$\mathbb{R}^3$ Index for Four-Dimensional $\mathcal{N}=2$ Field Theories

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In theories with $\mathcal{N}=2$ supersymmetry on $\mathbb{R}^{3,1}$, supersymmetric bound states can decay across walls of marginal stability in the space of Coulomb branch parameters, leading to discontinuities in the BPS indices $\Omega(\gamma,u)$. We consider a supersymmetric index $\mathcal{I}$ which receives contributions from 1/2-BPS states, generalizing the familiar Witten index $\mathrm{Tr}(-1)^F e^{-\beta H}$. We expect $\mathcal{I}$ to be smooth away from loci where massless particles appear, thanks to contributions from the continuum of multiparticle states. Taking inspiration from a similar phenomenon in the hypermultiplet moduli space of $\mathcal{N}=2$ string vacua, we conjecture a formula expressing $\mathcal{I}$ in terms of the BPS indices $\Omega(\gamma,u)$, which is continuous across the walls and exhibits the expected contributions from single particle states at large β. This gives a universal prediction for the contributions of multiparticle states to the index $\mathcal{I}$. This index is naturally a function on the moduli space after reduction on a circle, closely related to the canonical hyperkähler metric and hyperholomorphic connection on this space.

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It has been clear since the work of Seiberg and Witten [1] that extended supersymmetry gives enough control over four-dimensional quantum field theories to produce exact results on the *dynamics* of the theories, even when these theories are strongly interacting. Remarkably, such results are deeply related to some of the most interesting questions in the mathematics of algebraic geometry and differential geometry. As a significant example, the moduli space of a four-dimensional theory with $\mathcal{N}=2$ supersymmetry on a circle is a hyperkähler manifold (a special class of manifolds satisfying Einstein's equations), whose metric encodes both instanton corrections to gauge couplings and the spectrum of Bogomol'nyi-Prasad-Sommerfield (BPS) states in the four-dimensional theory [2]. In this Letter, we reinforce this connection, and construct a canonical function on the aforementioned moduli space, which, on the one hand, generates a solution to the self-dual Yang-Mills equations on this manifold, and, on the other hand, purportedly encodes interactions of BPS states in four dimensions.

BPS indices and the Witten index.—In four-dimensional field theories on $\mathbb{R}^{3,1}$ with $\mathcal{N}=2$ supersymmetry, the spectrum of BPS states in general strongly depends on the value of the Coulomb branch parameters. Part of this dependence can be removed by considering the BPS index

$$\Omega(\gamma,u)=-\frac{1}{2}\mathrm{Tr}_{\mathcal{H}_1(\gamma,u)}(-1)^{2J_3}(2J_3)^2, \tag{1}$$

where $\mathcal{H}_1(\gamma,u)$ is the Hilbert space of one-particle states with electromagnetic charge $\gamma\in\Gamma$ in the Coulomb vacuum u, J_3 is a component of the rotation group along a fixed axis, and $(-1)^{2J_3}$ is the fermionic parity by virtue of the spin statistics theorem. The BPS index $\Omega(\gamma,u)$, being sensitive only to short multiplets saturating the BPS bound [3], is a locally constant, integer valued function of u, but it is discontinuous across certain walls in moduli space, where some of the BPS bound states with charge γ decay into multiparticle BPS states [1,4]. The jump of $\Omega(\gamma,u)$ across the walls is governed by a universal wall-crossing formula [5], which can be derived by quantizing the configurational degrees of freedom of multicentered BPS states near the wall [6–8] (see, e.g., [9] for a review).

The present work addresses another apparently protected quantity, the Witten index

$$\mathcal{I}(\beta,u,C)=-\frac{1}{2}\mathrm{Tr}_{\mathcal{H}(u)}(-1)^{2J_3}(2J_3)^2\sigma e^{-\beta H-2\pi i\langle\gamma,C\rangle}, \tag{2}$$

where $\mathcal{H}(u)$ is the full Hilbert space of the four-dimensional theory on $\mathbb{R}^3$. Here, β is the inverse temperature, conjugate to the Hamiltonian H, C are chemical potentials conjugate to the electromagnetic charge γ, and σ is an operator on $\mathcal{H}(u)$ acting by a sign σ_γ in the sector with charge γ, such that $\sigma_\gamma\sigma_{\gamma'}=(-1)^{\langle\gamma,\gamma'\rangle}\sigma_{\gamma+\gamma'}$, where $\langle\gamma,\gamma'\rangle$ is the usual Dirac-Schwinger-Zwanziger product—this sign is crucial in ensuring the consistency of self-dual field theories [10–14]. For simplicity, we restrict our work to theories without flavor charges. The use of the canonical ensemble with respect to the electromagnetic charges is not essential, but facilitates the geometric interpretation of the index.

Most importantly, unlike the well-studied case of the index on S^3, the spectrum of the Hamiltonian on $\mathbb{R}^3$ is gapless, due to massless gauge bosons and their superpartners, and continuous, as it includes all multiparticle states made out of the discrete states in $\bigoplus_{\gamma\in\Gamma}\mathcal{H}_1(\gamma,u)$. The contribution of the point spectrum to the index (2) is controlled by the BPS indices $\Omega(\gamma,u)$, and is therefore discontinuous across walls of marginal stability. Multiparticle states, on the other hand, can also contribute despite the fact that they do not saturate the BPS bound, due to a possible spectral asymmetry between bosonic and fermionic states [15]. Our main assumption is that the Witten index (2) is continuous across walls of marginal stability, as a result of cancellations between discontinuities from single and multiparticle state contributions. This assumption is physically reasonable, since the path integral defining (2) suffers no phase transition across the wall. Under this assumption, we propose a formula for expressing (2) in terms of the BPS indices $\Omega(\gamma,u)$.

Our assumption is further supported by analogy with the case of framed BPS indices associated to line defects in $\mathcal{N}=2$ theories of class S. These indices are defined by a formula almost identical to (2) [without the insertion of $(2J_3)^2$], and are known to be smooth across BPS walls [16]. The Witten index (2) can be viewed as the extension of the framed index to the case of a trivial line defect.

Another class of examples where a Witten-type index is known to be a smooth function of the moduli arises in $\mathcal{N}=2$ supersymmetric massive theories in $1+1$ dimensions: the BPS indices Ω_{ab}, which count single-particle kinks interpolating between pairs ab of supersymmetric vacua, exhibit similar wall-crossing phenomena as in $3+1$ dimensions [17], while the Cecotti-Fendley-Intriligator-Vafa (CFIV) index $\mathrm{Tr}_{ab}(-1)^F F e^{-\beta H}$ is continuous across the walls, as a result of cancellations between single-particle and multiparticle contributions [18,19].

Yet another way to support our assumption is the general expectation that the Witten index controls quantum corrections to BPS-saturated couplings in the low-energy effective action. In the case of four-dimensional $\mathcal{N}=2$ theories, an appropriate coupling is the metric on the moduli space of the theory reduced on a spatial circle of radius R. In $2+1$ dimensions, Abelian gauge fields can be dualized into scalar fields, and the dynamics on the Coulomb branch can be formulated as a nonlinear sigma model. Its target space $\mathcal{M}_3(R)$ is a torus fibration over the Coulomb branch moduli space $\mathcal{M}_4$ in $3+1$ dimensions, with the torus fiber parametrizing the holonomies C of the electromagnetic gauge fields around the circle [20]. Supersymmetry requires the metric on $\mathcal{M}_3(R)$ to be hyperkähler (HK). In the limit $R\to\infty$ it is obtained from the special Kähler metric on $\mathcal{M}_4$ via the so-called “rigid c-map” procedure [21]. For finite radius, however, the metric on $\mathcal{M}_3(R)$ receives instanton corrections of order e^{-R} from BPS states in $3+1$ dimensions, whose Euclideanized worldline winds around the circle [2,20] (a supersymmetric version of a mechanism first envisaged in [22]). Although corrections to the metric components include an infinite series of multi-instanton corrections, they are entirely controlled by the BPS indices $\Omega(\gamma,u)$ counting single-particle states. Furthermore, it is manifest from the twistorial construction of $\mathcal{M}_3(R)$ [2] that the quantum corrected metric is regular across walls of marginal stability, with multi-instanton contributions on one side of the wall replacing the one-instanton correction on the other side (alternatively, the smoothness of the metric on $\mathcal{M}_3(R)$ provides a physical rationale for the wall-crossing formulas of [5]).

Since quantum corrections to the moduli space metric in theories with 8 supercharges are generally saturated by 1/2-BPS contributions, it is natural to expect a connection between the metric on $\mathcal{M}_3(R)$ and the Witten index $\mathcal{I}(\beta,u,C)$ for $\beta=2\pi R$. The goal of this Letter is to construct a natural function on the family of spaces $\mathcal{M}_3(R)$, continuous across the walls, which reproduces the expected contributions of single-particle states to the Witten index in the limit $R\to\infty$. We conjecture that these two functions are equal, which allows us to predict the contributions of the continuum of multiparticle states to $\mathcal{I}(\beta,u,C)$.

The clue for our construction comes from an analogous problem in superstring theory, namely, the vector multiplet moduli space $\tilde{\mathcal{M}}_3$ in type IIA/B string vacua of the form $\mathbb{R}^3\times S^1(R)\times\mathfrak{Y}$ where $\mathfrak{Y}$ is a Calabi-Yau threefold. After T duality on the circle and its decompactification, the same moduli space describes the hypermultiplet sector of the dual type IIB/A theory on $\mathbb{R}^4\times\mathfrak{Y}$ [23]. In contrast to the gauge theory case, $\tilde{\mathcal{M}}_3$ is a quaternion-Kähler (QK) manifold, where R appears as one of the coordinates. In the limit $R\to\infty$, the metric is obtained by the “local c-map” procedure from the vector multiplet moduli space $\tilde{\mathcal{M}}_4$ in type IIA/B on $\mathbb{R}^4\times\mathfrak{Y}$ [24], whereas for finite radius it receives $\mathcal{O}(e^{-R})$ corrections from four-dimensional BPS states winding around the circle (T dual to D-instantons). These instanton corrections can be incorporated through the twistor space construction [25,26] (see Refs. [27,28] for reviews). However, unlike the gauge theory setup, the instanton series is divergent due to the exponential growth of the BPS indices. Arguably, this is resolved by the existence of further gravitational (or NS5-brane) instanton corrections of order e^{-R^2} [29]. In the sector with zero NUT charge, which is insensitive to these additional instantons, the twistorial construction of [25,26] is formally isomorphic to the gauge-theoretic one [2], specialized to the case of theories with a nonanomalous $U(1)_R$ symmetry, described by a homogeneous prepotential $F(X)$ of degree two. This isomorphism was shown to be a particular instance of a general correspondence between QK metrics with quaternionic $U(1)$ action and HK metrics with $U(1)$ isometry rotating the complex structures [30–32] (the correspondence proceeds by lifting the $U(1)$ action on the QK manifold to the Swann bundle and then taking the HK quotient). Through this correspondence, the family of HK metrics $\mathcal{M}_3(R)$ inherits a canonical function, the moment map of the $U(1)_R$ action, which is smooth as long as the metric on $\mathcal{M}_3(R)$ is. On the QK side it appears as the "contact potential," which relates the $O(2)$-twisted canonical one-form to the holomorphic contact one-form [25,33]. As we explain in the Supplemental Material [34], which includes Ref. [35], a generalized contact potential can be defined even when the prepotential F is not homogeneous, and understood geometrically as the ratio of two Hermitian metrics on the canonical line bundle constructed in [36].

A family of smooth functions on $\mathcal{M}_3(R)$.—To define our candidate for the Witten index $\mathcal{I}$, let us first recall the twistorial construction of the HK metric on $\mathcal{M}_3(R)$ [2] (see [37] for a review). The twistor space $\mathcal{Z} = \mathbb{P}_t \times \mathcal{M}_3(R)$ carries a family of functions $\{\mathcal{X}_\gamma(t)\}_{\gamma\in\Gamma}$, holomorphic in complex structure $J(t)$, satisfying the integral equations [38]

$$\frac{\mathcal{X}_\gamma}{\mathcal{X}_\gamma^{\rm sf}} = \exp\left[\sum_{\gamma'}\frac{\Omega(\gamma')}{4\pi i}\langle\gamma,\gamma'\rangle\int_{\ell_{\gamma'}}\frac{dt'}{t'}\frac{t+t'}{t-t'}\log\left(1-\mathcal{X}_{\gamma'}(t')\right)\right], \tag{3}$$

where $\ell_{\gamma'}$ are the BPS rays $\{t'\in\mathbb{C}^\times : Z_{\gamma'}/t'\in i\mathbb{R}^-\}$ and $\mathcal{X}_\gamma^{\rm sf}$ provide the boundary conditions at $R\to\infty$,

$$\mathcal{X}_\gamma^{\rm sf} = \sigma_\gamma e^{-\pi i R(t^{-1}Z_\gamma - t\bar{Z}_\gamma) - 2\pi i\langle\gamma,C\rangle}. \tag{4}$$

Here $Z_\gamma = \langle\gamma, X\rangle$ is the central charge and $X = (X^\Lambda, F_\Lambda)$ is the holomorphic symplectic section on the special Kähler manifold $\mathcal{M}_4$ with $F_\Lambda = \partial_{X^\Lambda}F$ in special coordinates. In the limit $R\to\infty$, the system (3) can be solved iteratively, generating a sum of products of iterated integrals of $\mathcal{X}_\gamma^{\rm sf}$ interpreted as multi-instanton contributions. Given such a solution, the triplet of Kähler forms on $\mathcal{M}_3(R)$, which define the metric, is read off from the $\mathcal{O}(2)$-twisted holomorphic symplectic form on $\mathcal{Z}$,

$$\omega = it^{-1}\omega_+ + \omega_3 + it\omega_- = \frac{\epsilon^{ab}}{8\pi^2}\frac{d\mathcal{X}_{\gamma_a}}{\mathcal{X}_{\gamma_a}}\wedge\frac{d\mathcal{X}_{\gamma_b}}{\mathcal{X}_{\gamma_b}}, \tag{5}$$

where γ_a is a basis of Γ, and ϵ^{ab} is the inverse of $\langle\gamma_a,\gamma_b\rangle$.

With these notations in place, fix any smooth function $F_\gamma(t,u,C)$ on $\Gamma\times\mathcal{Z}$, linear in γ and define

$$\Phi(R,u,C) = \sum_\gamma \Omega(\gamma)\int_{\ell_\gamma}\frac{dt}{t}F_\gamma\log\left(1-\mathcal{X}_\gamma\right). \tag{6}$$

We claim that Φ is a smooth function on $\mathcal{M}_3(R)$, provided the BPS indices $\Omega(\gamma)$ jump across walls of marginal stability according to the standard wall-crossing formula [5]. Indeed, on a wall $W(\gamma_1,\gamma_2)$, where the central charges Z_{γ_1}, Z_{γ_2} associated to two primitive charge vectors become aligned in the complex plane, the BPS rays $\ell_{m\gamma_1+n\gamma_2}$ with $m,n\geq 0$ all coalesce into one ray ℓ, across which the potential discontinuity is given by

$$\Delta\Phi = \int_\ell\frac{dt}{t}\sum_\gamma F_\gamma[\Omega^+(\gamma)\log\left(1-\mathcal{X}_\gamma^+\right) - \Omega^-(\gamma)\log\left(1-\mathcal{X}_\gamma^-\right)], \tag{7}$$

where $\Omega^\pm(\gamma)$ and $\mathcal{X}_\gamma^\pm$ are the BPS indices and solutions of the corresponding Eqs. (3) on either side of the wall. Now, recall that the semiclassical limit of the motivic version of the wall-crossing formula implies the functional identity [31]

$$\sum_\gamma\Omega^+(\gamma)L_{\sigma_\gamma}(\mathcal{X}_\gamma^+) = \sum_\gamma\Omega^-(\gamma)L_{\sigma_\gamma}(\mathcal{X}_\gamma^-), \tag{8}$$

where $L_\epsilon(z)$ is a variant of the Rogers dilogarithm,

$$L_\epsilon(z) \equiv \mathrm{Li}_2(z) + \frac{1}{2}\log(\epsilon^{-1}z)\log(1-z). \tag{9}$$

The invariance of (8) under monodromies $M_\gamma : \mathcal{X}_{\gamma'} \mapsto e^{2\pi i\langle\gamma,\gamma'\rangle}\mathcal{X}_{\gamma'}$ leads to the Γ-valued identity

$$\sum_\gamma\gamma[\Omega^+(\gamma)\log\left(1-\mathcal{X}_\gamma^+\right) - \Omega^-(\gamma)\log\left(1-\mathcal{X}_\gamma^-\right)] = 0. \tag{10}$$

The vanishing of the discontinuity (7) then follows from (10) and from the linearity of F_γ with respect to γ.

A candidate for the Witten index.—Having constructed a family of smooth functions on $\mathcal{M}_3(R)$, we now aim for one that may plausibly be identified with the Witten index (2). For HK manifolds $\mathcal{M}_3(R)$ related to QK manifolds $\tilde{\mathcal{M}}_3$ by the QK-HK correspondence, a natural candidate is the contact potential on $\tilde{\mathcal{M}}_3$ [25,33], which translates on the HK side into

$$\mathcal{I} = \frac{R}{16i\pi^2}\sum_\gamma\Omega(\gamma)\int_{\ell_\gamma}\frac{dt}{t}(t^{-1}Z_\gamma - t\bar{Z}_\gamma)\log[1-\mathcal{X}_\gamma(t)]. \tag{11}$$

This function is a member of the family (6) with $F_\gamma(t)\propto t^{-1}Z_\gamma - t\bar{Z}_\gamma$, so it is smooth across walls of marginal stability. Its reality follows from the reality property $\overline{\mathcal{X}_\gamma(-1/\bar{t})} = \mathcal{X}_{-\gamma}(t)$ and the CPT relation $\Omega(-\gamma) = \Omega(\gamma)$.

In order to assess whether (11) qualifies to represent the Witten index, let us compute the formal multi-instanton expansion of $\mathcal{I}$, by substituting the iterated solution of (3) into (11). Up to second order, $\mathcal{I} = \sum_\gamma \mathcal{I}^{(1)}_\gamma + \sum_{\gamma,\gamma'} \mathcal{I}^{(2)}_{\gamma,\gamma'} + \cdots$ with

$$\mathcal{I}^{(1)}_\gamma = \frac{R}{4\pi^2}\sigma_\gamma \bar{\Omega}(\gamma)|Z_\gamma|K_1(2\pi R|Z_\gamma|)e^{-2\pi i\langle\gamma,C\rangle},$$

$$\mathcal{I}^{(2)}_{\gamma,\gamma'} = -\frac{R}{64\pi^3}\bar{\Omega}(\gamma)\bar{\Omega}(\gamma')\langle\gamma,\gamma'\rangle \int_{\ell_\gamma}\frac{dt}{t}\int_{\ell'_\gamma}\frac{dt'}{t'}\frac{t+t'}{t-t'} \times (t^{-1}Z_\gamma - t\bar{Z}_\gamma)\mathcal{X}^{\rm sf}_\gamma(t)\mathcal{X}^{\rm sf}_{\gamma'}(t'), \tag{12}$$

where $\bar{\Omega}(\gamma) = \sum_{d|\gamma}(1/d^2)\Omega(\gamma/d)$ denotes the rational index. Remarkably, for primitive charge vector γ the one-instanton contribution $\mathcal{I}^{(1)}_\gamma$ agrees with the contribution of a single-particle, relativistic BPS state of charge γ and mass $M = |Z_\gamma|$ to the Witten index (2). To see this, we use a Schwinger time parametrization to linearize the relativistic Hamiltonian $H = \sqrt{-\Delta + M^2}$, and introduce a nonzero chemical potential θ conjugate to J_3 and periodic boundary conditions $\psi(z) = \psi(z+L)$ along the z axis, with $L \gg 1/M$, to regulate infrared divergences. Denoting by $\chi_{\rm spin}(\theta)$ the $SU(2)$ character for the spin degrees of freedom, we have

$$\begin{aligned}\mathrm{Tr}e^{-2\pi RH+i\theta J_3} &= R\int_0^\infty \frac{dt}{t^{3/2}}\mathrm{Tr}e^{-\pi(R^2/t)-\pi(-\Delta+M^2)t+i\theta J_3} \\ &= R\int_0^\infty \frac{dt}{t^{3/2}}\frac{L}{2\pi\sqrt{t}}\frac{\chi_{\rm spin}(\theta)}{4\sin^2(\theta/2)}e^{-\pi\frac{R^2}{t}-\pi M^2 t} \\ &= \frac{L}{2\pi}\frac{\chi_{\rm spin}(\theta)}{4\sin^2(\theta/2)}2MK_1(2\pi MR).\end{aligned} \tag{13}$$

For a BPS multiplet of spin j, the spin character is

$$\chi_{\rm spin}(\theta) = \left(2 + 2\cos\frac{\theta}{2}\right)\frac{\sin[(j+\frac{1}{2})\theta]}{\sin(\theta/2)}, \tag{14}$$

corresponding to a BPS index $\Omega(\gamma) = 2\partial^2_\theta\chi_{\rm spin}(\theta)|_{\theta=2\pi} = (-1)^{2j}(2j+1)$. Comparing with the first line in (12) we find

$$\mathcal{I}^{(1)}_\gamma = 2R\lim_{\substack{\theta\to 2\pi\\ L\to\infty}}\partial^2_\theta\left[\frac{\sin^2(\theta/2)}{\pi L}\mathrm{Tr}(\sigma e^{-2\pi RH+i\theta J_3-2\pi i\langle\gamma,C\rangle})\right]. \tag{15}$$

The factor $\sin^2(\theta/2)/(2\pi L)$ can be understood as dividing by the regularized volume of $\mathbb{R}^3$.

Based on this agreement, and smoothness across walls of marginal stability, we conjecture that (11) in fact computes the Witten index (2), with the specific prescription given in (15) for regulating infrared divergences. If true, this implies that the two-instanton term $\mathcal{I}^{(2)}_{\gamma,\gamma'}$ in (12) should be identified with the contribution of the continuum of two-particle states, and similarly for higher $\mathcal{I}^{(n)}$'s.

Discussion.—In this article we conjectured a formula (11) for the generalized Witten index (2) in four-dimensional $\mathcal{N} = 2$ gauge theories. The formula is manifestly smooth across walls of marginal stability, and correctly reproduces the expected BPS bound states contributions. The evidence for this conjecture is admittedly weak, since within the class (6) of smooth functions on the Coulomb branch $\mathcal{M}_3(R)$ in three dimensions, one could easily find other functions which would differ only at higher order in the multiparticle expansion. As we explain in the Supplemental Material [34], our proposal is distinguished by the fact that $\mathcal{I}$ is related to the Kähler form and hyperholomorphic curvature on $\mathcal{M}_3(R)$, in accordance with the general slogan that corrections to the moduli space metric in theories with 8 supercharges are saturated by 1/2-BPS contributions. The function $\mathcal{I}$ has also appeared in the context of the analogy of the system (3) with TBA equations [2], where it is identified with the free energy of the corresponding integrable system [39], and in the context of minimal surfaces in AdS_5 [40,41]. It would be interesting to extend our construction to gauge theories with massive flavors [42].

If correct, our conjecture predicts that multiparticle state contributions to the Witten index are universal functions of the BPS indices $\Omega(\gamma)$ associated with the constituents. The predicted contribution of the continuum of two-particle states can be found in (12), while higher orders can be easily obtained by combining (11) with the iterated solution to the TBA-like system (3). It is a challenge to check these predictions from a direct computation of the difference of densities of bosonic and fermionic states of a system of n dyons. While the result near a wall of marginal stability can actually be deduced by analyzing the nonrelativistic electron-monopole system [43], the result (12) should hold throughout moduli space, where the constituents are relativistic.

Note also that our conjecture naturally extends to the case of $\mathcal{N} = 2$ string vacua, where the formula (11) computes instanton corrections to the contact potential on the QK moduli space $\tilde{\mathcal{M}}_3$ generated by multidyonic BPS black holes. Therefore, another check would be to reproduce the smooth, duality invariant partition function for two-centered D4-D2-D0 black holes constructed in [44], extending the arguments in [45] beyond the one-instanton level.

As we have mentioned, the generalized Witten index (2) may be viewed as the analog of the framed BPS index for a trivial line defect. One possible way to derive (2) would then be to study the fusion of two line defects whose operator product expansion contain the trivial line defect. This analogy also suggests the existence of a refined Witten index, which would arise in the fusion of framed protected spin characters. It is natural to conjecture that this refined index might be related to the CFIV index of the two-dimensional theory obtained by placing the four-dimensional theory on an Ω-background with $\epsilon_1 \neq 0$, $\epsilon_2 = 0$ [46].

Finally, our conjecture—if true—could reveal interesting and nontrivial information on BPS spectra which is not easily accessible by other means. For example, consider a theory of class S where the ultraviolet curve C is a compact Riemann surface with negative curvature. In this case $\mathcal{I}_{\rm tot}$ is just the moment map for the natural $U(1)$ action on Hitchin data and hence proportional to the L^2 norm square of the Higgs field [32]. The expression (11) is highly nontrivial already in the A_1 case. In this case one may be able to give a systematic large R expansion of the norm square of the Higgs field by solving the classical sinh-Gordon theory on C. Using the parametrization of Ref. [47], Eq. (13.14), it is easy to show that, on a real slice of moduli space one needs to expand

$$\mathcal{I}_{\rm tot} = \frac{iR^2}{4}\int_C \lambda\bar{\lambda}\cosh(2h) \tag{16}$$

at large R for solutions to the sinh-Gordon equation

$$\partial\bar{\partial}h - 2R^2\lambda\bar{\lambda}\sinh(2h) = 0, \tag{17}$$

with boundary condition $h \sim -\frac{1}{2}\log|z - z_a| + \cdots$ at the first order zeros $z = z_a$ of the quadratic differential λ^2. We hope to return to this problem in a future publication.

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