

On-shell constrained M_2 variables with applications to mass measurements and topology disambiguation

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ABSTRACT: We consider a class of on-shell constrained mass variables that are 3+1 dimensional generalizations of the Cambridge M_{T2} variable and that automatically incorporate various assumptions about the underlying event topology. The presence of additional on-shell constraints causes their kinematic distributions to exhibit sharper endpoints than the usual M_{T2} distribution. We study the mathematical properties of these new variables, e.g., the uniqueness of the solution selected by the minimization over the invisible particle 4-momenta. We then use this solution to reconstruct the masses of various particles along the decay chain. We propose several tests for validating the assumed event topology in missing energy events from new physics. The tests are able to determine: 1) whether the decays in the event are two-body or three-body, 2) if the decay is two-body, whether the intermediate resonances in the two decay chains are the same, and 3) the exact sequence in which the visible particles are emitted from each decay chain.

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1 Introduction

The measurement of particle properties in events with missing energy at hadron colliders is a challenging problem which has been receiving increased attention as of late (see [1] and [2] for reviews on mass and spin measurement methods, respectively). The difficulty arises because in most new physics models with dark matter candidates, some conserved, often Z_2 , parity is needed to make the dark matter stable. Particles which are charged with respect to this parity are pair produced; each such event contains at least two invisible (dark matter) particles whose energy and momenta are not measured. It is precisely this lack of information which makes the straightforward application of standard mass reconstruction techniques impossible.

In order to deal with the lack of knowledge about the invisible particle momenta, the following three approaches have been suggested:

- *Use variables built from measured momenta only.*

The best known example is the invariant mass of (sets of) visible particles observed in the detector. The measurement of kinematic endpoints in various invariant mass distributions is the classic method for mass determination in supersymmetry [3–8]. Other recently proposed variables include the contranverse mass variable M_{CT} [9, 10] and its variants $M_{CT\perp}$ and $M_{CT\parallel}$ [11], the ratio of visible transverse energies [12, 13], and the energy itself [14–16].

Of course, while the *individual* invisible momenta are unknown, the sum of their transverse components is measured as the missing transverse momentum \vec{P}_T of the event. Thus one could also consider variables which are functions of the visible momenta and \vec{P}_T , e.g., the transverse mass [17, 18], the effective mass M_{eff} [3, 19], the minimum partonic center-of-mass energy $\sqrt{\hat{s}_{\text{min}}}$ [20–22], the razor variables [23, 24], etc. Such variables provide a good global characterization of the event, and are useful for discriminating signal from background, measuring an overall scale, or determining a signal rate. However, since they are not very sensitive to the particular details of the event, they are far from ideal for the purposes of precision studies of the signal.

- *Calculate exactly the unknown individual momenta of the invisible particles.*

This is generally done by assuming a specific event topology and imposing a sufficient number of on-shell constraints [25–28]. If applicable, this method is very powerful, since the event kinematics is fully determined and one can easily move on to precision studies [29]. The main disadvantage of exact reconstruction techniques is that they require sufficiently long decay chains in order to provide the required number of mass-shell constraints. Otherwise, the system is underconstrained, and mass measurements are only possible on a statistical basis, by testing for consistency over the whole ensemble of signal events [30, 31].

- *Use a compromise approach.*

The third approach is a compromise between the previous two — one still constructs kinematic variables which depend on the invisible momenta, but one gives up on trying to determine those momenta exactly on an event-per-event basis. Instead, some kind of ansatz is used to assign values (consistent with the measured \vec{P}_T) to the individual momenta of the invisible particles in each event. The most celebrated variable of this class is the Cambridge M_{T2} variable [32, 33], which is calculated by fixing the *transverse* momenta of the invisible particles to minimize the resulting transverse mass of the (larger of the two) parent particles. The idea of fixing the unknown invisible momenta by minimizing a suitable mass function is very powerful, and many of the kinematic variables proposed in the literature can be reinterpreted that way [34]. The M_{T2} approach is very well developed by now — analytical formulas exist for the calculation of M_{T2} in a given event and for the interpretation of its endpoint [35–41]. Since the original M_{T2} proposal [32, 33], several other related variables have been suggested as well, e.g. $M_{T2\perp}$ and $M_{T2\parallel}$ [42], the asymmetric M_{T2} [43, 44], M_{CT2} [45, 46], and M_{T2}^{approx} [47].

Note that the M_{T2} prescription determines only the transverse components of the invisible momenta. In order to fix the longitudinal components, one could rely on additional measurements or assumptions. For example, in the M_{T2} -assisted on-shell (MAOS) reconstruction method, one uses the measured M_{T2} kinematic endpoint and enforces the on-shell condition for the mother particle, which allows one to solve for the longitudinal momenta [48, 49]. (The idea behind the M_{2C} variable [50, 51] is very similar.) A variation of this method arises if the invisible particles are neutrinos from W (or τ) decays — then one can use the known W -boson (or τ -lepton) mass as a constraint and again solve for the longitudinal momenta [52–57]. Since the on-shell constraints are nonlinear functions, the MAOS approach typically yields multiple solutions for the longitudinal momentum components, so one must also specify a prescription for handling this multiplicity.

An alternative approach to MAOS, which may avoid this ambiguity, was outlined in ref. [34], which pointed out that the M_{T2} variable and its friends allow a 3+1 dimensional formulation, in which one always deals with the *actual* instead of the transverse masses. The corresponding 3+1 dimensional analogue of M_{T2} was denoted simply as M_2 , omitting the transverse index.¹ The actual mass, being 3+1 dimensional, already carries dependence on both transverse and longitudinal momentum components, thus the minimization procedure required to obtain M_2 is expected to automatically assign unique values for *all* momentum components of *each* individual invisible particle. Since much of our discussion below will make crucial use of this property, we will discuss carefully the minimization procedure for the different M_2 -type variables and the uniqueness of the resulting solutions for the invisible momenta in section 3.

An important benefit from extending the transverse M_{T2} formalism to the 3+1-dimensional M_2 language was recently emphasized in [41]. In many practical applications of M_{T2} and similar kinematic variables, one has in mind a very specific signal topology, which in turn implies additional kinematic constraints on the (unknown) individual invisible momenta. For example, SUSY decay chains often proceed through intermediate on-shell resonances, the classic example being the decay of a heavy gluino through a lighter on-shell intermediate squark. While the mass of the intermediate resonance is a priori unknown, in symmetric event topologies the two decay chains are identical, so one may still impose the condition that the mass of the intermediate resonance (whatever its value) ends up being equal in the two decay chains [41] (for specific applications to $H \rightarrow \tau^+\tau^-$ and $H \rightarrow WW$ decay, see [54] and [56], respectively). Adding such on-shell constraints further restricts the allowed domain of values for the components of the individual invisible momenta and in general leads to a different outcome from the minimization procedure, resulting in a new set of kinematic variables.²

In this paper we shall extend the discussion from [41], which focused only on intermediate resonances, i.e., particles appearing in the decay chain in between the decaying parent

¹Supersymmetry aficionados should not confuse M_2 with the wino mass parameter.

²Note that it is not possible to add such constraints in the case of transverse variables like M_T , M_{T2} , M_{CT} , etc.

and the corresponding daughter. In particular, we shall allow ourselves to also consider resonances which appear “outside” the parent-daughter system, e.g., progenitor particles upstream from the parents, or descendant particles downstream from the daughters. The benefits from this generalization will become clear in the physics examples studied below.

In the paper, we study the mathematical properties of these on-shell constrained M_2 kinematic variables and propose several novel techniques for mass measurements and for disambiguating alternative event topologies. Our main results are:

- *We find that differential distributions of the constrained M_2 variables exhibit sharper kinematic endpoints, making them easier to measure in the presence of backgrounds.*

This is because, as expected, the addition of on-shell kinematic constraints generally increases the value of the corresponding M_2 variable, thus providing a more stringent lower bound on the mass of the parent. The sharper endpoints would ultimately lead to an improvement in the precision with which the parent masses can be determined experimentally.

- *We propose a new method for measuring the mass of a heavy resonance in a SUSY decay chain, by using the invisible momenta found during the M_2 minimization.*

The standard procedure so far has been to treat that resonance as a parent particle in a suitably defined subsystem of the event [39], then measure the upper kinematic endpoint of the corresponding M_{T2} distribution. Instead, here we treat the resonance as an on-shell constraint to be applied during the minimization process while calculating the M_2 variable for a suitably defined subsystem (which may or may not extend over the resonance itself). Since the M_2 minimization procedure selects a unique configuration for the individual invisible momenta, one has all the information required to reconstruct the mass of this hypothetical resonance directly. The key observation, supported in our examples in section 4.2 below, is that the peak of that mass distribution is very well correlated with the true mass of the resonance. The spirit of our method is similar to MAOS reconstruction [48, 49, 52, 53, 57, 63] and the M_{2C} approach [50]. The difference is that we do not rely on preliminary measurements of kinematic endpoints; the measurement is instead done from first principles.

- *We find that this new method, in combination with other standard techniques, can be used to determine the mass of the invisible (dark matter) particles.*

An interesting feature of the method just described is that the result exhibits a different functional dependence on the test daughter mass than the results from analogous methods based on M_{T2} or invariant mass kinematic endpoints. This means that one is able to obtain the true daughter mass by simply putting together the functional parent-daughter mass relationship obtained from our method and from the other canonical methods in the literature — the true answer is given by the crossing point of the different curves. This technique is complementary to the M_{T2} “kink” method where one looks for a kink instead of a crossing point [35–38, 43] (other techniques for measuring the absolute daughter mass are described in [11, 42, 58–60]).

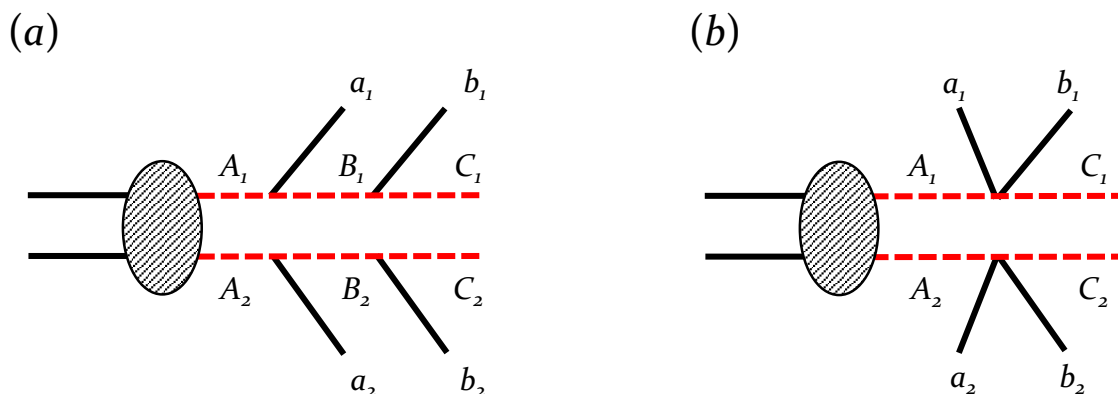


Figure 1. The decay topologies under consideration in this paper. In diagram (a), each parent particle, A_i , ($i = 1, 2$) decays to two visible particles, a_i and b_i , and an invisible daughter particle, C_i , through an intermediate on-shell resonance, B_i . In diagram (b), the intermediate state, B_i , is absent (or very heavy) and the $A_i \rightarrow a_i b_i C_i$ decay is a three body process.

- We propose methods for identifying the event topology and resolving combinatorial ambiguities.

The large variety of on-shell constrained M_2 variables allows us to address a long standing problem in SUSY phenomenology, namely, the question of identifying the correct event topology. There are two aspects of the problem — first, resolving the combinatorial ambiguities in assigning the observed final state particles to the hypothesized event topology [61–63], and second, validation of the hypothesized event topology itself, e.g., the partitioning into two decay chains [64, 65], the number of invisible particles [65–68], the number of intermediate on-shell resonances [64, 65], etc. We can use the fact that the different versions of our on-shell constrained M_2 variables have different assumptions about the underlying event topology built in. Thus, by comparing results obtained with different M_2 variables, we can test those assumptions, for example:

1. In section 5.1 we design a method which tests for the presence of intermediate on-shell resonances in the SUSY decay chain, i.e., distinguishes between a sequence of two 2-body decays and a single 3-body decay.
2. In section 5.2 we address the question of the proper sequence in which the visible particles get emitted along a SUSY decay chain. We use the invisible particle momenta selected by the M_2 minimization procedure to construct Dalitz-type plots involving invariant masses of suitable particle pairs. The correct ordering of the visible particle is then determined by comparing the characteristic shapes of those plots.
3. A similar idea, illustrated in section 5.3, can be used to test whether the events are symmetric, i.e., whether the two decay chains are the same [44].

The paper is organized as follows. In section 2, we specify the process studied (depicted in figure 1) and introduce our conventions and terminology. We then proceed to define all possible on-shell constrained M_2 variables for that process (a total of 12 variables altogether, listed in table 2). However, not all of those variables are independent — section 3 discusses the existing relationships among them, including the connection to the Cambridge M_{T2} variable.³ The subsequent sections demonstrate the utility of those variables for practical applications: mass measurements from kinematic endpoints (section 4.1), mass measurements from M_2 -assisted peak reconstruction (section 4.2), and topology disambiguation (section 5). Section 6 is reserved for our conclusions.

2 Notations and setup

2.1 The physics process

In this paper we shall consider the generic processes depicted in figure 1. We assume the pair production of two heavy particles, A_1 and A_2 , which decay in a similar fashion:

$$A_i \rightarrow a_i b_i C_i, \quad (i = 1, 2). \tag{2.1}$$

The process (2.1) may occur either through on-shell intermediate resonances, B_i , as in figure 1(a), or as a genuine three-body decay, as in figure 1(b). The particles, C_i , are invisible in the detector — in realistic models, their role is typically played by some dark matter candidate, e.g., the lightest supersymmetric particle (LSP) in supersymmetry. The particles, a_i and b_i , are SM particles which are visible in the detector, thus their 4-momenta $p_{a_1}^\mu$, $p_{b_1}^\mu$, $p_{a_2}^\mu$, and $p_{b_2}^\mu$ are measured known quantities. In contrast, the 4-momenta of the C_i , which we shall denote by q_i^μ , are a priori unknown,⁴ and are only constrained by the \vec{P}_T measurement:

$$\vec{q}_{1T} + \vec{q}_{2T} = \vec{P}_T. \tag{2.2}$$

The masses of the particles along the red dashed lines in figure 1 are denoted by m_{A_1} , m_{B_1} , \dots , m_{C_2} . The process (2.1) depicted in figure 1 covers a large class of physically interesting and motivated scenarios, including dilepton events from top pair production and decay, stop decays in supersymmetry ($\tilde{t} \rightarrow b\ell\tilde{\nu}_\ell$), and many more.

In what follows, we shall assume that all four visible particles a_i and b_i in figure 1 are distinguishable. As already mentioned in the introduction, depending on the nature of the visible particles a_i and b_i , various combinatorial issues may arise, e.g.:

1. Should the four visible particles be partitioned as 2+2, 1+3, or 0+4? This question can be answered relatively easily by studying suitable invariant mass distributions of the visible particles [64].

³Readers who are mostly interested in the practical applications of the M_2 variables and wish to skip over the math are invited to jump straight to section 3.4, where they will find a summary of the main results from section 3.

⁴Note that in our notation, the letter “p” is used for measured momenta, while the letter “q” refers to the unknown momenta of invisible particles. Since for the process of figure 1 there are only two invisible particles in the final state, we simplify the notation by using \vec{q}_i instead of the clumsier \vec{q}_{C_i} .

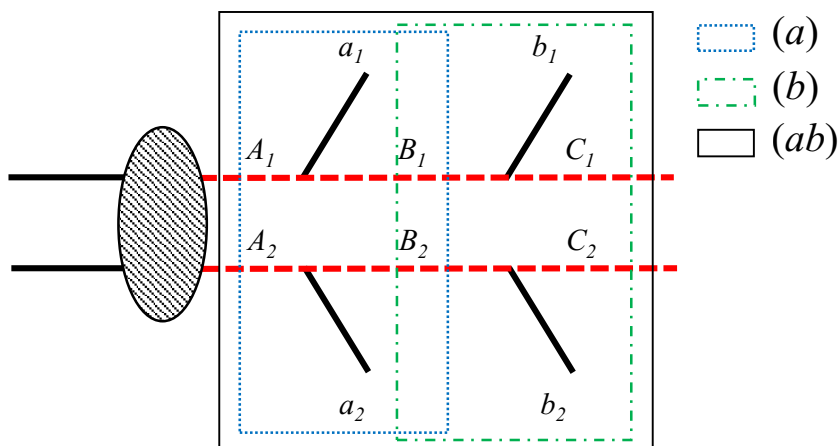


Figure 2. The decay process from figure 1(a) with the corresponding subsystems explicitly delineated. The blue dotted, green dot-dashed, and black solid lines indicate the subsystems (a), (b), and (ab), respectively.

2. Another question is, *which* visible particles belong to the first decay chain (a_1, b_1) and which belong to the second (a_2, b_2). Two possible approaches have been pursued: first, by applying suitable cuts, one could try to increase the chances of picking the correct pairwise assignment [61–63]. Alternatively, one could consider all possible assignments and then try to subtract out the contributions from wrong assignments (e.g., by the mixed event subtraction technique [3]).
3. Finally, when a_i is distinguishable from b_i , one could also ask which of these two particles was emitted first and which came second. The answer to this question will be the subject of section 5.2.

2.2 M_2 subsystems and the particle family tree

As first discussed in the context of the M_{T2} variable [39], one can proliferate the number of useful measurements by considering different subsystems within the original event. The subsystems are defined by the sets of visible particles which are used to construct an M_{T2} variable (see figure 2):

- The (ab) subsystem, indicated by the solid black box in figure 2. Here one uses both types of visible particles, a_i and b_i , treating A_i as parent particles and C_i as daughter particles.
- The (a) subsystem, shown by the blue dotted box in figure 2. Now one uses only the visible particles, a_i , but not b_i . The A_i particles are again treated as parents, but the daughters are now the B_i particles.
- The (b) subsystem, depicted by the green dot-dashed box in figure 2. Now the visible particles, b_i , are used, but not a_i . The parents are the B_i particles and the daughters are the C_i particles.

Subsystem	Parents P_i	Daughters D_i	Relatives R_i
(ab)	A_i	C_i	B_i
(a)	A_i	B_i	C_i
(b)	B_i	C_i	A_i

Table 1. The roles played by different particles depending on the subsystem under consideration.

In this paper, the M_{T2} variables corresponding to these three subsystems will be denoted as⁵ $M_{T2}(ab)$, $M_{T2}(a)$, and $M_{T2}(b)$; the same convention will be used for the M_2 variables defined below.

We see that, depending on our choice of subsystem, each particle from figure 1(a) can be classified into one of the following three categories (summarized also in table 1):

- **Parents.** These are the two particles at the top of the decay chains in a given subsystem. In the following, we shall denote the parents by P_i , ($i = 1, 2$) and their masses by M_{P_i} . The M_2 kinematic variables in section 2.3 below will be defined by a suitable minimization of the parent masses, M_{P_i} , over the unknown components of the invisible momenta [34].
- **Daughters.** These are the two particles at the end of the decay chains in a given subsystem. They may or may not be LSPs; see table 1. The daughters will be denoted by D_i and their masses by M_{D_i} . Each parent mass, M_{P_i} , is a function of the corresponding daughter mass, M_{D_i} , which is a priori unknown. Thus when calculating parent masses, one must always specify a test daughter mass parameter, which will be denoted by \tilde{m} throughout this paper. For the most part, we shall be considering “symmetric” events, i.e., events in which the two decay chains are the same, and thus there is a single test mass \tilde{m} . The generalization to the asymmetric case is straightforward [44] — one simply needs to introduce separate test masses, \tilde{m}_i , for the upper and the lower decay chains in figure 1.
- **Relatives.** These are particles which are neither parents nor daughters; see table 1. The relatives will be denoted by R_i and their masses by M_{R_i} . Since the decay chains in figure 1(a) involve only 3 new particles, there is always only one possible relative, which may appear upstream (as in the case of subsystem (b)), downstream (as in the case of subsystem (a)), or midstream (as in the case of subsystem (ab)). In other words, for the simple example of figure 1(a), the identity of the relative is uniquely fixed once we specify the subsystem under consideration, so we do not need to introduce any additional notation regarding the relatives. However, in more complicated examples with longer decay chains, there will be several relatives, and one would have to invent some notation to distinguish among them.

⁵Contrast this to the superscript notation previously used in [39, 69]: M_{T2}^{220} , M_{T2}^{221} , and M_{T2}^{210} .

2.3 Definition of the on-shell constrained M_2 variables

We start by reviewing the standard definition of the canonical M_{T2} variable [32]. Consider the transverse masses $M_{TP_i}(\vec{q}_{iT}, \tilde{m})$ of the two parent particles and then minimize the larger of them with respect to the transverse⁶ components of the invisible momenta, subject to the \vec{P}_T constraint, (2.2):

$$M_{T2}(\tilde{m}) \equiv \min_{\vec{q}_{1T}, \vec{q}_{2T}} \{ \max [M_{TP_1}(\vec{q}_{1T}, \tilde{m}), M_{TP_2}(\vec{q}_{2T}, \tilde{m})] \}. \tag{2.3}$$

$$\vec{q}_{1T} + \vec{q}_{2T} = \vec{P}_T$$

Following [34], one could instead start with the actual parent masses, M_{P_i} , and define the 3+1-dimensional analogue of (2.3) as

$$M_2(\tilde{m}) \equiv \min_{\vec{q}_1, \vec{q}_2} \{ \max [M_{P_1}(\vec{q}_1, \tilde{m}), M_{P_2}(\vec{q}_2, \tilde{m})] \}, \tag{2.4}$$

$$\vec{q}_{1T} + \vec{q}_{2T} = \vec{P}_T$$

where the minimization is performed over the 3-component momentum vectors \vec{q}_1 and \vec{q}_2 . As stated in [34, 50], the two definitions (2.3) and (2.4) are equivalent, in the sense that the resulting two variables, M_{T2} and M_2 , will have the same numerical value (a proof of this claim can be found in section 3.1 below). Nevertheless, for our purposes here, the definition (2.4) is much more convenient, for the following reasons:

- The minimization in (2.4) is done over the full 3-momentum vectors \vec{q}_1 and \vec{q}_2 , and thus it also selects their longitudinal components q_{1z} and q_{2z} . This completely fixes the kinematics of the event.
- The 3+1-dimensional language of eq. (2.4) makes it very easy to impose the additional on-shell constraints that arise in specific event topologies [41].

Given that here we are interested in the specific event topology of figure 1(a), it makes sense to consider additionally constrained versions of (2.4). There are two⁷ additional assumptions one can make: that the parents P_i are the same (or, more generally, that they have the same mass)

$$M_{P_1} = M_{P_2}, \tag{2.5}$$

or that the relatives have the same mass

$$M_{R_1} = M_{R_2}. \tag{2.6}$$

Of course, one could also impose (2.5) and (2.6) simultaneously, giving us a total of 4 possibilities. We choose to enumerate these 4 cases by adding two additional subscripts

⁶The longitudinal components q_{1z} and q_{2z} are irrelevant since they do not enter the definition of the transverse masses M_{TP_i} .

⁷Recall that throughout this paper we are already making the assumption that the daughters are the same.

on the M_2 variable to indicate whether the constraints (2.5) and (2.6) were imposed during the minimization or not. The first subscript always refers to the parents and their constraint, (2.5), while the second subscript always refers to the relatives and their constraint, (2.6). The value of the subscript will be “C” if the corresponding constraint is imposed and “X” otherwise. Altogether, we have the following four types of variables:

$$M_{2XX} \equiv \min_{\vec{q}_1, \vec{q}_2} \{ \max [M_{P_1}(\vec{q}_1, \tilde{m}), M_{P_2}(\vec{q}_2, \tilde{m})] \}, \quad (2.7)$$

$$\vec{q}_{1T} + \vec{q}_{2T} = \vec{P}_T$$

$$M_{2CX} \equiv \min_{\vec{q}_1, \vec{q}_2} \{ \max [M_{P_1}(\vec{q}_1, \tilde{m}), M_{P_2}(\vec{q}_2, \tilde{m})] \}, \quad (2.8)$$

$$\vec{q}_{1T} + \vec{q}_{2T} = \vec{P}_T$$

$$M_{P_1} = M_{P_2}$$

$$M_{2XC} \equiv \min_{\vec{q}_1, \vec{q}_2} \{ \max [M_{P_1}(\vec{q}_1, \tilde{m}), M_{P_2}(\vec{q}_2, \tilde{m})] \}, \quad (2.9)$$

$$\vec{q}_{1T} + \vec{q}_{2T} = \vec{P}_T$$

$$M_{R_1}^2 = M_{R_2}^2$$

$$M_{2CC} \equiv \min_{\vec{q}_1, \vec{q}_2} \{ \max [M_{P_1}(\vec{q}_1, \tilde{m}), M_{P_2}(\vec{q}_2, \tilde{m})] \}. \quad (2.10)$$

$$\vec{q}_{1T} + \vec{q}_{2T} = \vec{P}_T$$

$$M_{P_1} = M_{P_2}$$

$$M_{R_1}^2 = M_{R_2}^2$$

A few comments are in order. In the equations above, the masses of the parents, M_{P_i} , and the masses of the relatives, M_{R_i} , are always understood to be functions of the invisible 3-momenta \vec{q}_i . Thus the constraints $M_{P_1} = M_{P_2}$ and $M_{R_1} = M_{R_2}$ simply further restrict the allowed values for those momenta (in addition to the missing transverse momentum constraint, (2.2)). Obviously, the unrestricted variable M_{2XX} is nothing but the variable defined in (2.4), so in this sense the pair of indices “XX” may seem redundant. Nevertheless, given the existence of the other three choices (2.8)–(2.10), it seems wise to indicate explicitly the absence of any on-shell constraints in that case.

We note that while a parent mass squared is always positive, there is one case when the mass squared of a *relative* can be negative — for subsystem (a), the relative particle is C_i and its mass squared is $M_{R_i}^2 = (p_{B_i} - p_{b_i})^2$ (see figure 1(a)). Each of the 4-momenta $p_{B_i}^\mu$ and $p_{b_i}^\mu$ is time-like, but their difference may be time-like or space-like. Thus, in that situation, one has the option of additionally requiring positivity of the masses squared of relative particles. In this paper we shall not do that; we shall allow the relative masses squared obtained after the minimization to have either sign.⁸ This is why in eqs. (2.9) and (2.10), the constraint for the relatives is written as $M_{R_1}^2 = M_{R_2}^2$ instead of simply as $M_{R_1} = M_{R_2}$.

Applying (2.7)–(2.10) to the three possible subsystems of figure 2, we obtain a total of 12 on-shell constrained M_2 variables which are listed in table 2. Some of these variables

⁸The reason is that the momenta obtained in the minimization do not necessarily have to correspond to the momenta of any physical particles; as our reconstruction ansatz may not reflect the actual process. A

Subsystem (<i>ab</i>)		Subsystem (<i>a</i>)		Subsystem (<i>b</i>)	
variable	constraints	variable	constraints	variable	constraints
$M_{2XX}(ab)$	—	$M_{2XX}(a)$	—	$M_{2XX}(b)$	—
$M_{2CX}(ab)$	$M_{A_1}^2 = M_{A_2}^2$	$M_{2CX}(a)$	$M_{A_1}^2 = M_{A_2}^2$	$M_{2CX}(b)$	$M_{B_1}^2 = M_{B_2}^2$
$M_{2XC}(ab)$	$M_{B_1}^2 = M_{B_2}^2$	$M_{2XC}(a)$	$M_{C_1}^2 = M_{C_2}^2$	$M_{2XC}(b)$	$M_{A_1}^2 = M_{A_2}^2$
$M_{2CC}(ab)$	$M_{A_1}^2 = M_{A_2}^2$ $M_{B_1}^2 = M_{B_2}^2$	$M_{2CC}(a)$	$M_{A_1}^2 = M_{A_2}^2$ $M_{C_1}^2 = M_{C_2}^2$	$M_{2CC}(b)$	$M_{B_1}^2 = M_{B_2}^2$ $M_{A_1}^2 = M_{A_2}^2$

Table 2. A summary of the twelve M_2 variables defined in the text. For each of the three subsystems (*ab*), (*a*), and (*b*), one may choose to apply neither, one, or both of the constraints (2.5) and (2.6). In each case, the trial daughter masses are assumed to be the same, \tilde{m} .

(M_{2XX} and M_{2CX}) are simply 3+1 dimensional versions of M_{T2} [34, 41], while $M_{2XC}(ab)$ and $M_{2CC}(ab)$ were mentioned in [41]. The remaining 4 variables $M_{2XC}(a)$, $M_{2CC}(a)$, $M_{2XC}(b)$, and $M_{2CC}(b)$ are new. Notice that the meaning of a “C” index depends on both its position (first or second) and on the chosen subsystem. For example, a “C” index sitting in first position, $M_{2CX}(ab)$, implies equality of the parents: $M_{A_1}^2 = M_{A_2}^2$, while when sitting in second position, $M_{2XC}(ab)$, it indicates equality of the relatives: $M_{B_1}^2 = M_{B_2}^2$. Similarly, contrast analogous variables in the three subsystems: $M_{2XC}(ab)$ is calculated assuming $M_{B_1}^2 = M_{B_2}^2$; $M_{2XC}(a)$ is obtained with $M_{C_1}^2 = M_{C_2}^2$; while $M_{2XC}(b)$ implies $M_{A_1}^2 = M_{A_2}^2$.

At this point, it is instructive to consider a couple of specific examples, in order to better familiarize the reader with our notation. Consider, for example, $M_{2CC}(ab)$. It applies to the (*ab*) subsystem, where A_i are the parents, C_i are the daughters (with test masses \tilde{m}) and B_i are the relatives. Both indices are “on”, so the constraints (2.5) and (2.6) are applied. Explicitly, we have

$$\begin{aligned}
 M_{2CC}^2(ab) &\equiv \min_{\vec{q}_1, \vec{q}_2} \left\{ \max \left[(p_{a_1} + p_{b_1} + q_1)^2, (p_{a_2} + p_{b_2} + q_2)^2 \right] \right\}. & (2.11) \\
 q_1^2 &= \tilde{m}^2 \\
 q_2^2 &= \tilde{m}^2 \\
 \vec{q}_{1T} + \vec{q}_{2T} &= \vec{P}_T \\
 (p_{a_1} + p_{b_1} + q_1)^2 &= (p_{a_2} + p_{b_2} + q_2)^2 \\
 (p_{b_1} + q_1)^2 &= (p_{b_2} + q_2)^2
 \end{aligned}$$

As another example, consider $M_{2XC}(a)$. It applies to the (*a*) subsystem with A_i as parents, B_i as daughters, and C_i as relatives. Note that the test mass, \tilde{m} , now refers to m_{B_i} . The parents are not assumed to have equal masses, but the relatives are, thus

$$\begin{aligned}
 M_{2XC}^2(a) &\equiv \min_{\vec{q}_1, \vec{q}_2} \left\{ \max \left[(p_{a_1} + p_{b_1} + q_1)^2, (p_{a_2} + p_{b_2} + q_2)^2 \right] \right\}. & (2.12) \\
 (q_1 + p_{b_1})^2 &= \tilde{m}^2
 \end{aligned}$$

similar dilemma arises in the case of M_{T2} , when some invisible momenta found by the minimization may turn out to be anomalously large, well beyond the scale of the collider energy.

$$\begin{aligned}
 (q_2 + p_{b_2})^2 &= \tilde{m}^2 \\
 \vec{q}_{1T} + \vec{q}_{2T} &= \vec{P}_T \\
 q_1^2 &= q_2^2
 \end{aligned}$$

Our final example is $M_{2XC}(b)$, which reads

$$\begin{aligned}
 M_{2XC}^2(b) &\equiv \min_{\vec{q}_1, \vec{q}_2} \{ \max [(p_{b_1} + q_1)^2, (p_{b_2} + q_2)^2] \}. & (2.13) \\
 q_1^2 &= \tilde{m}^2 \\
 q_2^2 &= \tilde{m}^2 \\
 \vec{q}_{1T} + \vec{q}_{2T} &= \vec{P}_T \\
 (p_{a_1} + p_{b_1} + q_1)^2 &= (p_{a_2} + p_{b_2} + q_2)^2
 \end{aligned}$$

If it wasn't for the very last constraint, this would have been simply $M_{T2}(b)$, i.e., the M_{T2} variable for the (b) subsystem, in the presence of upstream momentum $p_{a_1} + p_{a_2}$. However, the constraint for the relatives $M_{A_1} = M_{A_2}$ is non-trivial and leads to a qualitatively new result.

3 Relations among the M_2 type variables and M_{T2}

In this section, we examine the relations among the four M_2 type variables defined in the preceding section and compare them to the conventional M_{T2} variable. For concreteness, we shall focus on the (ab) subsystem⁹ and consider the set

$$M_{T2}(ab), \quad M_{2XX}(ab), \quad M_{2CX}(ab), \quad M_{2XC}(ab), \quad M_{2CC}(ab). \quad (3.1)$$

We shall perform our study under the assumption that the intermediate particles, B_i , are *on-shell* as in figure 1(a). The *off-shell* scenario of figure 1(b) will be discussed in section 5 in the context of applications. In section 3.1, we first show that the three variables, $M_{2XX}(ab)$, $M_{2CX}(ab)$, and $M_{T2}(ab)$, have the same value event-by-event. Informed by this discussion, in section 3.2, we shall also discuss the question of the uniqueness of the invisible momentum configurations found in the process of minimization. Then, in section 3.3, we shall discuss the hierarchy among the three distinct variables on the list (3.1), namely $M_{2CX}(ab)$, $M_{2XC}(ab)$, and $M_{2CC}(ab)$. In section 3.4, we summarize the main results from section 3.

3.1 Equivalence theorem among M_{2XX} , M_{2CX} , and M_{T2}

Applying the general definition (2.3) to the (ab) subsystem, $M_{T2}(ab)$ can be expressed as follows [32]:

$$\begin{aligned}
 M_{T2}^2(ab) &= \min_{\vec{q}_{1T}, \vec{q}_{2T}} \{ \max [M_{TA_1}^2(\vec{q}_{1T}, \tilde{m}), M_{TA_2}^2(\vec{q}_{2T}, \tilde{m})] \} & (3.2) \\
 \vec{q}_{1T} + \vec{q}_{2T} &= \vec{P}_T
 \end{aligned}$$

⁹However, our results will hold for the other two subsystems as well; see the summary in section 3.4.

where

$$M_{TA_i}^2(\vec{q}_{iT}, \tilde{m}) = \tilde{m}^2 + m_{v_i}^2 + 2[E_{v_iT}E_{q_iT} - \vec{p}_{v_iT} \cdot \vec{q}_{iT}], \quad (3.3)$$

v_i is the visible state $a_i + b_i$ belonging to the i -th decay chain:

$$\vec{p}_{v_i} \equiv \vec{p}_{a_i} + \vec{p}_{b_i}, \quad (3.4)$$

and E_T denotes the transverse energy:

$$E_{v_iT} = \sqrt{m_{v_i}^2 + \vec{p}_{v_iT}^2}; \quad E_{q_iT} = \sqrt{\tilde{m}^2 + \vec{q}_{iT}^2}. \quad (3.5)$$

Using (2.7), we can construct $M_{2XX}(ab)$ in a similar manner:

$$M_{2XX}^2(ab) = \min_{\substack{\vec{q}_{1T}, \vec{q}_{2T} \\ q_{1z}, q_{2z}}} \left\{ \max [M_{A_1}^2(\vec{q}_{1T}, q_{1z}, \tilde{m}), M_{A_2}^2(\vec{q}_{2T}, q_{2z}, \tilde{m})] \right\}. \quad (3.6)$$

$$\vec{q}_{1T} + \vec{q}_{2T} = \vec{P}_T$$

The invariant masses of A_1 and A_2 can be written as

$$M_{A_i}^2(\vec{q}_{iT}, q_{iz}, \tilde{m}) = \tilde{m}^2 + m_{v_i}^2 + 2[E_{v_iT}E_{q_iT} \cosh(\Delta\eta_i) - \vec{p}_{v_iT} \cdot \vec{q}_{iT}], \quad (3.7)$$

where $\Delta\eta_i$ is the rapidity difference between the visible state v_i and particle C_i . The minimization of (3.6) over the transverse momenta, \vec{q}_{iT} , and the longitudinal momenta, q_{iz} , can in principle be done in any order, but it is much easier to minimize over q_{iz} first, since they do not enter the \vec{P}_T constraint. Furthermore, the longitudinal momenta are decoupled from each other, and thus the two minimizations can be performed independently. We can therefore rewrite (3.6) as

$$M_{2XX}^2(ab) = \min_{\vec{q}_{1T}, \vec{q}_{2T}} \left\{ \max \left[\min_{q_{1z}} \{M_{A_1}^2(\vec{q}_{1T}, q_{1z}, \tilde{m})\}, \min_{q_{2z}} \{M_{A_2}^2(\vec{q}_{2T}, q_{2z}, \tilde{m})\} \right] \right\}. \quad (3.8)$$

$$\vec{q}_{1T} + \vec{q}_{2T} = \vec{P}_T$$

where we have switched the order of the $\min_{q_{iz}} \{ \}$ and $\max \{ \}$ operations. The minimization over q_{iz} is equivalent to minimization over $\Delta\eta_i$. From (3.7) it is easy to see that the minimum is obtained for $\Delta\eta_i = 0$, which reduces (3.7) to (3.3), so that (3.8) becomes simply

$$M_{2XX}^2(ab) = \min_{\vec{q}_{1T}, \vec{q}_{2T}} \left\{ \max [M_{TA_1}^2(\vec{q}_{1T}, \tilde{m}), M_{TA_2}^2(\vec{q}_{2T}, \tilde{m})] \right\}. \quad (3.9)$$

$$\vec{q}_{1T} + \vec{q}_{2T} = \vec{P}_T$$

Comparing (3.9) with (3.2), we see that [34]

$$M_{2XX}^2(ab) = M_{T2}^2(ab). \quad (3.10)$$

Moving our attention to $M_{2CX}(ab)$, we see that the proof of its equivalence to $M_{T2}(ab)$ is not difficult either. A formal proof based on the method of Lagrange multipliers is presented in appendix A, so here we shall give just the heuristic argument.

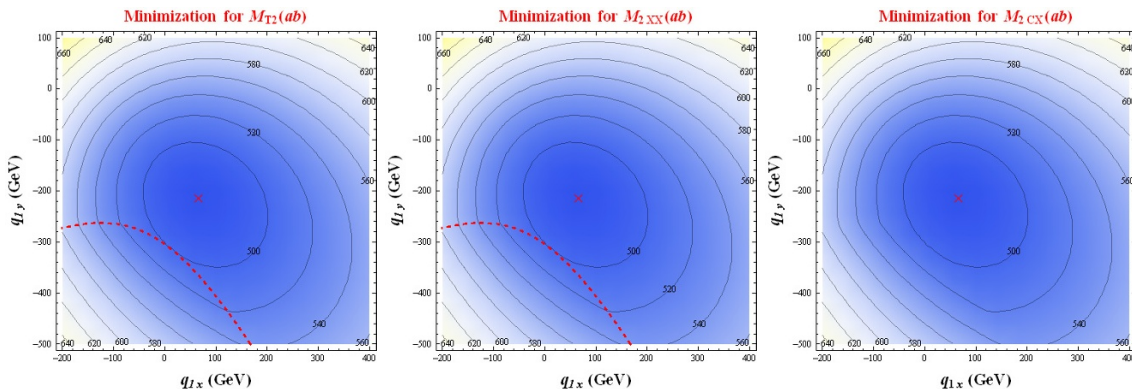


Figure 3. Contour plots of the functions $f_{T2}(\vec{q}_{1T})$ (left panel), $f_{2XX}(\vec{q}_{1T})$ (middle panel), and $f_{2CX}(\vec{q}_{1T})$ (right panel) in the plane of \vec{q}_{1T} . The chosen event leads to an unbalanced solution for $M_{T2}(ab)$. (The red dashed curve delineates the points with $M_{TA_1} = M_{TA_2}$.) The red \times symbol marks the global minimum of the function in each case. At the minimum, $M_{T2}(ab) = M_{2XX}(ab) = M_{2CX}(ab) = 483.71$ GeV, and the corresponding solution for \vec{q}_{1T} is given by $\vec{q}_{1T}^\times = (66.09, -212.90)$ GeV.

Starting from eq. (3.10), without any loss of generality we can assume that $M_{2XX}^2(ab)$ is obtained by minimizing $M_{A_1}^2$, i.e., that in the neighborhood of the minimum, we have $M_{A_2}^2 < M_{A_1}^2$, and thus the max function in the definition (3.6) picks up $M_{A_1}^2$ for the minimization. The parent constraint $M_{A_1} = M_{A_2}$ is clearly not satisfied, but this can be fixed without changing the value obtained in eq. (3.10). Keeping \vec{q}_{1T} , q_{1z} , and \vec{q}_{2T} fixed to their values at the $M_{2XX}^2(ab)$ minimum, we start varying q_{2z} in the direction of increasing M_{A_2} . Eventually, we will find a value for q_{2z} for which M_{A_2} will reach M_{A_1} and the parent constraint $M_{A_1} = M_{A_2}$ will be satisfied. In the meantime, nothing has changed regarding the $M_{A_1}^2$ function: since \vec{q}_{1T} and q_{1z} were kept the same as before, its value is still given by (3.10).

This simple exercise shows that by adjusting the longitudinal invisible momenta, one can always turn M_{2XX} into M_{2CX} :

$$M_{2CX}^2(ab) = M_{2XX}^2(ab). \tag{3.11}$$

The main lesson is that this comes at a price — the invisible momentum configuration selected by the M_{2XX} minimization may be different from the configuration obtained in the M_{2CX} minimization. We shall have much more to say about this in section 3.2 below.

Combining (3.11) with (3.10), we also trivially obtain the relation [41]

$$M_{2CX}^2(ab) = M_{T2}^2(ab). \tag{3.12}$$

In order to illustrate (3.10)–(3.12) pictorially, in figure 3 we plot the three functions

$$f_{T2}(\vec{q}_{1T}) \equiv \max \left[M_{TA_1}(\vec{q}_{1T}, \vec{m}), M_{TA_2}(\vec{P}_T - \vec{q}_{1T}, \vec{m}) \right], \tag{3.13}$$

$$f_{2XX}(\vec{q}_{1T}) \equiv \min_{q_{1z}, q_{2z}} \left\{ \max \left[M_{A_1}(\vec{q}_{1T}, q_{1z}, \vec{m}), M_{A_2}(\vec{P}_T - \vec{q}_{1T}, q_{2z}, \vec{m}) \right] \right\}, \tag{3.14}$$

$$f_{2CX}(\vec{q}_{1T}) \equiv \min_{\substack{q_{1z}, q_{2z} \\ M_{A_1} = M_{A_2}}} \left\{ \max \left[M_{A_1}(\vec{q}_{1T}, q_{1z}, \tilde{m}), M_{A_2}(\vec{P}_T - \vec{q}_{1T}, q_{2z}, \tilde{m}) \right] \right\} \quad (3.15)$$

in the \vec{q}_{1T} plane. (\vec{q}_{2T} is then determined from the \vec{P}_T constraint as $\vec{q}_{2T} = \vec{P}_T - \vec{q}_{1T}$.) These are precisely the functions which need to be minimized over \vec{q}_{1T} in order to obtain the variables $M_{T_2}(ab)$, $M_{2XX}(ab)$, and $M_{2CX}(ab)$, respectively. Note that these functions already contain different number of minimizations over longitudinal momenta: $f_{2XX}(\vec{q}_{1T})$ has two, $f_{2CX}(\vec{q}_{1T})$ has one (the other longitudinal degree of freedom is fixed by the $M_{A_1} = M_{A_2}$ constraint), while $f_{T_2}(\vec{q}_{1T})$ has none. The event chosen for figure 3 was selected such that the associated $M_{T_2}(ab)$ value comes from an unbalanced situation, i.e., the minimum of $f_{T_2}(\vec{q}_{1T})$, marked with the red \times symbol, is at $M_{TA_1} \neq M_{TA_2}$.

Figure 3 demonstrates that the three functions (3.13)–(3.15) are identical, thus justifying the identities (3.10)–(3.12). In other words, once the minimization over the longitudinal components is done for the $M_{2XX}(ab)$ and $M_{2CX}(ab)$ variables, the remaining functions $f_{2XX}(\vec{q}_{1T})$ and $f_{2CX}(\vec{q}_{1T})$ become identical to $f_{T_2}(\vec{q}_{1T})$, so the remaining minimization over \vec{q}_{1T} will converge to the common point marked with the \times symbol. This means that all three variables $M_{2XX}(ab)$, $M_{2CX}(ab)$, and $M_{T_2}(ab)$ not only have a common value, but also select the same *transverse* components \vec{q}_{iT} for the invisible momenta at their respective minima. However, this is not the case for the *longitudinal* invisible momenta, q_{iz} , which will be the subject of the next subsection.

3.2 Uniqueness of the longitudinal momenta found by M_{2XX} and M_{2CX}

As already mentioned in the Introduction, one of the main advantages of the M_2 -type variables over purely transverse analogues like M_{T_2} , M_{CT_2} etc., is that they supply values for not just the transverse, but also the longitudinal components of the invisible particle momenta. The knowledge of the full 4-momentum of each invisible particle enables us to reconstruct the mass of each particle along the decay chain, and in particular the relative particles; see section 4.2. One should keep in mind that the momenta found by the M_2 minimization are *not* the actual momenta of the invisible particles in the event. Nevertheless, the MAOS approach demonstrates that they can be successfully used for reconstruction [48, 49, 57].

Let us now investigate the solutions for q_{1z} and q_{2z} more closely. Consider the starting point of the M_{2XX} calculation, the function

$$G_{2XX}(\vec{q}_{1T}, q_{1z}, q_{2z}) \equiv \max \left[M_{A_1}(\vec{q}_{1T}, q_{1z}, \tilde{m}), M_{A_2}(\vec{P}_T - \vec{q}_{1T}, q_{2z}, \tilde{m}) \right]. \quad (3.16)$$

As we saw in section 3.1, its minimization along the transverse directions \vec{q}_{1T} results in unique solutions; we call them $\vec{q}_{1T}^{(\times)}$. (See the red \times symbols in figure 3). Therefore, for the purposes of discussing the minimization over the longitudinal momentum components, we can fix the transverse momenta, $\vec{q}_{1T} = \vec{q}_{1T}^{(\times)}$, and investigate the q_{iz} dependence of the function

$$g_{2XX}(q_{1z}, q_{2z}) \equiv \max \left[M_{A_1}(\vec{q}_{1T}^{(\times)}, q_{1z}, \tilde{m}), M_{A_2}(\vec{P}_T - \vec{q}_{1T}^{(\times)}, q_{2z}, \tilde{m}) \right]. \quad (3.17)$$

The unconstrained minimization of $g_{2XX}(q_{1z}, q_{2z})$ over q_{1z} and q_{2z} yields the value of M_{2XX} , while minimizing (3.17) subject to the parent constraint, $M_{A_1} = M_{A_2}$, gives the value of M_{2CX} .

Let us first study the effect of the parent constraint¹⁰

$$(p_{a_1} + p_{b_1} + q_1)^2 = (p_{a_2} + p_{b_2} + q_2)^2, \quad (3.18)$$

which can be solved for q_{2z} in terms of q_{1z} :

$$q_{2z} = \frac{p_{v_2z}K \pm E_{v_2}\sqrt{K^2 - E_{q_2T}^2(E_{v_2}^2 - p_{v_2z}^2)}}{E_{v_2}^2 - p_{v_2z}^2}, \quad (3.19)$$

where

$$K \equiv \frac{m_{v_1}^2 - m_{v_2}^2}{2} + E_{q_1}E_{v_1} - \vec{q}_{1T} \cdot \vec{p}_{v_1T} - q_{1z}p_{v_1z} + \vec{q}_{2T} \cdot \vec{p}_{v_2T}. \quad (3.20)$$

One can obtain an analogous expression for q_{1z} in terms of q_{2z} , by substituting $v_1 \leftrightarrow v_2$ and $q_1 \leftrightarrow q_2$ in eqs. (3.19) and (3.20).

A couple of observations can be made from these equations. First, one can easily see from (3.19) that the q_{2z} solution is not uniquely determined, i.e., q_{2z} has a twofold ambiguity for a fixed q_{1z} , unless the expression inside the square root, the *discriminant*, vanishes. The same argument can be made regarding the analogous expression giving q_{1z} in terms of q_{2z} . Then the question becomes whether both q_{1z} and q_{2z} have double roots for some \vec{q}_{1T} . This is where the second observation comes into play. It turns out that for the value of q_{1z} which minimizes the function (3.16), $q_{1z}^{(min)}$, the discriminant in eq. (3.19) is proportional to the difference between the transverse masses of A_1 and A_2 :

$$K^2 - E_{q_2T}^2(E_{v_2}^2 - p_{v_2z}^2)|_{q_{1z}^{(min)}} \propto (M_{TA_1}^2 - M_{TA_2}^2). \quad (3.21)$$

On the other hand, the discriminant that would appear in the expression analogous to (3.19) giving q_{1z} in terms of q_{2z} , will be proportional to $M_{TA_2}^2 - M_{TA_1}^2$, i.e., the difference of the same squared transverse masses, only taken in opposite order. This suggests an interesting complementarity, in which q_{1z} and q_{2z} do not suffer from twofold ambiguities simultaneously, i.e., if q_{2z} has two solutions in eq. (3.19), then q_{1z} is uniquely determined, and vice versa. This observation also reveals the necessary condition for both q_{1z} and q_{2z} to be uniquely determined simultaneously: the transverse masses of A_1 and A_2 must be the same, $M_{TA_1} = M_{TA_2}$, see the red dashed curves in figure 3.

Figure 4, which was made for the same unbalanced event used in figure 3, pictorially illustrates the above discussion. Let us call the two solutions of (3.19) $q_{2z(low)}$ (corresponding to the “−” sign) and $q_{2z(high)}$ (corresponding to the “+” sign). They are plotted in the lower two panels of figure 4 in the \vec{q}_{1T} plane. The remaining momenta are fixed as follows: at each point of the plane, \vec{q}_{2T} is given by the \vec{P}_T condition (2.2), while q_{1z} is chosen so that it minimizes the function (3.16): $q_{1z} = q_{1z}^{(min)}$. The upper two panels of figure 4 show

¹⁰Recall that throughout this section we have in mind the (ab) subsystem.

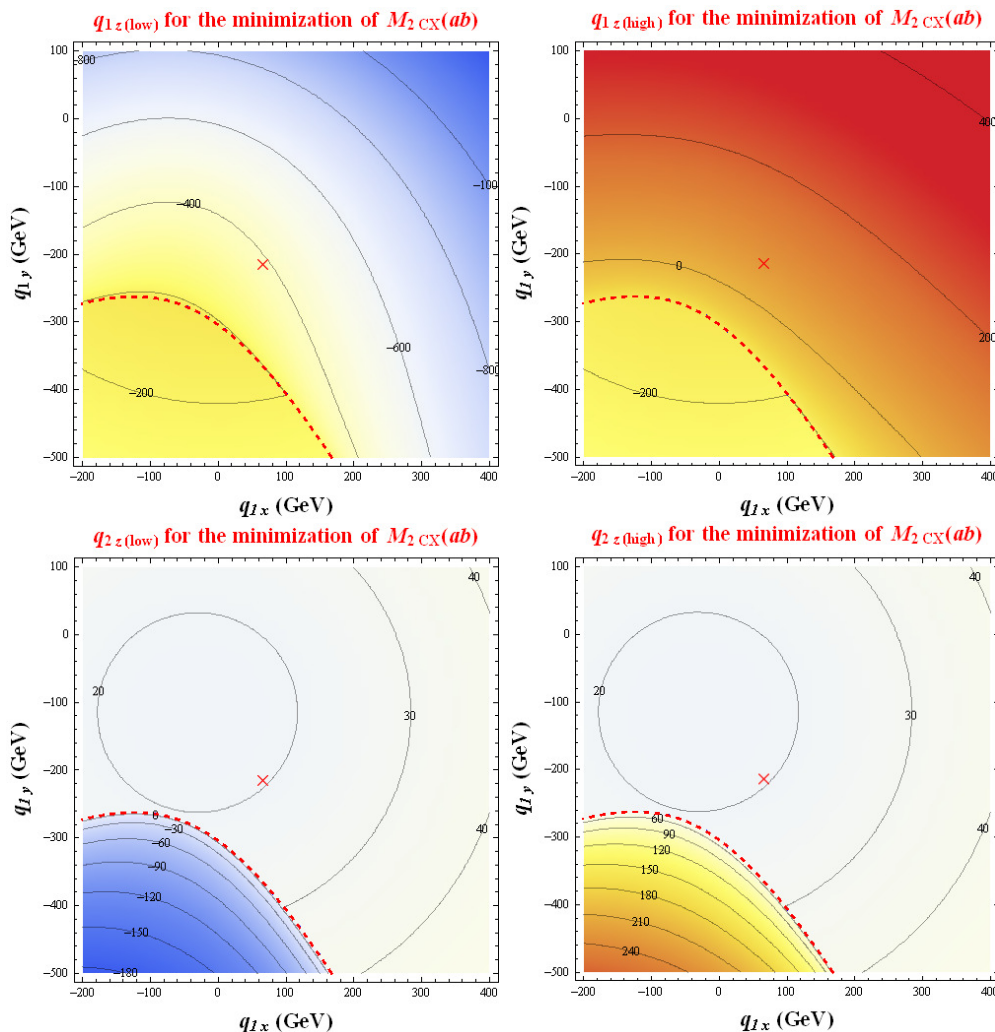


Figure 4. The values of the longitudinal invisible momenta $q_{iz(low)}$ (left panels) and $q_{iz(high)}$ (right panels) which solve the parent constraint, (3.18), in the \vec{q}_{1T} plane (\vec{q}_{2T} is then given by the \vec{P}_T condition (2.2)), for the same event shown in figure 3. The upper row shows $q_{1z(low)}$ and $q_{1z(high)}$ for $q_{2z} = q_{2z}^{(min)}$, while the lower row shows $q_{2z(low)}$ and $q_{2z(high)}$ for $q_{1z} = q_{1z}^{(min)}$. ($q_{iz}^{(min)}$ is always found by minimizing (3.16).) The red dashed curves denote the contours where the solutions to both q_{1z} and q_{2z} are unique.

the analogous plots where the roles of q_{1z} and q_{2z} are reversed — we find q_{2z} by minimizing (3.16), $q_{2z} = q_{2z}^{(min)}$, and then plot the two solutions for q_{1z} , $q_{1z(low)}$ and $q_{1z(high)}$. The red dashed lines delineate the points with balanced solutions for M_{T2} , $M_{TA1} = M_{TA2}$.

Figure 4 confirms that the red dashed line is a watershed boundary — in the region above and to the right of that line we always find two possible values for q_{1z} , but a single value for q_{2z} . Conversely, in the area below and to the left of that line there is always a unique solution for q_{1z} , but two solutions for q_{2z} instead. Now recall that the event depicted in figures 3 and 4 was unbalanced, i.e., the true global minimum was obtained at the red \times point, at which $M_{TA1} \neq M_{TA2}$. This point also happens to be located in the region where

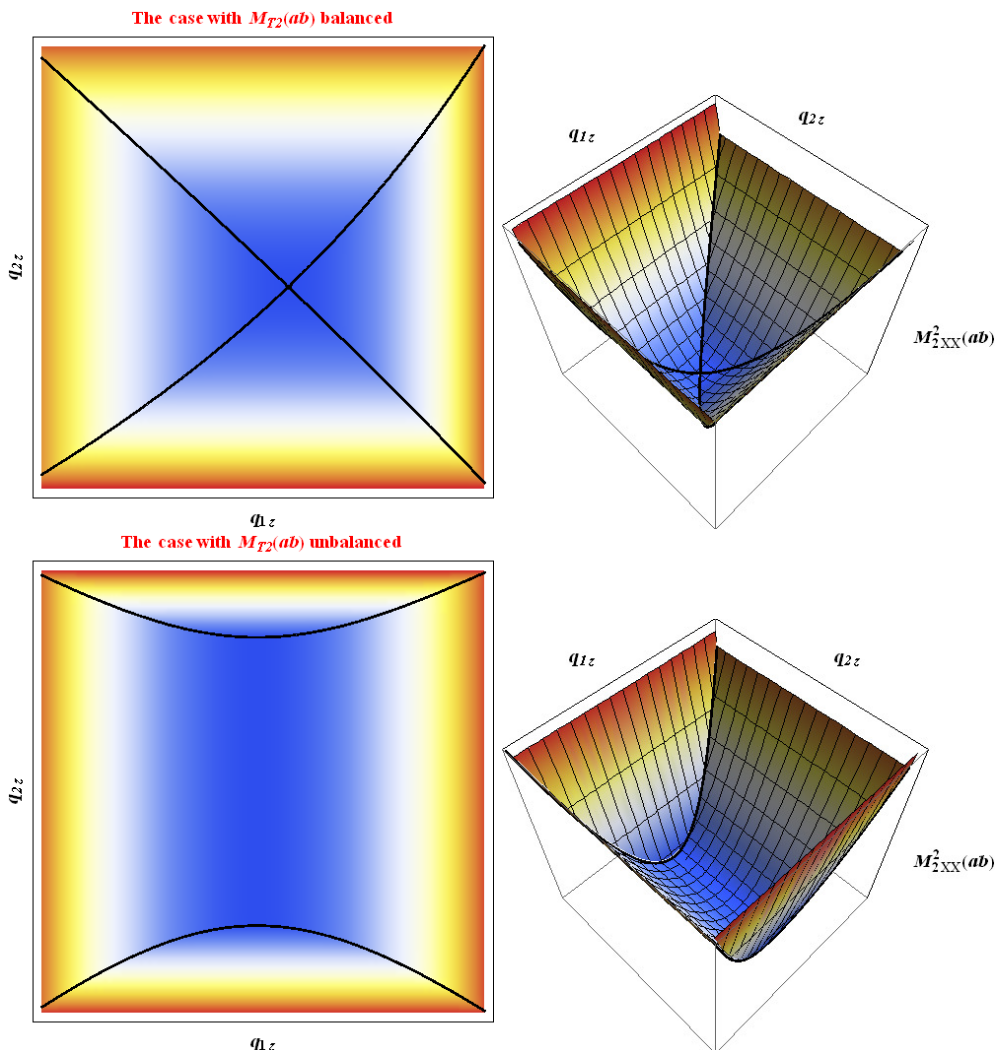


Figure 5. Plot of the function (3.17) in the (q_{1z}, q_{2z}) plane, for a balanced event with $M_{TA_1} = M_{TA_2}$ at the minimum (top row) and an unbalanced event with $M_{TA_1} \neq M_{TA_2}$ at the minimum (bottom row). The left panels are contour plots, while the right panels show the corresponding 3-dimensional view. The black solid curves mark the points satisfying the parent constraint, $M_{A_1} = M_{A_2}$.

the solution for q_{2z} is unique, but the solution for q_{1z} has a twofold ambiguity. On the other hand, if we had chosen a balanced event, the global minimum would fall somewhere on the red dashed line, and both q_{1z} and q_{2z} will be uniquely determined.

Having understood the minimization of $M_{2CX}(ab)$, it is easy to infer the corresponding solutions for q_{1z} and q_{2z} in the case of $M_{2XX}(ab)$. The ambiguity problem is now even more serious, because whenever $q_{iz(low)} \neq q_{iz(high)}$, any value of $q_{iz} \in (q_{iz(low)}, q_{iz(high)})$ is also allowed, i.e., the ambiguity is not just twofold, instead there is a flat direction. However, these ambiguities are present only for unbalanced events — for balanced events, $q_{iz(low)} = q_{iz(high)}$, and the solution for both q_{1z} and q_{2z} is unique. This is pictorially illustrated in figure 5, which shows the function (3.17) as a function of q_{1z} and q_{2z} . Since \vec{q}_{1T} is already

fixed to its correct value, $\vec{q}_{1T}^{(\times)}$, at the global minimum of (3.16), the global unconstrained minimum of the function (3.17) seen in figure 5 corresponds to M_{2XX} , while the constrained minimization along the black solid lines with $M_{A_1} = M_{A_2}$ yields the value of M_{2CX} . The two plots in the top row of figure 5 correspond to a balanced event, in which there is a single global minimum, and thus the longitudinal momentum configuration at the minimum is unique. Furthermore, the global minimum is at the intersection of the two black solid lines, implying that the parent constraint, $M_{A_1} = M_{A_2}$, is satisfied and therefore $M_{2XX} = M_{2CX}$, in agreement with the theorem from section 3.1. On the other hand, the bottom two plots show an unbalanced event, in which the unconstrained minimization reveals a flat direction along q_{2z} . Any value of q_{2z} along the bottom of that valley is acceptable and will give the correct value of M_{2XX} . If we now consider the constrained minimization along the black solid lines to obtain M_{2CX} , we find two degenerate global minima — one on the upper black solid curve and one on the lower black solid curve. Thus, as expected, there is a twofold ambiguity — in this case in the value of q_{2z} , while q_{1z} is unique. Again, the values of M_{2XX} and M_{2CX} are the same, since the function (3.17) is constant along the flat direction.

Since later on we shall be using the momenta found by the minimization for reconstruction purposes, the results from this subsection raise the question of how one should deal with unbalanced events, for which (some of) the momentum components are not uniquely determined. There can be several approaches:

- Restrict one’s attention to balanced events only, incurring some (minor) loss in statistical significance.
- Sum over all possible kinematic solutions (i.e., integrate over the flat direction in figure 5), and enter the results in histograms with correspondingly reduced weights.
- Instead of obtaining the momenta from M_{2XX} and M_{2CX} , use the variables with relative constraints, M_{2XC} and M_{2CC} , for which these ambiguities generally do not arise, see section 3.3.

3.3 The variables M_{2XC} and M_{2CC}

Having seen in section 3.1 that $M_{2XX}(ab)$ and $M_{2CX}(ab)$ are equivalent to $M_{T2}(ab)$, we now shift our focus to the new variables $M_{2XC}(ab)$ and $M_{2CC}(ab)$ and investigate their relationship with the other variables.

First we argue that the result from minimization with respect to \vec{q}_{1T} will be different in general when obtaining these new variables. For this purpose, let us assume the opposite, i.e., consider the function (3.17) in which \vec{q}_{1T} has been fixed to the result $\vec{q}_{1T}^{(\times)}$ found in the \vec{q}_{1T} minimization in figure 3. We then discuss its minimization in the (q_{1z}, q_{2z}) plane as in figure 5. The new element here is the presence of the relative constraint, $M_{B_1}^2 = M_{B_2}^2$, which can be written as

$$(p_{b_1} + q_1)^2 = (p_{b_2} + q_2)^2, \tag{3.22}$$

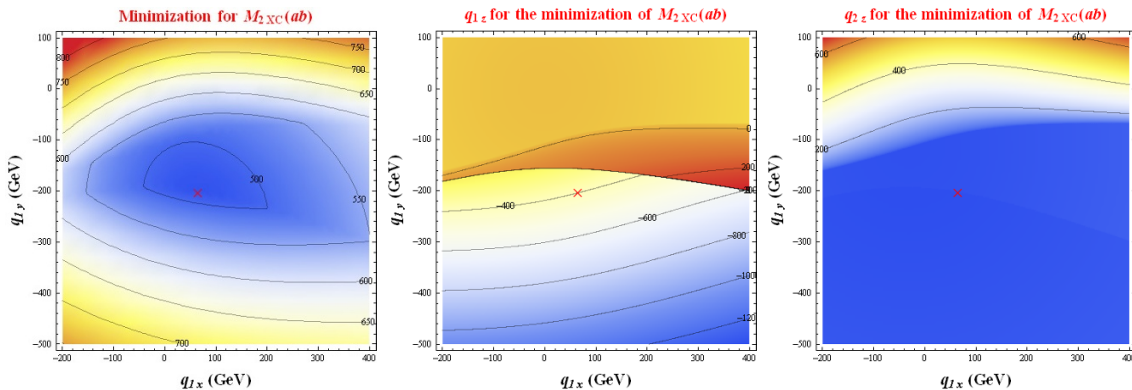


Figure 6. The analogues of figure 3 (left panel) and figure 4 (middle and right panels) for the case of M_{2XC} . The cross symbols show the location of the global minimum for M_{XC} , at which $M_{2XC}(ab) = 483.85$ GeV, and the invisible momenta are given by $(\vec{q}_{1T}, q_{1z}, q_{2z}) = (64.61, -202.37, -395.75, 19.07)$ GeV.

and which can be solved for q_{2z} in analogy to (3.19):

$$q_{2z} = \frac{p_{b_2z}K' \pm E_{b_2}\sqrt{K'^2 - E_{q_2T}^2 p_{b_2T}^2}}{p_{b_2T}^2}, \quad (3.23)$$

where

$$K' \equiv E_{q_1}E_{b_1} - \vec{q}_{1T} \cdot \vec{p}_{b_1T} - q_{1z}p_{b_1z} + \vec{q}_{2T} \cdot \vec{p}_{b_2T}. \quad (3.24)$$

A similar expression can be obtained for q_{1z} in terms of q_{2z} , with the replacements $q_{2z} \leftrightarrow q_{1z}$ and $b_1 \leftrightarrow b_2$ in (3.23), (3.24). Due to the “ \pm ” sign in (3.23), the relative constraint, (3.22), again implies two branches in the (q_{1z}, q_{2z}) plane, analogous to the black solid curves in figure 5. If at least one of these two curves passes through the global minimum point¹¹ found previously for the case of M_{2XX} , then M_{2XC} will turn out to be the same as $M_{2XX}(ab)$. However, the chances of a plane curve passing through a given point (or even a given finite line segment) are minimal, therefore we expect that, in general, the solution found previously for M_{2XX} will not obey the relative constraint, (3.22). This means that our choice of $\vec{q}_{1T} = \vec{q}_{1T}^{(\times)}$ was wrong, and that the minimum for M_{2XC} is obtained at a different value for \vec{q}_{1T} than the one found in figure 3. In particular, the constrained global minimum found by M_{2XC} will be higher than the corresponding unconstrained global minimum M_{2XX} :

$$M_{2XC}(ab) \geq M_{2XX}(ab) = M_{2CX}(ab) = M_{T2}(ab). \quad (3.25)$$

Figure 6 pictorially illustrates the above discussion. The left panel shows the function to be minimized when calculating M_{2XC} . As compared with the analogous figure 3 for the case of M_{2XX} , we see that the shape of the function is completely different, and as a result the global minimum (marked with a red \times symbol) is obtained at a different point in \vec{q}_T

¹¹Recall from figure 5 that balanced events lead to a unique global minimum as shown in the top panels while unbalanced events lead to a flat direction along a finite line segment as shown in the bottom panels.

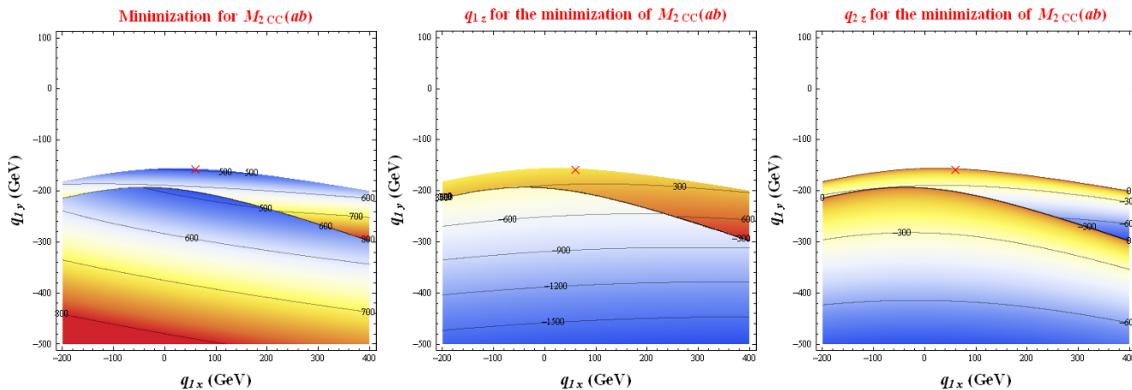


Figure 7. The same as figure 6, but for the case of $M_{2CC}(ab)$. At the global minimum (marked with the red \times symbol), $M_{2CC}(ab) = 487.86$ GeV, and the solution for the invisible momenta is given by $(\vec{q}_{1T}, q_{1z}, q_{2z}) = (60.11, -156.62, 121.35, 17.44)$ GeV. Within the white region, the constraints (3.18) and (3.22) cannot be simultaneously satisfied.

space, $\vec{q}_{1T} = (64.61, -202.37)$ GeV (as opposed to $\vec{q}_{1T}^\times = (66.09, -212.90)$ GeV, which was found in figure 3).

Another important lesson from the middle and right panels in figure 6 is that the solutions for q_{1z} and q_{2z} are now unique, unlike in the case of M_{2XX} and M_{2CX} exhibited in figure 4. We shall use this fact later on when reconstructing the mass of relative particles and studying the event topology.

Finally, it remains to discuss the variable $M_{2CC}(ab)$, where the parent and relative constraints, (3.18) and (3.22), are simultaneously applied. The analysis proceeds very similarly to the case of $M_{2XC}(ab)$ and the corresponding results are displayed in figure 7. Because of the additional constraint, the true global minimum for $M_{2CC}(ab)$ is now even greater than $M_{2XC}(ab)$. We thus arrive at our final result relating the variables (3.1):

$$M_{2CC}(ab) \geq M_{2XC}(ab) \geq M_{2XX}(ab) = M_{2CX}(ab) = M_{T2}(ab). \quad (3.26)$$

Note that there are large regions in invisible momentum parameter space (the white areas in figure 7), for which the on-shell kinematic constraints (3.18) and (3.22) cannot be simultaneously satisfied. As before, the red \times symbol marks the solution for \vec{q}_{1T} , which is found at a new location, $\vec{q}_{1T} = (60.11, -156.62)$ GeV. The corresponding $M_{2CC}(ab)$ value is 487.86 GeV, which is slightly larger than $M_{2XC}(ab) = 483.85$ GeV, in agreement with (3.26). The solutions for the longitudinal momenta are also unique (just as in the case of $M_{2XC}(ab)$ in figure 6), and are found at $(q_{1z}, q_{2z}) = (121.35, 17.44)$ GeV.

3.4 Summary of the properties of the on-shell constrained M_2 variables

We now collect our main results from section 2 and section 3 before moving on to the practical applications of the M_2 variables in the next few sections.

Variable	Balanced events		Unbalanced events	
	\vec{q}_{iT}	q_{iz}	\vec{q}_{iT}	q_{iz}
$M_{T_2}(ab)$	unique	NA	unique	NA
$M_{2XX}(ab)$	unique	unique	unique	flat direction
$M_{2CX}(ab)$	unique	unique	unique	twofold ambiguity
$M_{2XC}(ab)$	unique	unique	unique	unique
$M_{2CC}(ab)$	unique	unique	unique	unique

Table 3. Table summarizing the uniqueness of the invisible momentum configurations corresponding to the global minimum.

In section 2, we defined five different types of variables for each of the three subsystems in figure 2 (see table 2). The hierarchy among those variables is¹²

$$M_{2CC}(ab) \geq M_{2XC}(ab) \geq M_{2XX}(ab) = M_{2CX}(ab) = M_{T_2}(ab); \quad (3.27)$$

$$M_{2CC}(a) \geq M_{2XC}(a) \geq M_{2XX}(a) = M_{2CX}(a) = M_{T_2}(a); \quad (3.28)$$

$$M_{2CC}(b) \geq M_{2XC}(b) \geq M_{2XX}(b) = M_{2CX}(b) = M_{T_2}(b). \quad (3.29)$$

Thus, out of the fifteen variables seen in (3.27)–(3.29), there are only nine which are quantitatively different.

Each of the M_2 variables in table 2 is calculated by minimizing a suitably defined mass function in terms of the invisible momenta, see (2.7)–(2.10). The global minimum thus selects a special configuration of the invisible momenta which can be used for kinematical studies. In this section, we also investigated the uniqueness of the global minimum and consequently, the uniqueness of the associated invisible momenta. Our results are summarized in table 3. For completeness, in the table we also include the M_{T_2} variable, which, however, cannot determine the longitudinal components of the invisible momenta. In the case of balanced events, all four M_2 variables uniquely determine the invisible 3-momenta, while for unbalanced events, only M_{2XC} and M_{2CC} do so. Note that the twofold ambiguity in the case of M_{2CX} and the flat direction in the case of M_{2XX} are only with respect to *one* of the q_{iz} components, while the *other* q_{iz} component is uniquely determined.

4 Mass measurements

We now discuss several physics examples illustrating the potential uses and advantages of the M_2 variables. In this section, we first consider the simpler scenario where we have made the correct hypothesis about the true physics model and show how the use of M_2 variables can improve the precision of the mass measurements (in section 4.1) and provide a generalization of the MAOS technique [48] (in section 4.2). Then in section 5, we move

¹²Strictly speaking, in this section we only discussed the (ab) subsystem and the relations (3.27), but the analysis leading to (3.28) and (3.29) is very similar.

to the case where we are uncertain about which new physics model is correct. We show how one can then use the M_2 variables to rule out the incorrect model assumptions and hone in on the correct event topology.

The model to be studied in this section is the one depicted in figure 1(a), where the two decay chains are assumed to be identical:

$$m_{A_1} = m_{A_2} \equiv m_A, \quad m_{B_1} = m_{B_2} \equiv m_B, \quad m_{C_1} = m_{C_2} \equiv m_C. \quad (4.1)$$

In order to avoid confusion, from here on we shall use lowercase letters as in (4.1) to denote the true physical masses of the particles, reserving the corresponding uppercase letters M_A , M_B , etc., for masses which are *reconstructed* using kinematic information from the visible decay products in the event. Where necessary, input *test* masses (i.e., mass ansätze) will be denoted with a tilde. Throughout the paper, for our simulations we shall use event samples of $\sim 100,000$ events each, generated at threshold ($\sqrt{\hat{s}} = m_{A_1} + m_{A_2}$) without any spin correlations (i.e. we use pure “phase space” distributions).¹³

4.1 M_2 kinematic endpoints and parent mass measurements

The relations (3.27)–(3.29) imply that the on-shell constrained variables M_{2XC} and M_{2CC} can provide progressively better measurements of an upper kinematic endpoint, as compared with the conventional variables M_{T2} , M_{2XX} , and M_{2CX} . The reason is that the shapes of the M_{2XC} and M_{2CC} distributions will be skewed to the right, thus better populating the bins in the vicinity of the endpoint. This expectation is confirmed in figure 8, where we compare the distributions of these five variables for the example of (4.1) with mass spectrum $(m_A, m_B, m_C) = (500, 300, 200)$ GeV. For concreteness and simplicity, we choose the input trial mass, \tilde{m} , to be the same as the actual daughter mass in each case. Comparisons are made for each subsystem of figure 2: subsystem (ab) (upper left panel), subsystem (ab) but using only balanced events (upper right panel), subsystem (a) (lower left panel), and subsystem (b) (lower right panel). Although each panel shows results for five variables, only three distributions (at most) can be seen, because the distributions of M_{T2} , M_{2XX} , and M_{2CX} are identical, in accordance with the equivalence theorem from section 3.1. An interesting observation is that M_{2CC} and M_{2CX} also turn out to be the same for balanced events (i.e., events in which the transverse masses of the parents end up being equal for the momentum configuration obtained when minimizing the respective mass function). This observation is supported by the upper right plot in figure 8, which uses only events in which $M_{T2}(ab)$ is obtained from a balanced configuration,¹⁴ and by the two lower plots in figure 8, in which $M_{T2}(a)$ and $M_{T2}(b)$ always come from balanced configurations.

In the case of subsystem (ab), the M_{T2} distribution is already very sharp near the kinematic endpoint, and the improvement from replacing M_{T2} with M_{2XC} or M_{2CC} appears

¹³In general, depending on the details of the new physics model, the parents A_i will be produced with some non-zero boost, i.e., $\sqrt{\hat{s}} > m_{A_1} + m_{A_2}$. However, given the current LHC bounds, the parents A_i are expected to be heavy, so that they should be predominantly produced near threshold. We have also tested our methods below with more realistic event samples, including the effects from initial state radiation and proton structure, and found that our conclusions remain unchanged.

¹⁴In our sample, 64% (36%) of the events have balanced (unbalanced) solutions for $M_{T2}(ab)$.

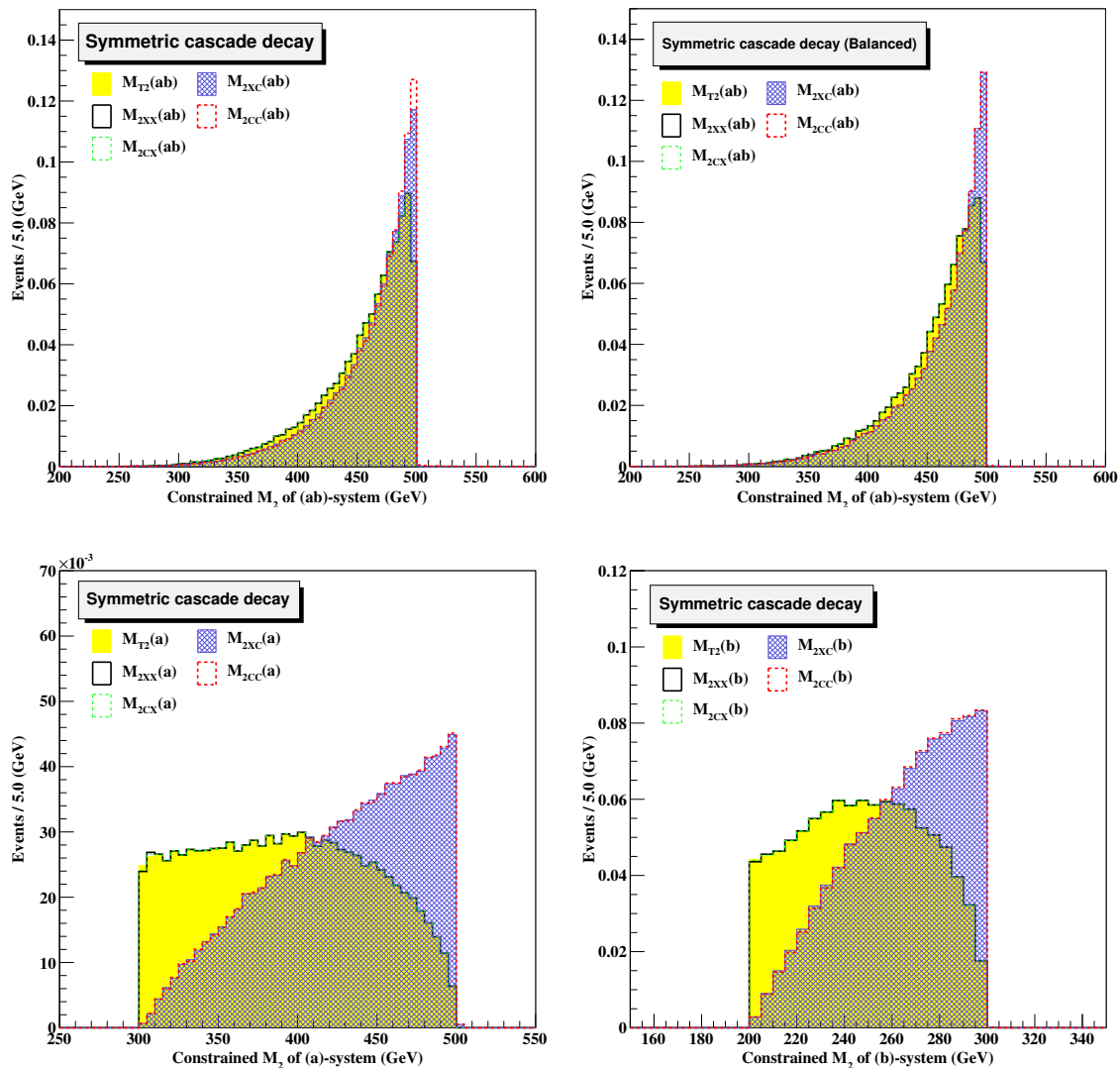


Figure 8. Unit-normalized differential distributions of the variables M_{T2} (yellow shaded histograms), M_{2XX} (black solid line), M_{2CX} (green dashed line), M_{2XC} (blue hatched histograms), and M_{2CC} (red dashed line) for the process of figure 1(a) with mass spectrum $(m_A, m_B, m_C) = (500, 300, 200)$ GeV. Results are shown for subsystem (ab) (upper left panel), subsystem (ab) with balanced events only (upper right panel), subsystem (a) (lower left panel), and subsystem (b) (lower right panel). The input trial mass is chosen to be the same as the true mass of the relevant daughter particle.

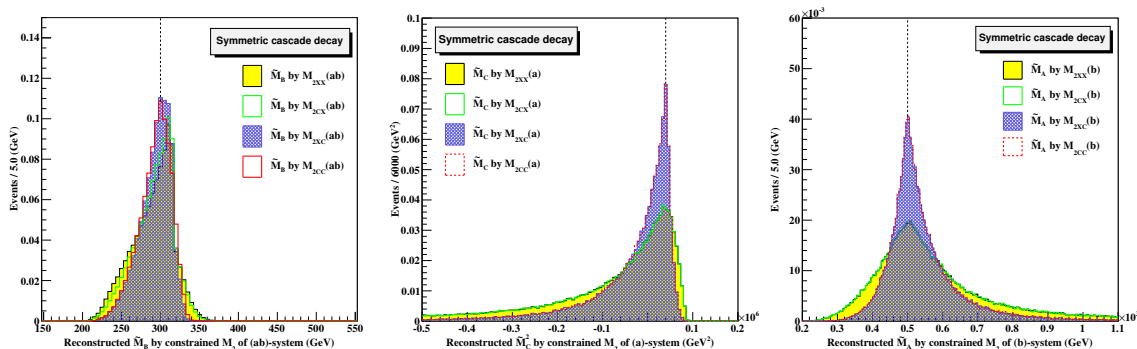


Figure 9. Reconstruction of the mass of the relative particle in the case of subsystem (ab) (left panel), subsystem (a) (middle panel), and subsystem (b) (right panel). The ansatz for the invisible particle momenta can be taken from the corresponding M_{2XX} variable (yellow-shaded histogram), M_{2CX} variable (green histogram), M_{2XC} variable (blue-shaded histogram), or M_{2CC} variable (red histogram). The true mass spectrum and trial masses are chosen as in figure 8. The vertical black dashed line in each plot denotes the true mass of the associated relative particle. The middle panel (for subsystem (a)) shows the mass *squared* of the relative particle, which can be negative at times.

marginal. However, the effect is very drastic in the case of subsystem (a) or subsystem (b) (the lower two plots in figure 8), where the M_{T2} distribution (the yellow-shaded histogram) has very few events near the kinematic endpoint. Now, using M_{2XC} or M_{2CC} in place of M_{T2} completely changes the character of the distribution, and the bins near the endpoint become the most populated ones. Notice the extremely sharp drop-off at the endpoint of the $M_{2XC}(a)$ and $M_{2XC}(b)$ distributions (the blue-shaded histograms). This feature should be easily observable over the background and would lead to more accurate endpoint measurements and extraction of masses.

4.2 M_2 -assisted mass reconstruction of relative peaks

As explained in the introduction, an attractive feature of the M_{T2} variable is that it provides an ansatz for the transverse momenta of the invisible particles. The M_2 variables, being 3+1 dimensional extensions of M_{T2} , take this one step further and extend the ansatz to the full 4-momenta of the invisible particles. This allows us to apply the MAOS method for mass reconstruction [48, 52, 53] in a pure form, i.e., without the need for additional assumptions in order to solve for the longitudinal momenta of the invisible particles — since those are already provided by the M_2 minimization itself.¹⁵ As shown in section 3, the variables M_{2CC} and M_{2XC} are somewhat better suited for our purpose (in comparison to M_{2XX} and M_{2CX}), since they provide a unique ansatz for the invisible particle momenta in the case of unbalanced events. Of course, for balanced events, any of our four types of M_2 variables can be used.

Figure 9 shows the results for the reconstruction of the masses of the relative particles in each of the three subsystems from figure 2. In the left panel of figure 9, we use the

¹⁵Thus in our case, the MAOS abbreviation should perhaps be thought of as “ M_2 -assisted on-shell” reconstruction.

invisible momenta obtained from various $M_2(ab)$ -type variables to reconstruct the mass,¹⁶ \tilde{M}_B , of the relative particle, B , in subsystem (ab) ; in the middle panel we use the momenta obtained from $M_2(a)$ -type variables to find the mass *squared*, \tilde{M}_C^2 , of the relative particle, C , in subsystem (a) ; and finally, in the right panel, we use the momenta from $M_2(b)$ -type variables to reconstruct the mass, \tilde{M}_A , of the relative particle, A , in subsystem (b) . Each distribution in figure 9 is color coded according to the type of M_2 variable supplying the invisible momenta: yellow-shaded histograms for the case of M_{2XX} , green histograms for M_{2CX} , blue-shaded histogram for M_{2XC} , and red histograms for M_{2CC} . The events are generated with the mass spectrum from eq. (4.1) and the test mass was always chosen to be the true mass of the relevant daughter particle: $\tilde{m} = m_C$ for subsystem (ab) (left panel), $\tilde{m} = m_B$ for subsystem (a) (middle panel), and $\tilde{m} = m_C$ for subsystem (b) (right panel).

The most interesting feature of the plots in figure 9 is that the distributions always peak close to the true mass of the relative particle (denoted by the vertical black dashed line in each plot). This suggests a new technique for measuring the mass of a relative particle — by using the location of the peak of the reconstructed relative mass distribution as shown in figure 9. A closer inspection of figure 9 reveals another advantage of the M_2 variables that incorporate on-shell kinematic constraints for relative particles in their definition. Note that in each panel, all four distributions peak near the true relative mass, but in the case of M_{2XC} and (especially) M_{2CC} , the peak is much more narrow, and, more importantly, the peak location is very close to the true value of the mass of the respective relative particle. We therefore anticipate that the precision of the new technique will be much better when using M_{2CC} (and M_{2XC}) as opposed to M_{2CX} or M_{2XX} .

This technique is in principle independent of (and complementary to) the previous methods in which masses are measured from upper kinematic endpoints. For example, consider particle B (the intermediate particle in the decay chains of figure 1). It is known that its mass can be measured (as a function of $\tilde{m} \equiv \tilde{m}_C$) from the upper kinematic endpoint $M_{T_2}^{\max}(b)$ of the $M_{T_2}(b)$ distribution in subsystem b , where B_i is treated as a parent [33, 39]

$$\tilde{m}_B(\tilde{m}_C) = M_{T_2}^{\max}(b)(\tilde{m}_C). \tag{4.2}$$

Using the correct value for the daughter particle mass, m_C , in (4.2) yields the correct value of the parent mass, m_B :

$$m_B = M_{T_2}^{\max}(b)(m_C). \tag{4.3}$$

We now propose to consider subsystem (ab) instead, where B_i is treated as a relative, and extract $\tilde{m}_B(\tilde{m}_C)$ from the location of the peak $\tilde{M}_B^{\text{peak}}$ of one of the \tilde{M}_B distributions in the left panel of figure 9, e.g., the one where the invisible momenta are fixed by $M_{2CC}(ab)$:

$$\tilde{m}_B(\tilde{m}_C) = \tilde{M}_B^{\text{peak}}(ab)(\tilde{m}_C). \tag{4.4}$$

The procedure is pictorially illustrated in figure 10. The \tilde{M}_B distribution from figure 9 can now be re-obtained without the “cheat” of fixing $\tilde{m} = m_C$. Instead, we can now simply vary the input test mass, \tilde{m}_C , and read off the location of the \tilde{M}_B peak for each

¹⁶From here on, a tilde over a quantity implies that it is a function of the test mass \tilde{m} .

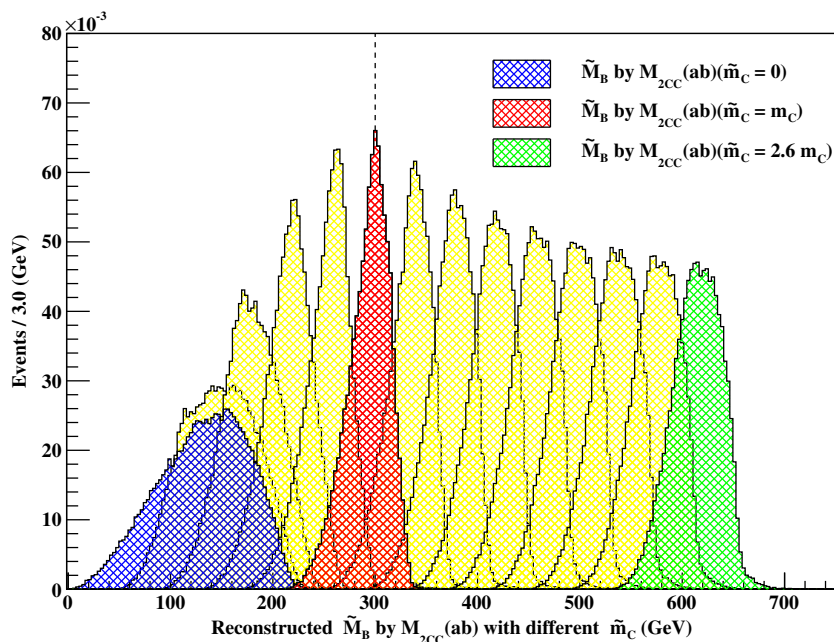


Figure 10. Unit-normalized distributions of the reconstructed mass, \tilde{M}_B , of the relative particle in subsystem (ab), using invisible momenta from $M_{2CC}(ab)$, and picking a series of different values for the input test mass, \tilde{m}_C , from $\tilde{m}_C = 0$ (blue histogram) to $\tilde{m}_C = 2.6m_C$ (green histogram). The red shaded distribution corresponds to the true value, $\tilde{m}_C = m_C$, and is the same as the red histogram in the left panel of figure 9. The vertical black dashed line marks the true mass, $m_B = 300$ GeV, in our example.

\tilde{m}_C value, thus experimentally determining the function (4.4). This method relies on the fact demonstrated by the red shaded histogram in figure 10 — that for the correct value, m_C , of the test daughter mass the peak of the \tilde{M}_B distribution matches the correct value, m_B , of the mass for the relative particle:¹⁷

$$m_B = \tilde{M}_B^{\text{peak}}(ab)(m_C). \tag{4.5}$$

Notice the analogy between the relationships (4.2) and (4.4) — they both relate the mass of particle B_i with the mass of particle C_i . The difference is that the correlation (4.2) is derived from a kinematic endpoint in subsystem (b), while the correlation (4.4) is derived from the peak of a distribution within subsystem (ab). Also one should keep in mind that while (4.3) is a mathematical identity, the relation (4.5) at this point is a conjecture supported by the numerical results from figures 9 and 10. (Compare to the similar conjecture relating the peak of the $\sqrt{\hat{s}_{\text{min}}}$ distribution to the mass of the corresponding parents [20].)

¹⁷The careful reader might notice some other interesting features of the red-shaded histogram in figure 10 — it appears to be the most localized distribution and, correspondingly, has the highest peak among all distributions shown in figure 10. However, we do not pursue further this observation, since figure 12 below provides a counterexample in which the highest peak is obtained for the wrong value of the test mass.

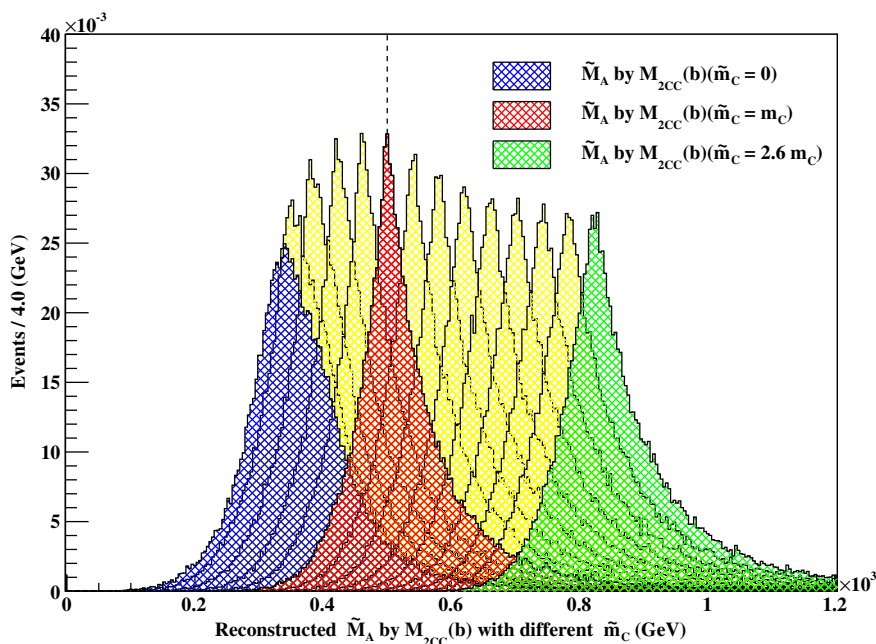


Figure 11. The same as figure 10, this time reconstructing the mass, \tilde{M}_A , of the relative particle in subsystem (b) for several values of \tilde{m}_C , using invisible momenta from $M_{2CC}(b)$. The red shaded distribution corresponds to the true value, $\tilde{m}_C = m_C$, and is the same as the red histogram in the right panel of figure 9. The vertical black dashed line marks the true mass, $m_A = 500$ GeV, in our example.

Similar logic can be applied to particle A. It is known that its mass can be measured from the upper kinematic endpoint of the $M_{T2}(ab)$ distribution in subsystem (ab), as a function of the input test mass, \tilde{m}_C , in complete analogy to (4.2) [35, 38]:

$$\tilde{m}_A(\tilde{m}_C) = M_{T2}^{\max}(ab)(\tilde{m}_C). \quad (4.6)$$

Alternatively, it can be measured from the upper kinematic endpoint of the $M_{T2}(a)$ distribution in subsystem (a), this time as a function of the test mass, \tilde{m}_B [33, 38, 39]:

$$\tilde{m}_A(\tilde{m}_B) = M_{T2}^{\max}(a)(\tilde{m}_B). \quad (4.7)$$

We now propose a third way of measuring the mass of A_i , by treating it as a relative particle in subsystem (b): using the invisible momenta from the $M_{2CC}(b)$ calculation, we can reconstruct the mass of the relative, \tilde{M}_A , and read off the location of the peak, $\tilde{M}_A^{\text{peak}}$, in analogy to (4.4)

$$\tilde{m}_A(\tilde{m}_C) = \tilde{M}_A^{\text{peak}}(b)(\tilde{m}_C). \quad (4.8)$$

The function, (4.8), can be experimentally derived as shown in figure 11 — one varies the test mass, \tilde{m}_C , and forms a series of \tilde{M}_A distributions. The location of the peak of each distribution represents the value of \tilde{m}_A for the given hypothesized value of \tilde{m}_C . The

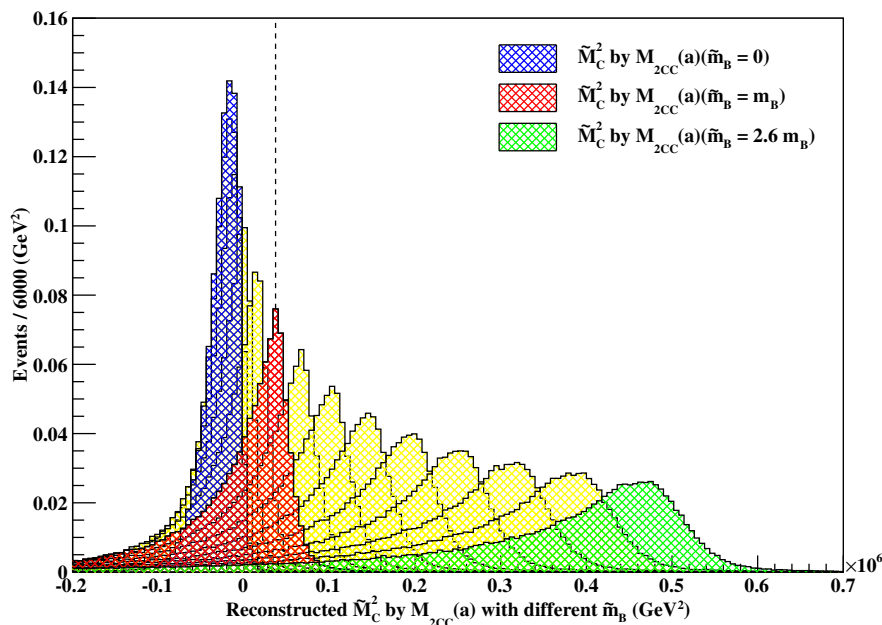


Figure 12. The same as figure 10, this time reconstructing the mass squared, \tilde{M}_C^2 , of the relative particle in subsystem (a) for several values of \tilde{m}_B , using invisible momenta from $M_{2CC}(a)$. The red shaded distribution corresponds to the true value, $\tilde{m}_B = m_B$, and is the same as the red histogram in the middle panel of figure 9. The vertical black dashed line marks the value of the true mass squared, $m_C^2 = 40,000 \text{ GeV}^2$, in our example.

red shaded histogram in figure 11 corresponds to the true value of $\tilde{m}_C = m_C$ and again peaks at the correct value of the mass, m_A , of the relative particle:

$$m_A = \tilde{M}_A^{\text{peak}}(b)(m_C). \tag{4.9}$$

Finally, one may also consider the subsystem (a) and study the distributions of the reconstructed relative mass, \tilde{M}_C , shown in the middle panel of figure 9. This establishes the relation

$$\tilde{m}_C(\tilde{m}_B) = \tilde{M}_C^{\text{peak}}(a)(\tilde{m}_B). \tag{4.10}$$

The procedure is illustrated in figure 12, where we have used $M_{2CC}(a)$ to fix the momenta of the invisible particles before computing \tilde{M}_C^2 . A peculiar feature of figure 12 is that for low enough values of the test mass, \tilde{m}_B , the peak of the distribution is found at negative values of \tilde{M}_C^2 , which is why we do not take a square root and instead use the mass squared in the plot. Nevertheless, the important feature of figure 12 is that, just like in figures 10 and 11, for the correct choice of the test mass, $\tilde{m}_B = m_B$ (see red histogram), the peak reveals the true value, m_C , of the relative particle (in this case C_i).

Before concluding, in figure 13 we summarize the different mass determination methods discussed in this section. The existing method relies on measuring M_{T2} kinematic endpoints in the three subsystems of figure 2, establishing the three relationships (4.2), (4.6),

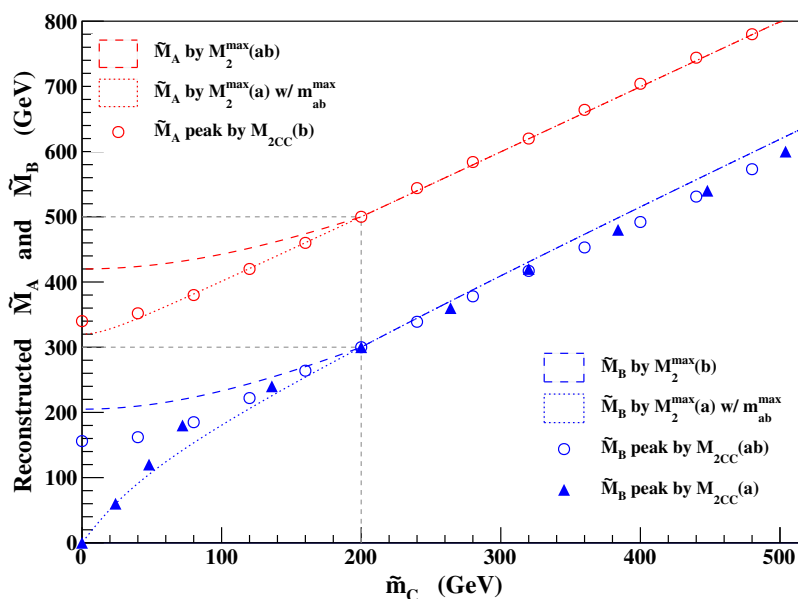


Figure 13. A summary of the different mass correlation methods discussed in the text: (4.11) is represented by a blue dashed line, (4.12) is given by a red dashed line, (4.15) is shown by the red open circles, (4.16) is denoted by the blue open circles, while (4.17) is marked by the blue triangles. The red dotted line represents the relationship between \tilde{m}_A and \tilde{m}_C which is obtained by eliminating \tilde{m}_B from (4.13) and (4.14), while the blue dotted line shows the orthogonal relationship among \tilde{m}_B and \tilde{m}_C resulting from eliminating \tilde{m}_A from (4.13) and (4.14).

and (4.7). In section 4.1, we proposed to measure the sharper M_{2CC} kinematic endpoints instead, resulting in three analogous relations

$$\tilde{m}_B(\tilde{m}_C) = M_2^{\max}(b)(\tilde{m}_C), \quad (4.11)$$

$$\tilde{m}_A(\tilde{m}_C) = M_2^{\max}(ab)(\tilde{m}_C), \quad (4.12)$$

$$\tilde{m}_A(\tilde{m}_B) = M_2^{\max}(a)(\tilde{m}_B). \quad (4.13)$$

These can be supplemented with the classic measurement of the kinematic endpoint of the invariant mass, M_{ab} , of the two visible particles, a_i and b_i , in each decay chain

$$M_{ab}^{\max} = \sqrt{\frac{(\tilde{m}_A^2 - \tilde{m}_B^2)(\tilde{m}_B^2 - \tilde{m}_C^2)}{\tilde{m}_B^2}}, \quad (4.14)$$

which provides a constraint among all three masses \tilde{m}_A , \tilde{m}_B , and \tilde{m}_C . The four measurements (4.11)–(4.14) are already sufficient to determine the three unknowns \tilde{m}_A , \tilde{m}_B , and \tilde{m}_C [39]. The new measurements proposed in section 4.2 are the peak determinations

$$\tilde{m}_A(\tilde{m}_C) = \tilde{M}_A^{\text{peak}}(b)(\tilde{m}_C), \quad (4.15)$$

$$\tilde{m}_B(\tilde{m}_C) = \tilde{M}_B^{\text{peak}}(ab)(\tilde{m}_C), \quad (4.16)$$

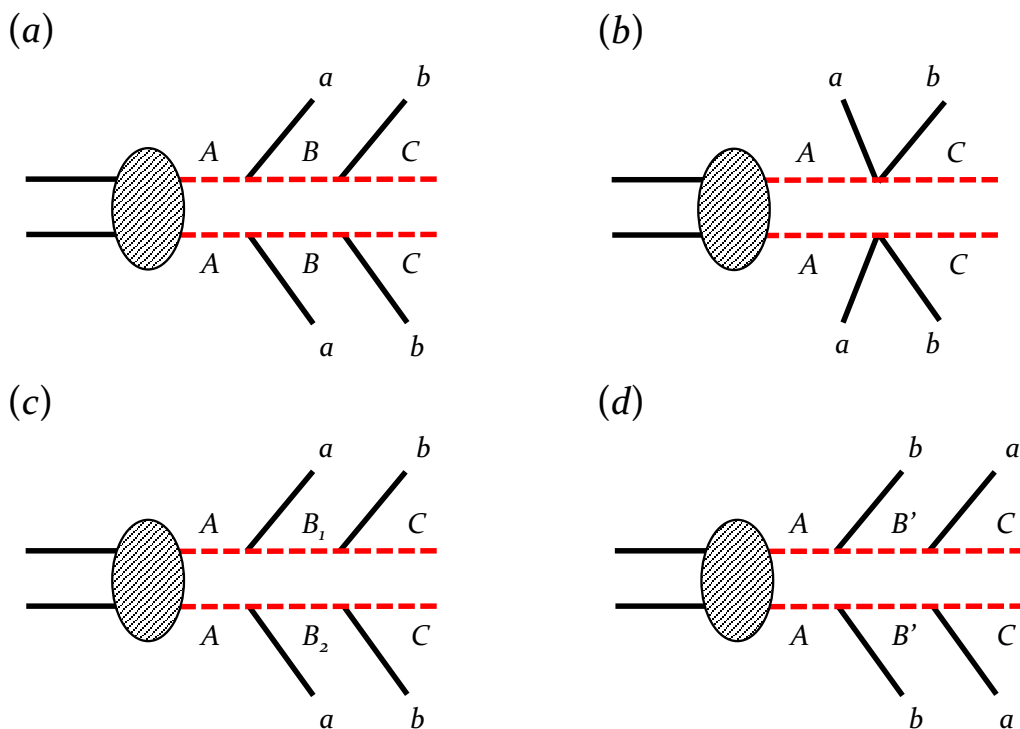


Figure 14. The four benchmark decay topologies studied in section 5.

$$\tilde{m}_C(\tilde{m}_B) = \tilde{M}_C^{\text{peak}}(a)(\tilde{m}_B). \quad (4.17)$$

The seven relations (4.11)–(4.17) are pictorially illustrated in figure 13. In order to display all seven relations on the same plot, we first plot (4.11)–(4.12) and (4.15)–(4.17) directly, then from the remaining two relations (4.13) and (4.14) we either eliminate \tilde{m}_B to obtain \tilde{m}_A as a function of \tilde{m}_C (red dotted line), or eliminate \tilde{m}_A to obtain \tilde{m}_B as a function of \tilde{m}_C (blue dotted line). All seven correlations (4.11)–(4.17) agree for the correct values for m_A , m_B and m_C , marked with the black dotted lines in figure 13. What is more interesting is that they *disagree* for the wrong values of the test input mass, \tilde{m}_C . This is particularly noticeable in the region $\tilde{m}_C < m_C$. Figure 13 suggests that by combining the results from all the different methods (4.11)–(4.17) one can determine the true value of m_C as the location of the crossing point of the different curves shown in the figure. Our method is complementary to other methods in the literature for determining the absolute value of m_C [11, 13, 35–38, 42, 43, 58–60, 70].

5 Using M_2 variables for topology disambiguation

Up to this point, we have been studying events under the correct assumption about the event topology. However, in a real experiment, there is no prior indication as to what the correct event topology is for any given observed final state, and one should consider (and test for) all possible alternatives. This is exactly what we set out to do in this section.

Given our observed final state of two visible a particles, two visible b particles and missing transverse momentum, a number of event topologies are possible; four of which are shown in figure 14. Figure 14(a) shows our nominal scenario, (4.1), considered so far, in which there is an on-shell B_i resonance in each chain and furthermore, the two B resonances are the same: $B_1 = B_2 \equiv B$. Figure 14(b) represents the off-shell scenario in which the intermediate B resonance is very heavy and the decays are three-body. Figure 14(c) is the same as figure 14(a), but with a slight modification — now the two intermediate resonances, B_i , are different: $B_1 \neq B_2$. Finally, figure 14(d) is the analogue of figure 14(a) in which the visible particles, a and b , are switched, i.e., the decay to b takes place first, followed by the decay to a .

In this section, we shall design several tests which discriminate among the alternative possibilities depicted in figure 14. The tests make crucial use of the constrained M_2 variables introduced in section 2.

5.1 Endpoint test

We first design a test to distinguish among the three event topologies shown in figure 14(a), figure 14(b), and figure 14(c). (This test will not be able to discriminate among figure 14(a) and figure 14(d).) The basic idea is very simple. Recall that the $M_{2XC}(ab)$ and $M_{2CC}(ab)$ variables from section 2 were defined under the assumption of a common relative particle. I.e.,

- there is an intermediate B_i resonance in each decay chain, and
- the two B_i particles are the same, so that $m_{B_1} = m_{B_2}$.

If either of these two assumptions is incorrect, the definition of $M_{2XC}(ab)$ and $M_{2CC}(ab)$ loses its physical meaning, and as a result something will go wrong. Therefore, by testing for the consistency of $M_{2XC}(ab)$ and $M_{2CC}(ab)$ with another, topology-independent, variable like $M_{2XX}(ab)$, we can verify the above two assumptions. Note that relaxing the first assumption leads to the event topology of figure 14(b), while dropping the second assumption leads to the event topology of figure 14(c).

How can one test for the consistency of $M_{2XC}(ab)$ and $M_{2CC}(ab)$? Recall that the basic property of all M_2 variables is that they provide a lower bound on the mass of the corresponding parent, and their upper kinematic endpoints saturate that bound, revealing the mass of the parent (as a function of the test daughter mass). Now consider the relevant variables (3.1) for subsystem (ab) . They bound the mass of *the same* parent A , the only difference is that they have various assumptions about the event topology built in. Therefore, if all those assumptions are correct, the kinematic endpoints of all the variables should agree as well.¹⁸

$$M_{T2}^{\max} = M_{2XX}^{\max} = M_{2CX}^{\max} = M_{2XC}^{\max} = M_{2CC}^{\max}. \tag{5.1}$$

¹⁸Of course, due to the equivalence theorem discussed in section 3.1, the first two equalities in eq. (5.1) are trivially satisfied, so that the actual test involves only the last two equalities in eq. (5.1).

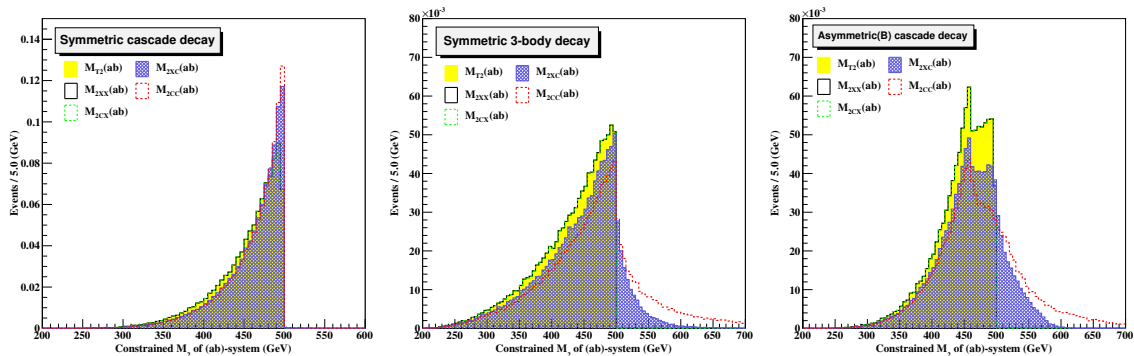


Figure 15. The same as the upper left panel of figure 8, but using events from three different scenarios. The left panel shows the nominal event topology from figure 14(a) with $(m_A, m_B, m_C) = (500, 300, 200)$ GeV. The middle panel corresponds to the off-shell case of figure 14(b) with $(m_A, m_C) = (500, 200)$ GeV. The right panel represents the asymmetric event topology from figure 14(c) with $(m_A, m_{B_2}, m_{B_1}, m_C) = (500, 400, 300, 200)$ GeV. The test mass was always chosen to be $\tilde{m}_C = 200$ GeV.

Conversely, if some of the assumptions are not satisfied, (5.1) will be violated — there will be a certain number of events in which the values of $M_{2XC}(ab)$ and $M_{2CC}(ab)$ will violate the upper kinematic endpoint M_{2XX}^{\max} of the topology-independent variable $M_{2XX}(ab)$.

The test is performed in figure 15, where we compare the distributions of the five variables in the (ab) subsystem, as in the upper left panel of figure 8. In the left panel of figure 15, we first consider the case of our nominal event topology from figure 14(a) with the mass spectrum from (4.1). As already observed in figure 8, the distributions may have slightly different shapes, but their endpoints are exactly the same. Therefore, this case passes the endpoint test, (5.1), as expected.

We next consider the off-shell case of figure 14(b) with $(m_A, m_C) = (500, 200)$ GeV and plot the results in the middle panel of figure 15. In accordance with the equivalence theorem from section 3.1, the distributions of M_{T2} , M_{2XX} , and M_{2CX} are identical, and their common endpoint provides a reference value, M_{2XX}^{\max} , to be compared against the endpoints of M_{2XC} and M_{2CC} . The plot clearly shows that the distributions of M_{2XC} and M_{2CC} develop long tails beyond M_{2XX}^{\max} , thus violating (5.1) and failing the endpoint test. The violation is more severe in the case of M_{2CC} (the red histograms in figure 15), where a larger number of events have migrated beyond the anticipated endpoint M_{2XX}^{\max} . The reason for this violation is easy to understand — in the off-shell case of figure 14(b) there are no intermediate resonances, B_1 and B_2 . Thus when we enforce the relative constraint, $M_{B_1} = M_{B_2}$, in constructing the M_{2XC} and M_{2CC} variables, we unnecessarily restrict the range of allowed values of the invisible momenta during the minimization, and thus arrive at an unphysical global minimum. Based on the results from the middle panel of figure 15, we can therefore safely rule out the on-shell event topology of figure 14(a) as being the source of these events.

The right panel in figure 15 shows the case of the asymmetric event topology from figure 14(c) with $(m_A, m_{B_2}, m_{B_1}, m_C) = (500, 400, 300, 200)$ GeV. This time, the interme-

diate resonances, B_1 and B_2 , are present, but their masses are not equal: $m_{B_1} = 300$ GeV, while $m_{B_2} = 400$ GeV. Thus applying the relative constraint, $M_{B_1} = M_{B_2}$, during the minimization for M_{2XC} and M_{2CC} once again leads to an unphysical situation. As a result, the M_{2XC} and M_{2CC} distributions again develop tails beyond M_{2XX}^{\max} , failing the test (5.1) and ruling out the on-shell event topology of figure 14(a) as being the source of these events.

Note that in the last two cases, when the endpoint test failed, it simply told us which event topology is wrong, but it did not specify the correct answer. For this, we must develop further tests as in the next two subsections. However, notice the distinctive shape of the distributions in the right panel of figure 15 in comparison with the middle panel. One might hope to use this shape difference to further discriminate among the event topologies of figure 14(b) and figure 14(c). However, such detailed shape analysis is beyond the scope of this paper.

Note that the ability to discriminate among two alternative event topologies suggests an interesting application of the constrained M_2 variables in discriminating the signal from irreducible backgrounds [71]. The SM backgrounds have known event topologies, for which the corresponding on-shell constraints can be readily applied; the resulting distributions will still have the same endpoint. With a suitably chosen cut above this expected SM endpoint, one would be able to remove most, if not all, background events. On the other hand, the signal event topology is generally different, and the signal events will migrate to higher values of M_2 once the kinematic constraints are imposed, leading to a higher signal efficiency when using M_2 in place of M_{T2} .

Before concluding this subsection we comment on the applicability of our method in the presence of backgrounds (either from Standard Model processes resulting in the same final state, or from signal combinatorial backgrounds). First, one should keep in mind that precision kinematic measurements like the ones discussed in this paper will be attempted long after the initial discovery, when large statistics samples with large signal to background ratios are available. (Reducible backgrounds can be easily suppressed by other cuts unrelated to the M_2 variables.) Unfortunately, the background cannot be eliminated completely. In particular, some background events might populate the region above the M_2 endpoints for a number of reasons, e.g., a different event topology, smearing, or combinatorics ambiguities. Thus one may wonder if the failure of the endpoint test can be attributed to such background contamination. We believe that this is unlikely to be the case, since only a (small) fraction of the background events are actually affecting the endpoint test. Indeed, the background events fall into three categories. The first¹⁹ contains background events whose M_{2XX} , M_{2XC} and M_{2CC} values are all below the observed signal M_2 endpoint. Such events will remain in the bulk of the M_2 distributions and will not have any impact on the endpoint test. The second category contains background events for which *all three variables* M_{2XX} , M_{2XC} and M_{2CC} exceed the observed signal M_2 endpoint. In spite of such endpoint violations, these events also have no relevance to the endpoint test, since they will be recognized as background already from the M_{2XX} distribution alone. It is only the

¹⁹In practice, this is the most populous of the three categories, because for the signal mass spectra of interest, the M_2 endpoint for the background is typically below the M_2 endpoint for the signal.

third category of background events which could be problematic — those are the events where M_{2XX} is below the signal M_2 endpoint, while M_{2XC} and M_{2CC} happen to migrate above the signal M_2 endpoint. The expected number of such events for SM background events can be estimated from Monte Carlo and therefore will not bias the test. Given that this third category encompasses only a small fraction of all background events (which are themselves assumed to be small compared to the total number of signal events), we do not expect that such background migration would invalidate the proposed endpoint test.

5.2 Dalitz plot test

In this subsection, we develop a Dalitz plot test which enables us to discriminate the event topology in figure 14(a) from those in 14(b) and 14(d). The idea is to use the invisible momenta obtained in the M_2 minimization to form invariant mass combinations involving the final state invisible particles, C_i .

To see how the method works, let us assume that the signal comes from the event topology of figure 14(a). First consider the ideal case when we have exact knowledge of the four momenta of the invisible particles, C_i . Since there are three particles in the final state of each decay chain, a_i , b_i , and C_i , and we know their 4-momenta, we can form three invariant mass combinations, M_{ab} , M_{bC} , and M_{aC} . Since particles b_i and C_i originate from the same mother particle, B_i , M_{bC} simply equals the mass, m_B , of that mother particle, regardless of the value of M_{ab} . Therefore, the Dalitz plot in the (M_{bC}^2, M_{ab}^2) plane is characterized by a single vertical line:

$$M_{bC}^2 = m_B^2 \quad \text{for any } M_{ab}^2 \in [0, (M_{ab}^{\max})^2], \tag{5.2}$$

with M_{ab}^{\max} given by (4.14). On the other hand, M_{aC}^2 takes values within a given range consistent with the sum rule

$$M_{aC}^2 = m_A^2 - m_B^2 + m_C^2 - M_{ab}^2 \quad \text{for any } M_{ab}^2 \in [0, (M_{ab}^{\max})^2], \tag{5.3}$$

which is nothing but a straight line with a negative slope in the plane of (M_{aC}^2, M_{ab}^2) . The predictions (5.2) and (5.3) in this idealized case are illustrated in the upper left panel of figure 16, where the vertical line corresponds to (5.2), and the slanted line corresponds to (5.3).

We are now ready to consider the more realistic case in which we do not have exact knowledge of the individual momenta of the invisible particles, C_i , but instead obtain them from the M_{2CC} ansatz. The corresponding results are shown in the remaining two plots in the top row of figure 16 — the middle panel shows a scatter plot in the (M_{aC}^2, M_{ab}^2) plane, while the right panel shows a scatter plot in the (M_{bC}^2, M_{ab}^2) plane. Since the invisible momenta are only approximated, the correlations are not exactly linear, but nevertheless they tend to follow the general trends given by (5.2) and (5.3).

Let us now move on to the event topology of figure 14(b). This case is illustrated in the middle row of figure 16. Since the intermediate B_i resonance is absent, the visible particles, a_i and b_i , arise from the same vertex and are on equal footing. Thus, we expect the associated Dalitz plots in the (M_{aC}^2, M_{ab}^2) , and (M_{bC}^2, M_{ab}^2) planes to be very similar, and indeed this is what we observe by comparing the middle and right panels of the middle

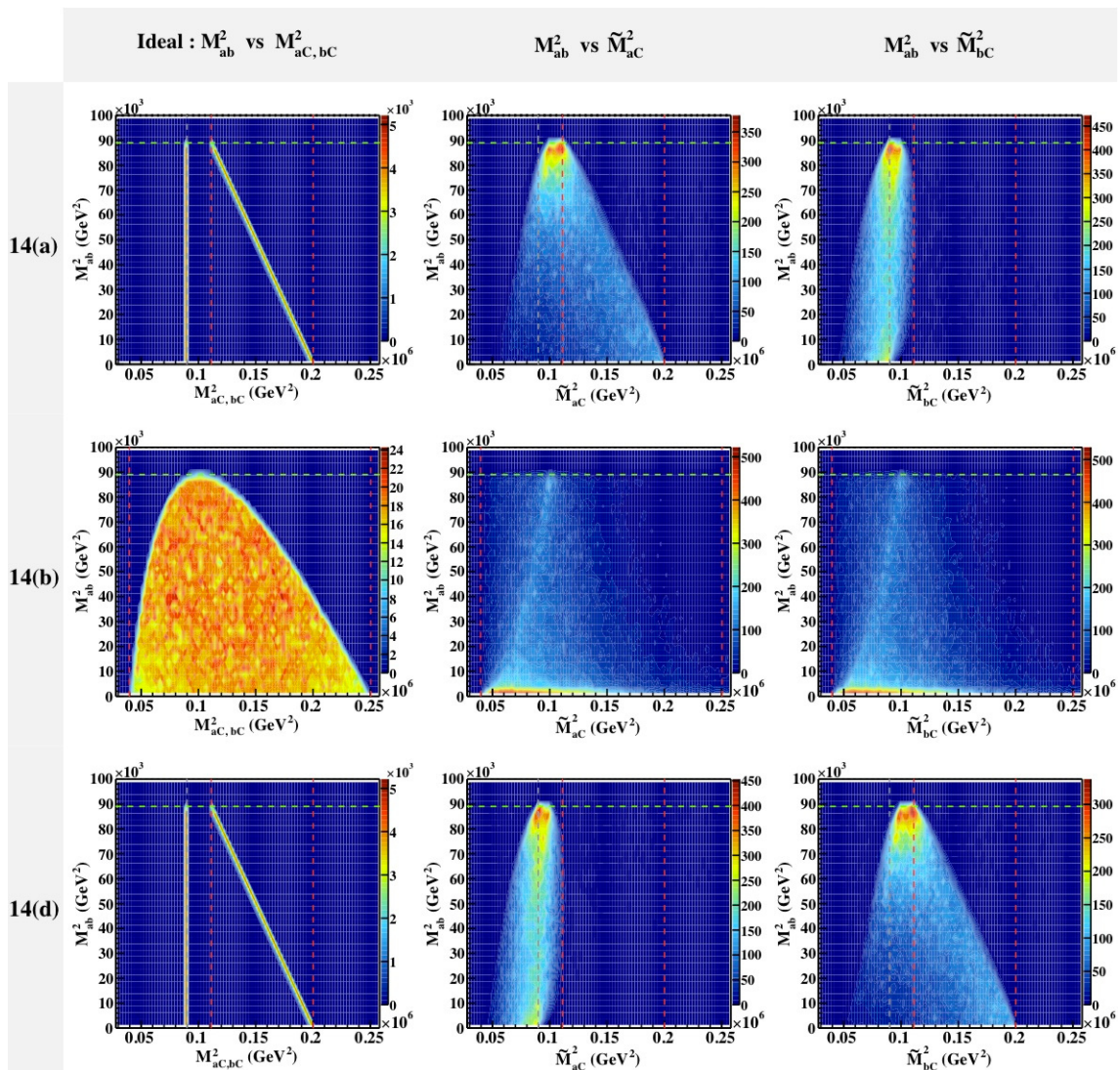


Figure 16. Dalitz plots for the event topologies of figure 14(a) (top row), figure 14(b) (middle row), and figure 14(d) (bottom row). Using the invisible particle momenta obtained from $M_{2CC}(ab)$, we show scatter plots of \tilde{M}_{aC}^2 versus M_{ab}^2 (middle column) and \tilde{M}_{bC}^2 versus M_{ab}^2 (right column). The left column shows the corresponding results in the ideal case when we use the *true* momenta of the invisible particles in the event. The mass spectrum is fixed as in (4.1).

row. We therefore conclude that the similarity between the two Dalitz plots is an indication of an off-shell scenario as in figure 14(b).

Finally, the bottom row in figure 16 represents the case of the event topology from figure 14(d), which again has a pair of identical intermediate resonances, B_i , only now the visible particles, a_i and b_i , are emitted in the opposite order — b_i comes first and a_i comes second.²⁰ Comparing this to the decay topology of figure 14(a), we see that the only

²⁰In this way we are trying to resolve the combinatorial ambiguity associated with the assignment of visible particles within a given decay chain.

difference is that the roles of the visible particles, a_i and b_i , are reversed. Therefore our previous analysis leading up to eqs. (5.2) and (5.3) still applies, only now the two trends are interchanged — the correlation in the (M_{aC}^2, M_{ab}^2) plane is expected to be a vertical straight line, while the correlation in the (M_{bC}^2, M_{ab}^2) plane is expected to be a slanted straight line. This ideal case with perfect knowledge of the invisible momenta is shown in the left bottom panel of figure 16. The more realistic case, in which the invisible momenta are taken from the $M_{2CC}(ab)$ minimization, is presented in the middle and right bottom panels of figure 16. As expected, the behavior is exactly the opposite of what we observed in the corresponding plots in the upper row of figure 16. Our conclusion, therefore, is that whenever the two scatter plots are different, the visible particle in the scatter plot with the vertical correlation is the one which is emitted second, while the visible particle in the slanted scatter plot is the one which is emitted first.

5.3 Resonance scatter plot test

Finally, we describe a test aimed at detecting and identifying any intermediate resonances, B_i . In particular, we shall revisit the event topologies from figures 14(a), 14(b), and 14(c) and attempt to answer the questions:

- Is there an intermediate B_i resonance in each decay chain?
- If so, are the two B_i particles the same or not?

Once again, the idea is to use the invisible momenta found by one of the M_2 -type minimizations and then reconstruct the masses of the hypothesized B_i resonances. As already discussed in section 3, the novel advantage of the M_2 -type variables (e.g., over transverse variables like M_{T2}) is that they supply the full 3-momenta of the invisible particles, including the longitudinal components. Thus, it becomes possible to carry out the direct reconstruction of any heavy particles along the decay chain. In our case, to form the mass of particle B_i , we simply use the measured 4-momentum of b_i and the momentum of C_i obtained in the minimization of $M_{2CX}(ab)$.²¹ In order to avoid the two-fold ambiguity discussed in section 3.2, we use only “balanced” events, for which the invisible momentum configuration is unique.

Since each event contains two decay chains, we will obtain two reconstructed values per event, \tilde{M}_{B_1} and \tilde{M}_{B_2} , which we order as usual as

$$\tilde{M}_B^> = \max \left\{ \tilde{M}_{B_1}, \tilde{M}_{B_2} \right\}, \tag{5.4}$$

$$\tilde{M}_B^< = \min \left\{ \tilde{M}_{B_1}, \tilde{M}_{B_2} \right\}. \tag{5.5}$$

We then investigate the resonance structure of the corresponding scatter plot in the $(\tilde{M}_B^>, \tilde{M}_B^<)$ plane, as shown in figure 17.

The left panel in figure 17 represents the case of the event topology from figure 14(a), which has two identical intermediate resonances, B_1 and B_2 . Correspondingly, the scatter

²¹Here we prefer to avoid any bias from using momenta from $M_{2XC}(ab)$ or $M_{2CC}(ab)$, which assume the presence of identical intermediate resonances from the outset.

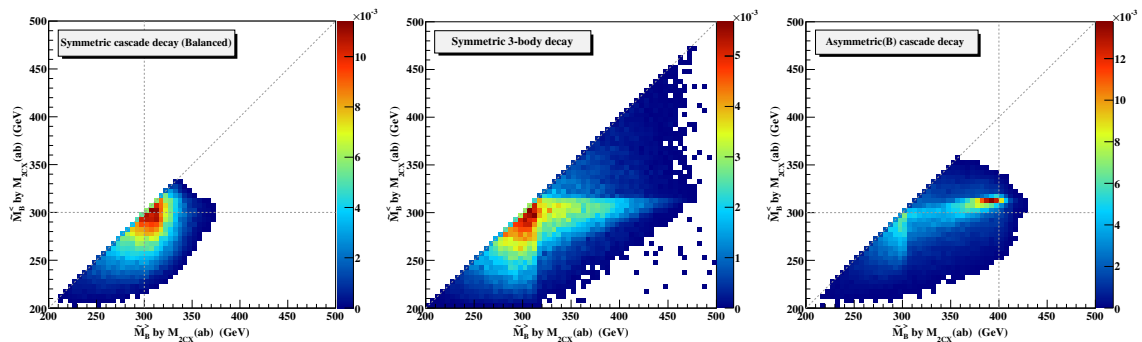


Figure 17. Scatter plots of the reconstructed masses of the two intermediate resonances, \tilde{M}_{B_1} and \tilde{M}_{B_2} , for the three scenarios from figure 15, with invisible momenta taken from the $M_{2CX}(ab)$ minimization. The larger of the two reconstructed masses, $\tilde{M}_B^{>}$, is plotted on the x -axis, while the smaller of the two reconstructed masses, $\tilde{M}_B^{<}$, is plotted on the y -axis. The vertical and horizontal black dashed lines denote the true masses of the associated relative particles.

plot exhibits a distinct clustering of events near the diagonal line ($\tilde{M}_B^{>} = \tilde{M}_B^{<}$), indicating the presence of such identical resonances. Furthermore, we can also roughly read the mass scale as $M_{B_i} \sim 300$ GeV (compare with the true values marked with the black dashed lines). Now contrast this situation with the case of the event topology from figure 14(c), which is shown in the rightmost panel of figure 17. Again, we find a narrow clustering of points, indicating the presence of intermediate B_i resonances. Now, however, the cluster lies significantly far from the diagonal line, implying that the intermediate resonances are *different*. The location of the cluster is also consistent with the input mass spectrum ($m_{B_1} = 300$ GeV, $m_{B_2} = 400$ GeV, as indicated with the black dashed lines).

The third example, shown in the middle panel of figure 17, is the event topology from figure 14(b). The two decay chains are the same, so we expect most of the events to end up near the diagonal line $\tilde{M}_B^{>} = \tilde{M}_B^{<}$. However, since there are no intermediate resonances, we do not expect a significant clustering in any particular location and instead would expect a broader distribution that in the previous two resonant cases. These expectations are confirmed in figure 17 — the middle panel exhibits a large population near the diagonal line whose structure differs from that in the left panel, allowing us to distinguish the topology of figure 14(b) from the topology of figure 14(a). Again, we defer a more detailed shape analysis to future work.

6 Conclusions and outlook

The main goal of this paper is to advocate a wider use of the 3+1-dimensional M_2 -type variables, which so far have been used only sporadically [34, 41, 50, 51]. In contrast, transverse mass variables like M_T , M_{T2} , M_{CT} , etc. have found widespread application in both precision measurements [69, 72] and in searches for new physics [73, 74]. There are two main advantages of the 3+1-dimensional formulation in terms of M_2 :

1. It is very easy to impose various additional assumptions about the underlying event topology [41]. In this paper we illustrated this feature with the addition of on-shell

constraints for the relative particles, which led us to two new variables, M_{2XC} and M_{2CC} . The benefits from M_{2XC} and M_{2CC} are twofold — first, the solution for the longitudinal invisible momenta is unique, as discussed in section 3.3, and second, their distributions exhibit much sharper endpoints, as demonstrated in section 4.1.

2. The minimization procedure required to calculate the value of M_2 fixes *all* components of the invisible particle momenta, including the longitudinal components. This gives an event with fully determined kinematics, opening the door for a number of precision reconstruction studies. As an illustration, in section 4.2, we reconstructed the mass of the relative particle and showed that the peak of the resulting distribution is nicely correlated with the true mass of the relative particle. This provides a new technique for mass measurements in missing energy events, which is complementary to the existing methods based on measuring kinematic endpoints.

It is interesting to note that even for a process as simple as the one studied here (see figure 1), we were able to define a relatively large number of M_2 -type variables, summarized in table 2. While the casual reader might feel intimidated by this proliferation of kinematic mass variables, we emphasize that there is a great benefit in having such a large arsenal of kinematic variables at one's disposal. The main reason why there are so many variables is that each involves different levels of assumptions. Thus, by testing for consistency of the results obtained with two different variables, we are essentially checking the validity of the assumptions that are present in one of the variables but not the other.

Following this idea, we developed several tests for distinguishing among the alternative event topologies of figure 14, which lead to the same final state:

- *Endpoint test.* In section 5.1, we proposed a test which compares the endpoints of the distributions of M_2 variables with and without relative constraints. If the constraints are satisfied in the event sample, the kinematic endpoints would match (even though the shapes of the distributions are generally different). Conversely, if the mass constraints are not satisfied, the endpoints will be different, which is an indication that our hypothesis regarding the event topology is wrong. We have checked that this test is applicable even when one does not have precise knowledge of the daughter mass.
- *Dalitz plot test.* In section 5.2, we used the fact that the computation of the M_2 variables supplies values for the 4-momenta of the invisible particles and proposed to build Dalitz-type plots of invariant mass combinations which include the invisible particles themselves (see figure 16). We showed that the distinctive shape of the Dalitz scatter plots can be used to ascertain the presence of intermediate resonances and to resolve the combinatorial ambiguity related to the ordering of the visible final state particles along the decay chain.
- *Resonance scatter plot test.* The invisible momenta supplied by M_2 found another application in section 5.3, where we were able to test for the symmetry of the events, i.e., whether the two decay chains are the same or not (see figure 17).

These are just a few of the many potential applications of the M_2 variables — for example, one could imagine spin measurements along the lines of [29, 48, 57], using the invisible particle momenta supplied by the M_2 minimizations. It is also possible to further extend the set of variables from table 2 to more complicated event topologies — e.g., decay chains with more than one relative particle, decay chains with relatives of known mass, etc. One technical problem which will need to be addressed in the near future is the lack of a public code for the calculation of the on-shell constrained M_2 variables. The availability of such code would certainly encourage more experimentalists to make use of these variables whose benefits seem undeniable.

Acknowledgments

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A Proof that $M_{T2} = M_{2CX}$ with the method of Lagrange multipliers

In this appendix, we show the equivalence between M_{T2} and M_{2CX} using the method of Lagrange multipliers. For concreteness, the formal proof is presented for the (ab) subsystem, but the same argument can be applied to the other subsystems, (a) and (b) , as well.

In order to calculate $M_{2CX}^2(ab)$, we must perform the minimization of the function

$$\max [M_{A_1}^2(\vec{q}_{1T}, \Delta\eta_1; \tilde{m}), M_{A_2}^2(\vec{q}_{2T}, \Delta\eta_2; \tilde{m})], \quad (\text{A.1})$$

subject to the two constraints

$$M_{A_1}^2(\vec{q}_{1T}, \Delta\eta_1; \tilde{m}) = M_{A_2}^2(\vec{q}_{2T}, \Delta\eta_2; \tilde{m}), \quad (\text{A.2})$$

$$\vec{q}_{1T} + \vec{q}_{2T} = \vec{P}_T. \quad (\text{A.3})$$

Here we have already assumed that the hypothesized masses of the daughter particles, C_i , are the same, so that there is a single input test mass, \tilde{m} . We have also expressed the parent masses, M_{A_i} , as functions of $\Delta\eta_i$ instead of q_{iz} , as in (3.7).

We can use the method of Lagrange multipliers to reformulate the problem as the unconstrained minimization of a new target function

$$\begin{aligned} & f(\vec{q}_{1T}, \vec{q}_{2T}, \Delta\eta_1, \Delta\eta_2, \vec{\lambda}_T, \lambda_\eta; \tilde{m}) \\ &= \frac{1}{2} \{ E_{v_1T} E_{q_1T} \cosh \Delta\eta_1 - \vec{p}_{v_1T} \cdot \vec{q}_{1T} + E_{v_2T} E_{q_2T} \cosh \Delta\eta_2 - \vec{p}_{v_2T} \cdot \vec{q}_{2T} \} \\ & \quad + \lambda_\eta \{ E_{v_1T} E_{q_1T} \cosh \Delta\eta_1 - \vec{p}_{v_1T} \cdot \vec{q}_{1T} - E_{v_2T} E_{q_2T} \cosh \Delta\eta_2 + \vec{p}_{v_2T} \cdot \vec{q}_{2T} \} \\ & \quad + \vec{\lambda}_T \cdot (\vec{q}_{1T} + \vec{q}_{2T} - \vec{P}_T), \end{aligned} \quad (\text{A.4})$$

which needs to be minimized over all of its arguments: \vec{q}_{1T} , \vec{q}_{2T} , $\Delta\eta_1$, $\Delta\eta_2$, $\vec{\lambda}_T$, and λ_η . The constraint (A.2) is implemented through the Lagrange multiplier λ_η , while the

constraint (A.3) is incorporated through the Lagrange multiplier $\vec{\lambda}_T$. In view of the constraint (A.2) in the first term we have replaced (A.1) with the average of $M_{A_1}^2$ and $M_{A_2}^2$.

The extremum conditions for \vec{q}_{iT} and $\Delta\eta_i$ read:

$$\vec{\nabla}_{q_{1T}} f = \left(\frac{1}{2} + \lambda_\eta\right) \left(\frac{E_{v_1T}}{E_{q_1T}} \cosh \Delta\eta_1 \vec{q}_{1T} - \vec{p}_{v_1T}\right) + \vec{\lambda}_T = 0, \quad (\text{A.5})$$

$$\vec{\nabla}_{q_{2T}} f = \left(\frac{1}{2} - \lambda_\eta\right) \left(\frac{E_{v_2T}}{E_{q_2T}} \cosh \Delta\eta_2 \vec{q}_{2T} - \vec{p}_{v_2T}\right) + \vec{\lambda}_T = 0, \quad (\text{A.6})$$

$$\frac{\partial f}{\partial \Delta\eta_1} = \left(\frac{1}{2} + \lambda_\eta\right) E_{v_1T} E_{q_1T} \sinh \Delta\eta_1 = 0, \quad (\text{A.7})$$

$$\frac{\partial f}{\partial \Delta\eta_2} = \left(\frac{1}{2} - \lambda_\eta\right) E_{v_2T} E_{q_2T} \sinh \Delta\eta_2 = 0. \quad (\text{A.8})$$

There are two cases which can be considered separately: i) $\lambda_\eta = 1/2$ (or $\lambda_\eta = -1/2$), and ii) $\lambda_\eta \neq \pm 1/2$.

i) $\lambda_\eta = 1/2$: since $\lambda_\eta = 1/2$, (A.8) is automatically solved, so $\Delta\eta_2$ remains arbitrary. Then (A.6) implies $\vec{\lambda}_T = 0$, and from (A.7) it follows that $\Delta\eta_1 = 0$. Finally, (A.5) leads to $\vec{q}_{1T} = \frac{E_{q_1T}}{E_{v_1T}} \vec{p}_{v_1T}$. Substituting these results into (A.4), we get

$$f = E_{v_1T} E_{q_1T} - \vec{p}_{v_1T} \cdot \vec{q}_{1T} = M_{TA_1}^2(ab), \quad (\text{A.9})$$

which is nothing but the transverse mass of A_1 as given in (3.3). This implies that minimizing f is equivalent to minimizing the transverse mass of A_1 .

The same logic can be applied to the case $\lambda_\eta = -1/2$, where one finds that the problem reduces to the minimization of the transverse mass of A_2 . Thus we conclude that these two cases with $|\lambda| = \frac{1}{2}$ simply correspond to the *unbalanced* configuration of the $M_{T2}(ab)$ variable.

ii) $\lambda_\eta \neq \pm 1/2$: since $\lambda_\eta \neq \pm 1/2$, the only way to satisfy eqs. (A.7) and (A.8) is to have $\Delta\eta_i = 0$. This reduces the function (A.4) to

$$f = \frac{1}{2}(M_{TA_1}^2(ab) + M_{TA_2}^2(ab)) + \lambda_\eta(M_{TA_1}^2(ab) - M_{TA_2}^2(ab)) + \vec{\lambda}_T \cdot (\vec{q}_{1T} + \vec{q}_{2T} - \vec{P}_T), \quad (\text{A.10})$$

which is nothing but the Lagrange function associated with the *balanced* solution of the corresponding M_{T2} variable.

From i) and ii) we see that $M_{2CX}(ab)$ includes both the balanced and the unbalanced configurations of $M_{T2}(ab)$, thus it follows that $M_{2CX}(ab) = M_{T2}(ab)$.

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