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CUT-OFF VERSUS DIMENSIONAL REGULARIZATION IN THE LIGHT-CONE GAUGE

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ABSTRACT

The conventional cut-off method is applied to massless light-cone gauge Feynman integrals. Despite the presence of non-local terms in the unintegrated expression for the Yang-Mills self-energy, the cut-off procedure yields the same ultra-violet behaviour as the lengthier technique of dimensional regularization.

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1. - INTRODUCTION

The purpose of this note is to show that the cut-off method[1),2)] may also be applied to two-point functions in the light-cone gauge. The cut-off procedure is a simple technique which extracts the ultra-violet behaviour from a given integral and is, strictly speaking, only applicable at the one-loop level. To date, the method has been employed in both covariant gauges and non-covariant gauges, where it was found to yield the same results as the lengthier, albeit more general, technique of dimensional regularization.

In view of the ever-increasing number of light-cone gauge computations in such popular areas as supersymmetric Yang-Mills theory, supergravity and superstring theories, it is clearly desirable to have at one's disposal another, preferably shorter, means of attacking the standard integrals. As a test case we have applied the cut-off procedure to the Yang-Mills self-energy, obtained previously with the aid of dimensional regularization. The problem is not entirely trivial, since application of the correct light-cone prescription[3),4)] is known to yield non-local factors already at the one-loop level[4)]. These potentially dangerous factors are absent in the axial and planar gauges, since the latter do not manifestly break Lorentz invariance.

2. - YANG-MILLS SELF-ENERGY TO ONE LOOP

Consider the Lagrangian density

$$L_{YM} = -\frac{1}{4}\left(F^a_{\mu\nu}\right)^2 - \frac{1}{2\alpha}\left(n\cdot A^a\right)^2, \quad \alpha \text{ gauge parameter,} \tag{1}$$

where A^a_μ is a massless gauge field, $\mu = 0,1,2,3$, and n_μ an arbitrary constant four-vector. The light-cone gauge is specified by

$$n^\mu A^a_\mu = 0, \quad n^2 = 0. \tag{2}$$

The unphysical singularities of $(q\cdot n)^{-1}$ in the gauge field propagator $(\alpha \to 0)$

$$G^{ab}_{\mu\nu}(q) = \frac{-i\,\delta^{ab}}{(2\pi)^4(q^2+i\varepsilon)}\left[\delta_{\mu\nu} - \frac{(q_\mu n_\nu + q_\nu n_\mu)}{q\cdot n}\right], \varepsilon > 0, \tag{3}$$

are treated with the prescription[4)] [for an equivalent procedure, see Mandelstam[3)]]

$$\frac{1}{q\cdot n} \Longrightarrow \lim_{\varepsilon\to 0}\left(\frac{q\cdot n^*}{q\cdot n\, q\cdot n^* + i\varepsilon}\right),\ n_\mu = (n_0,\vec{n}),\ n^*_\mu = (n_0,-\vec{n}). \quad (4)$$

Using standard Feynman rules, we obtain the following expression for the one-loop Yang-Mills self-energy:

$$\Pi^{ab}_{\mu\nu}(p) = C^{ab}\Big[\frac{11}{3}\left(p^2\delta_{\mu\nu} - p_\mu p_\nu\right)R + \delta_{\mu\nu}\,D + E_{\mu\nu} + F_{\mu\nu} + G_{\mu\nu} + H_{\mu\nu}\Big],$$

$$D = 2\,p\cdot n\int d^4q\left(\frac{p^2}{q^2(q+p)^2\,q\cdot n} - \frac{p^2}{q^2(q-p)^2\,q\cdot n}\right),$$

$$E_{\mu\nu} = p\cdot n\left[p_\mu\int d^4q\left(\frac{q_\nu}{q^2(q+p)^2\,q\cdot n} + \frac{q_\nu}{q^2(q-p)^2\,q\cdot n}\right) + \mu\leftrightarrow\nu\right], \quad (5)$$

$$F_{\mu\nu} = \frac{1}{2}\left[p_\mu n_\nu\int d^4q\left(\frac{1}{q^2(q-p)\cdot n} - \frac{1}{(q-p)^2\,q\cdot n}\right) + \mu\leftrightarrow\nu\right],$$

$$G_{\mu\nu} = -p^2\left[n_\mu\int\frac{d^4q\; q_\nu}{q^2(q-p)^2}\left(\frac{1}{(q-p)\cdot n} + \frac{1}{q\cdot n}\right) + \mu\leftrightarrow\nu\right],$$

$$H_{\mu\nu} = n_\mu n_\nu\,\frac{p^2}{p\cdot n}\int d^4q\left(\frac{1}{(q-p)^2\,q\cdot n} - \frac{1}{q^2(q-p)\cdot n}\right),$$

where p_μ is the external momentum, and only potentially ultra-violet divergent integrals are shown explicitly. Moreover, $C^{ab} \equiv C_{YM}\delta^{ab}g^2$, $\delta^{ab}C_{YM} = f^{acd}f^{bcd}$, and R denotes the basic integral in the cut-off method,

$$R = \int^{\Lambda} d^4q/q^4 = \pi^2 \ln(\Lambda^2/\mu^2), \tag{6}$$

where Λ is the cut-off (in Euclidean space) and μ an arbitrary mass scale. Finally, notice that the last term $H_{\mu\nu}$ is <u>non-local</u> in p_μ.

3. - AN EXAMPLE

Before we extract the ultra-violet behaviour by the cut-off method from each of the four-dimensional integrals in Eq. (5), we shall illustrate the technique by evaluating the light-cone gauge integral

$$I_\mu(p) = \int \frac{d^4q \; q_\mu}{(q-p)^2 \, q\cdot n}, \tag{7}$$

which is clearly ultra-violet divergent. The standard procedure in the cut-off approach is to act on (7) with the operator $\mathcal{M}$[2),5)]

$$\mathcal{M} \equiv \frac{1}{2} p_\alpha p_\beta \left(\frac{\partial^2}{\partial p_\alpha \partial p_\beta} \bigg|_{p=0} \right), \tag{8}$$

giving

$$\mathcal{M} I_\mu = p_\alpha p_\beta \left[-\delta_{\alpha\beta} \int \frac{d^4q \; q_\mu}{q^4 \, q\cdot n} + 4 \int \frac{d^4q \; q_\alpha q_\beta q_\mu}{q^6 \; q\cdot n} \right], \tag{9}$$

and then to compute the two light-cone gauge integrals separately.

The first integral in (9) is calculated by making the ansatz[6)]

$$\int \frac{d^4q\; q_\mu}{q^4\, q\cdot n} = A\, n_\mu + B\, n^*_\mu\;, \tag{10}$$

and then determining the coefficients A, B. Multiplication of (10) by n_μ gives ($n^2 = 0$)

$$\int d^4q / q^4 = R = B\, n\cdot n^*,$$

so that

$$B = R / n\cdot n^*\,. \tag{11}$$

The coefficient A follows, for example, from dimensional analysis by observing that $\int d^4q\; q_\mu (q\cdot n q^4)^{-1} \sim O(1/n)$, so that the dimension of A, written [A], is $[A] = 1/n^2$. But $1/n^2$ is not an admissible invariant in the light-cone gauge, so A must vanish, A = 0, and

$$\int d^4q\, q_\mu (q^4 q\cdot n)^{-1} = n^*_\mu\;\; R / n\cdot n^*\,. \tag{12}$$

To evaluate the second term in Eq. (9), we observe that the integral

$$I_{\alpha\beta\mu} = \int d^4q\;\; q_\alpha q_\beta q_\mu\, (q^6 q\cdot n)^{-1} \tag{13}$$

is of order $O(1/n)$, suggesting on dimensional grounds the ansatz

$$\begin{aligned} I_{\alpha\beta\mu} = {} & A_0 \left(n^*_\alpha \delta_{\beta\mu} + n^*_\beta \delta_{\alpha\mu} + n^*_\mu \delta_{\alpha\beta} \right) \\ & + B_0 \left(n^*_\alpha n^*_\beta n_\mu + n^*_\alpha n^*_\mu n_\beta + n^*_\beta n^*_\mu n_\alpha \right). \end{aligned} \tag{14}$$

Multiplication of (14) by n_μ yields the relation $B_0 = -A_0/n\cdot n^*$, while contraction of the indices α, β gives $\int d^4q\; q_\mu (q\cdot nq^4)^{-1} = 4\, A_0 n^*_\mu$; hence from (12), $A_0 = R/(4n\cdot n^*)$, so that

$$\int \frac{d^4q\, q_\alpha q_\beta q_\mu}{q^6\, q\cdot n} = (4\, n\cdot n^*)^{-1}\Big[n^*_\alpha \delta_{\beta\mu} + n^*_\beta \delta_{\alpha\mu} + n^*_\mu \delta_{\alpha\beta} - (n\cdot n^*)^{-1}\big(n^*_\alpha n^*_\beta n_\mu + n^*_\mu n^*_\alpha n_\beta + n^*_\beta n^*_\mu n_\alpha \big)\Big] R\,. \tag{15}$$

Substitution of (12) and (15) into (9) gives the answer

$$I_\mu(p) = \frac{1}{n\cdot n^*}\left[2p\cdot n^* p_\mu - \frac{(p\cdot n^*)^2}{n\cdot n^*} n_\mu - \frac{2p\cdot n\; p\cdot n^* n^*_\mu}{n\cdot n^*}\right] R\,; \tag{16}$$

this result agrees with Eq. (3.6) of Ref. 7), because the mass term m^2 is zero here and the factor R corresponds to $\bar{I}$. The integral $\bar{I}$ corresponds, in Euclidean space, to

$$\bar{I} = \text{divergent part of} \int \frac{d^{2\omega} q}{q^2 (q-p)^2} \overset{\omega \to 2}{=} \pi^2 \Gamma(2-\omega) = \pi^2/(2-\omega).$$

4. - THE OPERATOR $\mathcal{M}$

Applying the operator (8) first to the local integrals in the self-energy $\Pi^{ab}_{\mu\nu}$, we find that

$$\mathcal{M}\left[2p\cdot n \int \frac{d^4q\;\; p^2}{q^2 (q+p)^2\, q\cdot n}\right] = 0\,, \tag{17a}$$

$$\mathcal{M}\left[p\cdot n\, p_\mu \int \frac{d^4q}{q^2}\left(\frac{q_\nu}{(q+p)^2 q\cdot n} + \frac{q_\nu}{(q-p)^2 q\cdot n}\right)\right] \tag{17b}$$
$$= 2 p_\mu n_\nu^* \;\; p\cdot n\, R / n\cdot n^* ,$$

$$\mathcal{M}\left[\tfrac{1}{2}\, p_\mu n_\nu \int d^4q \left(\frac{1}{q^2 (q-p)\cdot n} - \frac{1}{(q-p)^2 q\cdot n}\right)\right] \tag{17c}$$

$$= -2 p_\mu n_\nu \;\; p\cdot n^* R / n\cdot n^* ,$$

$$\mathcal{M}\left[- p^2 n_\mu \int \frac{d^4q\; q_\nu}{q^2 (q-p)^2}\left(\frac{1}{(q-p)\cdot n} + \frac{1}{q\cdot n}\right)\right] \tag{17d}$$
$$= -2 n_\mu n_\nu^* \;\; p^2 R / n\cdot n^* ,$$

in which case

$$\mathcal{M} D = 0 , \tag{18a}$$

$$\mathcal{M} E_{\mu\nu} = 2 (p\cdot n / n\cdot n^*)(p_\mu n_\nu^* + p_\nu n_\mu^*)\, R , \tag{18b}$$

$$\mathcal{M} F_{\mu\nu} = -2 (p\cdot n^* / n\cdot n^*)(p_\mu n_\nu + p_\nu n_\mu)\, R , \tag{18c}$$

$$\mathcal{M} G_{\mu\nu} = -2 (p^2 / n\cdot n^*)(n_\mu n_\nu^* + n_\nu n_\mu^*)\, R . \tag{18d}$$

The non-local expression $H_{\mu\nu}$ in (5) is peculiar to the light-cone gauge and requires special care. As it stands, $H_{\mu\nu}$ is non-local in the external momentum p_μ, but can be massaged into local form by using light-cone variables defined by

$$p^\pm = 2^{-1/2}(p^0 \pm p^3) , \qquad p^\mu = (p^0, p^1, p^2, p^3),$$

$$p_T = 2^{-1/2}(p^1 + i p^2)\ ,\ \bar{p}_T = 2^{-1/2}(p^1 - i p^2), \tag{19}$$
$$p\cdot p = 2(p^+ p^- - p_T \bar{p}_T),$$
$$p\cdot n = p^+ n_- + p^- n_+ - p_T \bar{n}_T - \bar{p}_T n_T ,$$

and working in the special frame where $p^\mu = (p^0, 0, 0, p^3)$, and $n_\mu = (1,0,0,1)$. Then $n_T = p_T = n_- = 0$, $n_+ = \sqrt{2}$, $p\cdot n = \sqrt{2}\ p^-$ and $p^2/n\cdot p = \sqrt{2}\ p^+$. Hence $H_{\mu\nu}$ reduces to

$$H_{\mu\nu} = n_\mu n_\nu \sqrt{2}\, p^+ \int d^4q \left(\frac{1}{(q-p)^2 q\cdot n} - \frac{1}{q^2 (q-p)\cdot n} \right). \tag{20}$$

Now let f define the integral

$$f \equiv \frac{p^2}{p\cdot n} \int d^4q\, [(q-p)^2 q\cdot n]^{-1}$$

and consider $\mathcal{M}f$:

$$\mathcal{M} f = \frac{1}{2}\Big\{ (p^+)^2 [(\partial^+)^2 f]_{p=0} + 2p^+p^- [\partial^+\partial^- f]_{p=0} + (p^-)^2 [(\partial^-)^2 f]_{p=0} \Big\} \tag{21}$$

$$= 2\int \frac{d^4q}{q^-}\left[(p^+)^2 \frac{q^-}{q^4} + p^+ p^- \frac{q^+}{q^4} \right] \tag{22}$$
$$= 2p^\mu \frac{p^2}{p\cdot n} \int \frac{d^4q\ q_\mu}{q^4\ q\cdot n}$$
$$= 2p^2\, p\cdot n^* (p\cdot n\, n\cdot n^*)^{-1} R\ ; \tag{23}$$

consequently,

$$\mathcal{M}\, H_{\mu\nu} = n_\mu n_\nu \, \frac{4p^2\, p\cdot n^*}{p\cdot n\; n\cdot n^*}\, R \; . \tag{24}$$

Thus we have successfully extracted the ultra-violet pole parts from the terms D, $E_{\mu\nu}$, ..., $H_{\mu\nu}$. Substituting the right-hand sides of Eqs. (18) and (24) into Eq. (5), we obtain

$$\begin{aligned}\Pi^{ab}_{\mu\nu}(p) = C^{ab}\Big[& \frac{11}{3}\left(p^2\delta_{\mu\nu} - p_\mu p_\nu\right) + \frac{2p\cdot n}{n\cdot n^*}\left(p_\mu n^*_\nu + p_\nu n^*_\mu\right) \\ & - \frac{2p\cdot n^*}{n\cdot n^*}\left(p_\mu n_\nu + p_\nu n_\mu\right) - \frac{2p^2}{n\cdot n^*}\left(n_\mu n^*_\nu + n_\nu n^*_\mu\right) \\ & + \frac{4\,p^2\, p\cdot n^*}{p\cdot n\; n\cdot n^*}\, n_\mu n_\nu \Big] R \; ,\end{aligned} \tag{25}$$

which agrees identically with Eq. (17) of Ref. 4) provided we identify R with $\pi^2(2-\omega)$, 2ω being the dimensionality of complex space-time.

5. - SUMMARY

We have demonstrated that the simple cut-off method may also be used in the light-cone gauge to extract the ultra-violet behaviour of one-loop Feynman integrals. In particular, the method was shown capable of handling the unavoidable non-local terms in the Yang-Mills self-energy. Although the cut-off method and the technique of dimensional regularization yield different expressions for individual integrals, the two procedures give identical results (as far as the ultra-violet behaviour is concerned) for a specific Feynman diagram.

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REFERENCES

1) C. Itzykson and J.-B. Zuber, Quantum Field Theory (McGraw-Hill, New York, 1980);
T.P. Beven and R. Delbourgo, Lett. Nuovo Cimento 23 (1978) 433; 27 (1980) 565.

2) A. Andraši and J.C. Taylor, Nucl. Phys. B192 (1981) 283; B227 (1983) 494.

3) S. Mandelstam, Nucl. Phys. B213 (1983) 149.

4) G. Leibbrandt, Phys. Rev. D29 (1984) 1699.

5) W. Kainz, W. Kummer and M. Schweda, Nucl. Phys. B79 (1974) 484.

6) G. Leibbrandt, Phys. Rev. D30 (1984) 2167.

7) G. Leibbrandt and S.-L. Nyeo, Phys. Lett. 140B (1984) 417.