{\text{sc}}(\{\tilde{p}\}, \tilde{k}_1, \dots, \tilde{k}_{n+1})}{v} \right] \right) \right) \right) \times \\ &\times \exp \left(-R'_{\text{NLL}} \ln \max \left[1, \lim_{v \rightarrow 0} \frac{V_{\text{sc}}(\{\tilde{p}\}, \tilde{k}_1, \dots, \tilde{k}_{n+1})}{v} \right] \right) \Bigg] . \end{aligned} \quad (\text{D.10})$$

Of course, the NNLL correction $\delta \mathcal{F}_{\text{sc}} = \delta \mathcal{F}_{\text{sc}}^< + \delta \mathcal{F}_{\text{sc}}^>$.

D.3 The function $\delta \mathcal{F}_{\text{hc}}$

We start from eq. (3.32), and select k_1 , the emission with the largest value among the ζ_i .

We consider first the case $\zeta < \zeta_1$. This gives

$$\delta \mathcal{F}_{\text{hc}}^< = \sum_{\ell_1=1,2} R'_{\text{NLL},\ell_1} \int_0^\infty \frac{d\zeta_1}{\zeta_1} \zeta_1^{R'_{\text{NLL}}} \int_0^{2\pi} \frac{d\phi_1}{2\pi} \int_0^{\zeta_1} \frac{d\zeta}{\zeta} \int_0^{2\pi} \frac{d\phi}{2\pi} \sum_{\ell=1,2} \frac{\alpha_s(v^{a+b_\ell} Q)}{\alpha_s(Q)(a+b_\ell)} \int_0^1 \frac{dz}{z} (z p_\ell(z) - 2C_\ell) \times$$

$$\begin{aligned}
 & \times \left[\epsilon^{R'_{\text{NLL}}} \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{i=2}^{n+1} \sum_{\ell_i=1,2} R'_{\text{NLL},\ell_i} \int_{\epsilon\zeta_1}^{\zeta_1} \frac{d\zeta_i}{\zeta_i} \int_0^{2\pi} \frac{d\phi_i}{2\pi} \right] \times \\
 & \times \left[\Theta \left(1 - \lim_{v \rightarrow 0} \frac{V_{\text{sc}}(\{\tilde{p}\}, k, k_1, \dots, k_{n+1})}{v} \right) - \Theta(1-\zeta) \Theta \left(1 - \lim_{v \rightarrow 0} \frac{V_{\text{sc}}(\{\tilde{p}\}, k_1, \dots, k_{n+1})}{v} \right) \right]. \quad (\text{D.11})
 \end{aligned}$$

We now define $\tilde{\zeta} = \zeta/\zeta_1$ and $\tilde{\zeta}_i = \zeta_i/\zeta_1$. Using the rIRC safety properties of the observable we get

$$\begin{aligned}
 \delta\mathcal{F}_{\text{hc}}^< &= \sum_{\ell_1=1,2} R'_{\text{NLL},\ell_1} \int_0^{\infty} \frac{d\zeta_1}{\zeta_1} \zeta_1^{R'_{\text{NLL}}} \int_0^{2\pi} \frac{d\phi_1}{2\pi} \int_0^1 \frac{d\tilde{\zeta}}{\tilde{\zeta}} \int_0^{2\pi} \frac{d\phi}{2\pi} \sum_{\ell=1,2} \frac{\alpha_s(v^{\frac{1}{a+b_\ell}} Q)}{\alpha_s(Q)(a+b_\ell)} \int_0^1 \frac{dz}{z} (z p_\ell(z) - 2C_\ell) \times \\
 & \times \left[\epsilon^{R'_{\text{NLL}}} \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{i=2}^{n+1} \sum_{\ell_i=1,2} R'_{\text{NLL},\ell_i} \int_\epsilon^1 \frac{d\tilde{\zeta}_i}{\tilde{\zeta}_i} \int_0^{2\pi} \frac{d\phi_i}{2\pi} \right] \times \quad (\text{D.12}) \\
 & \times \left[\Theta \left(1 - \zeta_1 \lim_{v \rightarrow 0} \frac{V_{\text{sc}}(\{\tilde{p}\}, \tilde{k}, \tilde{k}_1, \dots, \tilde{k}_{n+1})}{v} \right) - \Theta(1-\zeta_1\tilde{\zeta}) \Theta \left(1 - \zeta_1 \lim_{v \rightarrow 0} \frac{V_{\text{sc}}(\{\tilde{p}\}, \tilde{k}_1, \dots, \tilde{k}_{n+1})}{v} \right) \right].
 \end{aligned}$$

This allows us to perform the integration with respect to ζ_1 and obtain

$$\begin{aligned}
 \delta\mathcal{F}_{\text{hc}}^< &= \sum_{\ell_1=1,2} \frac{R'_{\text{NLL},\ell_1}}{R'_{\text{NLL}}} \int_0^{2\pi} \frac{d\phi_1}{2\pi} \int_0^1 \frac{d\tilde{\zeta}}{\tilde{\zeta}} \int_0^{2\pi} \frac{d\phi}{2\pi} \sum_{\ell=1,2} \frac{\alpha_s(v^{\frac{1}{a+b_\ell}} Q)}{\alpha_s(Q)(a+b_\ell)} \int_0^1 \frac{dz}{z} (z p_\ell(z) - 2C_\ell) \times \\
 & \times \left[\epsilon^{R'_{\text{NLL}}} \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{i=2}^{n+1} \sum_{\ell_i=1,2} R'_{\text{NLL},\ell_i} \int_\epsilon^1 \frac{d\tilde{\zeta}_i}{\tilde{\zeta}_i} \int_0^{2\pi} \frac{d\phi_i}{2\pi} \right] \times \\
 & \times \left[\exp \left(-R'_{\text{NLL}} \ln \lim_{v \rightarrow 0} \frac{V_{\text{sc}}(\{\tilde{p}\}, \tilde{k}, \tilde{k}_1, \dots, \tilde{k}_{n+1})}{v} \right) \right. \\
 & \left. - \exp \left(-R'_{\text{NLL}} \ln \max \left[\tilde{\zeta}, \lim_{v \rightarrow 0} \frac{V_{\text{sc}}(\{\tilde{p}\}, \tilde{k}_1, \dots, \tilde{k}_{n+1})}{v} \right] \right) \right]. \quad (\text{D.13})
 \end{aligned}$$

We next consider the case $\zeta > \zeta_1$. This gives

$$\begin{aligned}
 \delta\mathcal{F}_{\text{hc}}^> &= \int_0^\infty \frac{d\zeta}{\zeta} \zeta^{R'_{\text{NLL}}} \int_0^{2\pi} \frac{d\phi}{2\pi} \sum_{\ell=1,2} \frac{\alpha_s(v^{\frac{1}{a+b_\ell}} Q)}{\alpha_s(Q)(a+b_\ell)} \int_0^1 \frac{dz}{z} (z p_\ell(z) - 2C_\ell) \times \\
 & \times \left[\epsilon^{R'_{\text{NLL}}} \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{i=1}^n \sum_{\ell_i=1,2} R'_{\text{NLL},\ell_i} \int_{\epsilon\zeta}^\zeta \frac{d\zeta_i}{\zeta_i} \int_0^{2\pi} \frac{d\phi_i}{2\pi} \right] \times \quad (\text{D.14}) \\
 & \times \left[\Theta \left(1 - \lim_{v \rightarrow 0} \frac{V_{\text{sc}}(\{\tilde{p}\}, k, k_1, \dots, k_n)}{v} \right) - \Theta(1-\zeta) \Theta \left(1 - \lim_{v \rightarrow 0} \frac{V_{\text{sc}}(\{\tilde{p}\}, k_1, \dots, k_n)}{v} \right) \right].
 \end{aligned}$$

Defining $\tilde{\zeta}_i = \zeta_i/\zeta$ and exploiting the rIRC safety properties of the observable, we find

$$\begin{aligned}
 \delta\mathcal{F}_{\text{hc}}^> &= \int_0^\infty \frac{d\zeta}{\zeta} \zeta^{R'_{\text{NLL}}} \int_0^{2\pi} \frac{d\phi}{2\pi} \sum_{\ell=1,2} \frac{\alpha_s(v^{\frac{1}{a+b_\ell}} Q)}{\alpha_s(Q)(a+b_\ell)} \int_0^1 \frac{dz}{z} (z p_\ell(z) - 2C_\ell) \times \\
 & \times \left[\epsilon^{R'_{\text{NLL}}} \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{i=1}^n \sum_{\ell_i=1,2} R'_{\text{NLL},\ell_i} \int_\epsilon^1 \frac{d\tilde{\zeta}_i}{\tilde{\zeta}_i} \int_0^{2\pi} \frac{d\phi_i}{2\pi} \right] \times \quad (\text{D.15}) \\
 & \times \left[\Theta \left(1 - \zeta \lim_{v \rightarrow 0} \frac{V_{\text{sc}}(\{\tilde{p}\}, \tilde{k}, \tilde{k}_1, \dots, \tilde{k}_n)}{v} \right) - \Theta(1-\zeta) \Theta \left(1 - \zeta \lim_{v \rightarrow 0} \frac{V_{\text{sc}}(\{\tilde{p}\}, \tilde{k}_1, \dots, \tilde{k}_n)}{v} \right) \right].
 \end{aligned}$$

This allows us to perform the integration with respect to ζ , to obtain

$$\begin{aligned}
 \delta\mathcal{F}_{\text{hc}}^{\gt} &= \frac{1}{R'_{\text{NLL}}} \int_0^{2\pi} \frac{d\phi}{2\pi} \sum_{\ell=1,2} \frac{\alpha_s(v^{\frac{1}{a+b_\ell}} Q)}{\alpha_s(Q)(a+b_\ell)} \int_0^1 \frac{dz}{z} (z p_\ell(z) - 2C_\ell) \times \\
 &\times \left[\epsilon^{R'_{\text{NLL}}} \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{i=1}^n \sum_{\ell_i=1,2} R'_{\text{NLL},\ell_i} \int_\epsilon^1 \frac{d\tilde{\zeta}_i}{\tilde{\zeta}_i} \int_0^{2\pi} \frac{d\phi_i}{2\pi} \right] \times \\
 &\times \left[\exp \left(-R'_{\text{NLL}} \ln \lim_{v \rightarrow 0} \frac{V_{\text{sc}}(\{\tilde{p}\}, \tilde{k}, \tilde{k}_1, \dots, \tilde{k}_n)}{v} \right) \right. \\
 &\left. - \exp \left(-R'_{\text{NLL}} \ln \max \left[1, \lim_{v \rightarrow 0} \frac{V_{\text{sc}}(\{\tilde{p}\}, \tilde{k}_1, \dots, \tilde{k}_n)}{v} \right] \right) \right].
 \end{aligned} \tag{D.16}$$

D.4 The function $\delta\mathcal{F}_{\text{rec}}$

We start from eq. (3.28), and again pick up k_1 , the emission with the largest value among the ζ_i . For $\zeta < \zeta_1$ we have

$$\begin{aligned}
 \delta\mathcal{F}_{\text{rec}}^{\lt} &= \sum_{\ell_1=1,2} R'_{\text{NLL},\ell_1} \int_0^\infty \frac{d\zeta_1}{\zeta_1} \zeta_1^{R'_{\text{NLL}}} \int_0^{2\pi} \frac{d\phi_1}{2\pi} \sum_{\ell=1,2} \frac{\alpha_s(v^{1/(a+b_\ell)} Q)}{\alpha_s(Q)(a+b_\ell)} \int_0^1 dz p_\ell(z) \int_0^{\zeta_1} \frac{d\zeta}{\zeta} \int_0^{2\pi} \frac{d\phi}{2\pi} \times \\
 &\times \left[\epsilon^{R'_{\text{NLL}}} \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{i=2}^{n+1} \sum_{\ell_i=1,2} R'_{\text{NLL},\ell_i} \int_{\epsilon\zeta_1}^{\zeta_1} \frac{d\zeta_i}{\zeta_i} \int_0^{2\pi} \frac{d\phi_i}{2\pi} \right] \times \\
 &\times \left[\Theta \left(1 - \lim_{v \rightarrow 0} \frac{V_{\text{hc}}^{(k')}(\{\tilde{p}'\}, k', k_1, \dots, k_{n+1})}{v} \right) - \Theta \left(1 - \lim_{v \rightarrow 0} \frac{V_{\text{sc}}(\{\tilde{p}\}, k, k_1, \dots, k_{n+1})}{v} \right) \right].
 \end{aligned} \tag{D.17}$$

As usual, defining $\tilde{\zeta}_i = \zeta_i/\zeta_1$, and integrating over ζ_1 , we get

$$\begin{aligned}
 \delta\mathcal{F}_{\text{rec}}^{\lt} &= \sum_{\ell_1=1,2} \frac{R'_{\text{NLL},\ell_1}}{R'_{\text{NLL}}} \int_0^{2\pi} \frac{d\phi_1}{2\pi} \sum_{\ell=1,2} \frac{\alpha_s(v^{1/(a+b_\ell)} Q)}{\alpha_s(Q)(a+b_\ell)} \int_0^1 dz p_\ell(z) \int_0^1 \frac{d\tilde{\zeta}}{\tilde{\zeta}} \int_0^{2\pi} \frac{d\phi}{2\pi} \times \\
 &\times \left[\epsilon^{R'_{\text{NLL}}} \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{i=2}^{n+1} \sum_{\ell_i=1,2} R'_{\text{NLL},\ell_i} \int_\epsilon^1 \frac{d\tilde{\zeta}_i}{\tilde{\zeta}_i} \int_0^{2\pi} \frac{d\phi_i}{2\pi} \right] \times \\
 &\times \left[\exp \left(-R'_{\text{NLL}} \ln \lim_{v \rightarrow 0} \frac{V_{\text{hc}}^{(k')}(\{\tilde{p}'\}, \tilde{k}', \tilde{k}_1, \dots, \tilde{k}_{n+1})}{v} \right) \right. \\
 &\left. - \exp \left(-R'_{\text{NLL}} \ln \lim_{v \rightarrow 0} \frac{V_{\text{sc}}(\{\tilde{p}\}, \tilde{k}, \tilde{k}_1, \dots, \tilde{k}_{n+1})}{v} \right) \right].
 \end{aligned} \tag{D.18}$$

Similarly, for $\zeta > \zeta_1$ we define $\tilde{\zeta}_i = \zeta_i/\zeta$ and integrate over ζ , thus obtaining

$$\begin{aligned} \delta\mathcal{F}_{\text{rec}}^> &= \frac{1}{R'_{\text{NLL}}} \sum_{\ell=1,2} \frac{\alpha_s(v^{1/(a+b_\ell)}Q)}{\alpha_s(Q)(a+b_\ell)} \int_0^1 dz p_\ell(z) \int_0^{2\pi} \frac{d\phi}{2\pi} \times \\ &\times \left[\epsilon^{R'_{\text{NLL}}} \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{i=1}^n \sum_{\ell_i=1,2} R'_{\text{NLL},\ell_i} \int_\epsilon^1 \frac{d\tilde{\zeta}_i}{\tilde{\zeta}_i} \int_0^{2\pi} \frac{d\phi_i}{2\pi} \right] \times \\ &\times \left[\exp\left(-R'_{\text{NLL}} \ln \lim_{v \rightarrow 0} \frac{V_{\text{hc}}^{(k')}(\{\tilde{p}'\}, \tilde{k}', \tilde{k}_1, \dots, \tilde{k}_n)}{v}\right) \right. \\ &\left. - \exp\left(-R'_{\text{NLL}} \ln \lim_{v \rightarrow 0} \frac{V_{\text{sc}}(\{\tilde{p}\}, \tilde{k}, \tilde{k}_1, \dots, \tilde{k}_n)}{v}\right) \right]. \end{aligned} \quad (\text{D.19})$$

D.5 The function $\delta\mathcal{F}_{\text{wa}}$

We start from eq. (3.38), and repeat the above procedure obtaining the two contributions

$$\begin{aligned} \delta\mathcal{F}_{\text{wa}}^< &= \sum_{\ell_1=1,2} \frac{R'_{\text{NLL},\ell_1}}{R'_{\text{NLL}}} \int_0^{2\pi} \frac{d\phi_1}{2\pi} \int_0^1 \frac{d\tilde{\zeta}}{\tilde{\zeta}} \int_0^{2\pi} \frac{d\phi}{2\pi} \frac{2C_F}{a} \frac{\alpha_s(v^{1/a}Q)}{\alpha_s(Q)} \int_{-\infty}^{\infty} d\eta \\ &\times \left[\epsilon^{R'_{\text{NLL}}} \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{i=2}^{n+1} \sum_{\ell_i=1,2} R'_{\text{NLL},\ell_i} \int_\epsilon^1 \frac{d\tilde{\zeta}_i}{\tilde{\zeta}_i} \int_0^{2\pi} \frac{d\phi_i}{2\pi} \right] \\ &\times \left[\exp\left(-R_{\text{NLL}} \ln \lim_{v \rightarrow 0} \frac{V_{\text{wa}}^{(k)}(\{\tilde{p}\}, \tilde{k}, \tilde{k}_1, \dots, \tilde{k}_{n+1})}{v}\right) \right. \\ &\left. - \exp\left(-R'_{\text{NLL}} \ln \lim_{v \rightarrow 0} \frac{V_{\text{sc}}(\{\tilde{p}\}, \tilde{k}, \tilde{k}_1, \dots, \tilde{k}_{n+1})}{v}\right) \right], \end{aligned} \quad (\text{D.20})$$

and

$$\begin{aligned} \delta\mathcal{F}_{\text{wa}}^> &= \frac{1}{R'_{\text{NLL}}} \int_0^{2\pi} \frac{d\phi}{2\pi} \frac{2C_F}{a} \frac{\alpha_s(v^{1/a}Q)}{\alpha_s(Q)} \int_{-\infty}^{\infty} d\eta \\ &\times \left[\epsilon^{R'_{\text{NLL}}} \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{i=1}^n \sum_{\ell_i=1,2} R'_{\text{NLL},\ell_i} \int_\epsilon^1 \frac{d\tilde{\zeta}_i}{\tilde{\zeta}_i} \int_0^{2\pi} \frac{d\phi_i}{2\pi} \right] \\ &\times \left[\exp\left(-R_{\text{NLL}} \ln \lim_{v \rightarrow 0} \frac{V_{\text{wa}}^{(k)}(\{\tilde{p}\}, \tilde{k}, \tilde{k}_1, \dots, \tilde{k}_n)}{v}\right) \right. \\ &\left. - \exp\left(-R'_{\text{NLL}} \ln \lim_{v \rightarrow 0} \frac{V_{\text{sc}}(\{\tilde{p}\}, \tilde{k}, \tilde{k}_1, \dots, \tilde{k}_n)}{v}\right) \right]. \end{aligned} \quad (\text{D.21})$$

D.6 The function $\delta\mathcal{F}_{\text{correl}}$

We start from eq. (3.47), and pick up k_1 , the emission with the largest value among the ζ_i . We also restrict κ to be less than one, getting rid of the factor $1/2!$ in front of $C_{ab}(\kappa, \eta, \phi)$.

We consider first the case $\zeta_a < \zeta_1$. This gives

$$\delta\mathcal{F}_{\text{correl}}^< = \int_0^{\infty} \frac{d\zeta_1}{\zeta_1} \zeta_1^{R'_{\text{NLL}}} \int_0^{2\pi} \frac{d\phi_1}{2\pi} \sum_{\ell_1=1,2} R'_{\text{NLL},\ell_1} \int_0^{\zeta_1} \frac{d\zeta_a}{\zeta_a} \int_0^{2\pi} \frac{d\phi_a}{2\pi} \sum_{\ell_a=1,2} \frac{2C_{\ell_a} \lambda R''_{\ell_a}(v)}{a\beta_0 \alpha_s(Q)} \times$$

$$\begin{aligned}
 & \times \int_0^1 \frac{d\kappa}{\kappa} \int_{-\infty}^{\infty} d\eta \int_0^{2\pi} \frac{d\phi}{2\pi} C_{ab}(\kappa, \eta, \phi) \times \\
 & \times \left[\epsilon^{R'_{\text{NLL}}} \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{i=2}^{n+1} \sum_{\ell_i=1,2} R'_{\text{NLL}, \ell_i} \int_{\epsilon \zeta_1}^{\zeta_1} \frac{d\zeta_i}{\zeta_i} \int_0^{2\pi} \frac{d\phi_i}{2\pi} \right] \times \\
 & \times [\Theta(v - V_{\text{sc}}(\{\tilde{p}\}, k_a, k_b, k_1, \dots, k_{n+1})) - (v - V_{\text{sc}}(\{\tilde{p}\}, k_a + k_b, k_1, \dots, k_{n+1}))].
 \end{aligned} \tag{D.22}$$

We now define $\tilde{\zeta}_a = \zeta_a/\zeta_1$, and $\tilde{\zeta}_i = \zeta_i/\zeta_1$, and correspondingly we define the rescaled momenta \tilde{k}_a, \tilde{k}_b and \tilde{k}_i . Notice that κ, η and ϕ stay unchanged in the rescaling process. Integrating over ζ_1 we get

$$\begin{aligned}
 \delta \mathcal{F}_{\text{correl}}^{\leq} &= \int_0^{2\pi} \frac{d\phi_1}{2\pi} \sum_{\ell_1=1,2} \frac{R'_{\text{NLL}, \ell_1}}{R'_{\text{NLL}}} \int_0^1 \frac{d\tilde{\zeta}_a}{\tilde{\zeta}_a} \int_0^{2\pi} \frac{d\phi_a}{2\pi} \sum_{\ell_a=1,2} \frac{2C_{\ell_a} \lambda R''_{\ell_a}(v)}{a\beta_0 \alpha_s(Q)} \times \\
 & \times \int_0^1 \frac{d\kappa}{\kappa} \int_{-\infty}^{\infty} d\eta \int_0^{2\pi} \frac{d\phi}{2\pi} C_{ab}(\zeta, \eta, \phi) \times \\
 & \times \left[\epsilon^{R'_{\text{NLL}}} \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{i=2}^{n+1} \sum_{\ell_i=1,2} R'_{\ell_i} \int_{\epsilon}^1 \frac{d\tilde{\zeta}_i}{\tilde{\zeta}_i} \int_0^{2\pi} \frac{d\phi_i}{2\pi} \right] \times \\
 & \times \left[\exp\left(-R'_{\text{NLL}} \ln \lim_{v \rightarrow 0} \frac{V_{\text{sc}}(\{\tilde{p}\}, \tilde{k}_a, \tilde{k}_b, \tilde{k}_1, \dots, \tilde{k}_{n+1})}{v}\right) \right. \\
 & \left. - \exp\left(-R'_{\text{NLL}} \ln \lim_{v \rightarrow 0} \frac{V_{\text{sc}}(\{\tilde{p}\}, \tilde{k}_a + \tilde{k}_b, \tilde{k}_1, \dots, \tilde{k}_{n+1})}{v}\right) \right].
 \end{aligned} \tag{D.23}$$

Similarly, for $\zeta_a > \zeta_1$ we define $\tilde{\zeta}_i = \zeta_i/\zeta_a$, and integrate over ζ_a to obtain

$$\begin{aligned}
 \delta \mathcal{F}_{\text{correl}}^{\geq} &= \frac{1}{R'_{\text{NLL}}} \sum_{\ell_a=1,2} \frac{2C_{\ell_a} \lambda R''_{\ell_a}}{a\beta_0 \alpha_s(Q)} \int_0^{2\pi} \frac{d\phi_a}{2\pi} \int_0^1 \frac{d\kappa}{\kappa} \int_{-\infty}^{\infty} d\eta \int_0^{2\pi} \frac{d\phi}{2\pi} C_{ab}(\zeta, \eta, \phi) \times \\
 & \times \left[\epsilon^{R'} \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{i=1}^n \sum_{\ell_i=1,2} R'_{\text{NLL}, \ell_i} \int_{\epsilon}^1 \frac{d\tilde{\zeta}_i}{\tilde{\zeta}_i} \int_0^{2\pi} \frac{d\phi_i}{2\pi} \right] \times \\
 & \times \left[\exp\left(-R'_{\text{NLL}} \ln \lim_{v \rightarrow 0} \frac{V_{\text{sc}}(\{\tilde{p}\}, \tilde{k}_a, \tilde{k}_b, \tilde{k}_1, \dots, \tilde{k}_n)}{v}\right) \right. \\
 & \left. - \exp\left(-R'_{\text{NLL}} \ln \lim_{v \rightarrow 0} \frac{V_{\text{sc}}(\{\tilde{p}\}, \tilde{k}_a + \tilde{k}_b, \tilde{k}_1, \dots, \tilde{k}_n)}{v}\right) \right].
 \end{aligned} \tag{D.24}$$

E Expansion to $\mathcal{O}(\alpha_s^3)$

To conclude, we give the numerical expansion of the multiple emissions function for the observables analysed in the article. We recall the form of the resummed cross section

$$\Sigma(v) = e^{Lg_1(\lambda) + g_2(\lambda) + \frac{\alpha_s(Q)}{\pi} g_3(\lambda)} \left[\mathcal{F}_{\text{NLL}}(\lambda) + \frac{\alpha_s(Q)}{\pi} \delta \mathcal{F}_{\text{NNLL}}(\lambda) \right], \tag{E.1}$$

	T	C	ρ_H	B_T	B_W	T_M	O
\mathcal{F}_{22}	-23.394(6)	-23.394(6)	-11.697(4)	-74.121(6)	-27.332(7)	-53.287(7)	42.975(9)
\mathcal{F}_{33}	-208.252(3)	-208.252(3)	-119.324(2)	-724.49(2)	-371.76(2)	-563.24(7)	513.96(8)
\mathcal{F}_{10}	-5.4396	-1.0532	-5.4396	0	0	0	0
\mathcal{F}_{21}	-19.951(7)	-70.157(1)	-20.401(9)	61.45(2)	59.65(2)	-10.080(9)	80.79(5)
\mathcal{F}_{32}	-463.51(6)	-1427.72(5)	-247.79(4)	-717.1(1)	335.8(9)	-1287.0(8)	-79.(5)

Table 1. Expansion coefficients for the multiple emissions function at NLL (\mathcal{F}_{22} , \mathcal{F}_{33}), and NNLL (\mathcal{F}_{10} , \mathcal{F}_{21} , and \mathcal{F}_{32}) up to $\mathcal{O}(\alpha_s^3)$. The error is meant to be on the digit in brackets. The numbers shown are just indicative, and the numerical precision can be increased.

where we expand the multiple emissions contribution as

$$\mathcal{F}_{\text{NLL}}(\lambda) + \frac{\alpha_s(Q)}{\pi} \delta\mathcal{F}_{\text{NNLL}}(\lambda) = \sum_{i,j} \mathcal{F}_{ij} \left(\frac{\alpha_s}{2\pi}\right)^i L^j. \quad (\text{E.2})$$

In order to perform the matching to NNLO, we need the \mathcal{F}_{ij} coefficients up to $\mathcal{O}(\alpha_s^3)$. The results are summarized in table 1.

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