

# Scalar potentials, propagators and global symmetries in AdS/CFT

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## Abstract

We study the transition of a scalar field in a fixed  $AdS_{d+1}$  background between an extremum and a minimum of a potential. We first prove that two conditions must be met for the solution to exist. First, the potential involved cannot be generic, i.e. a fine-tuning of their parameters is mandatory. Second, at least in some region its second derivative must have a negative upper limit which depends only on the dimensionality  $d$ . We then calculate the boundary propagator for small momenta in two different ways: first in a WKB approximation, and second with the usual matching method, generalizing the known calculation to arbitrary order. Finally, we study a system with spontaneously broken non-Abelian global symmetry, and show in the holographic language why the Goldstone modes appear.

## 1 Introduction

The simplest example of AdS-CFT correspondence [1–3] is gravity plus a real scalar field system in asymptotic anti-de-Sitter (AdS) space (for a partial list see [4–10]).

Apart from some special cases (see for example [11]) it is expected that the even simplified version of such systems, i.e. the no-back-reaction limit where the gravitational coupling  $\kappa \rightarrow 0$ , would give the relevant information (for some reviews on this subject see for example [12–15]). Recently this has been done in [16], where the potential of the real scalar field has been approximated by a piece-wise quadratic potential in order to allow analytic treatment. It has been then shown that: a) in order for the solution between the UV extremum and the IR minimum to exist, there must be some non-trivial constraint among parameters in the potential; b) at least one region needs  $V'' < -d^2/4$ , where  $d$  is the dimension of the boundary; c) a solution of such a system has vanishing action and d) the propagator in the boundary theory exhibits a simple  $1/q^2$  pole as predicted by the Goldstone theorem applied to the spontaneously broken dilatation invariance [17].

The last two points has been considered in more detail in [18] (see also [19] and [20]) following an inspiring paper [21], where it was explicitly shown that even such a simplified system has a BPS type solution which exhibits the Goldstone theorem for a spontaneously broken conformal invariance in subtle way, i.e. mixing the normalizable and non-normalizable modes in the bulk at the next-to-leading order of the matching method.

The purpose of this contribution is twofold. First, we would like to shed more light on the first two issues, i.e. on the constraints the potential must satisfy for allowing a solution. Second, we would like to see the  $1/q^2$  propagator of the Goldstone in a different way, generalize the matching method at all orders, and present few examples of its use.

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The plan of the paper is the following. After setting the notation and main formulae in Section 2, we summarize in Section 3 the wall solution found in [16]. In Section 4 we then explain the reason for a fine-tuning of the potential parameters and explicitly show how one can find BPS-type solutions to the first order equation of motion even in this no-backreaction limit, i.e. generalize the usual  $\kappa \neq 0$  expression of the potential through the superpotential to the  $\kappa \rightarrow 0$  limit. With it we can prove in Section 5 in complete generality that for the solution to exist, the second derivative of the scalar potential must be smaller than  $-d^2/4$  in at least some region. In section 6 we find the same  $1/q^2$  propagator in the  $q \rightarrow 0$  limit of the dilaton using then WKB approximation, while a long Section 7 is devoted to a detailed analysis of the matching method to all orders. This is then used in Section 8 to show in an explicit example what exactly makes the Goldstone boson of a global symmetry massless in the holographic language. We conclude in section 9 with a brief summary of the results.

## 2 The no back-reaction limit ( $\kappa \rightarrow 0$ )

We will consider in most of this paper a real scalar field  $t$  in  $d + 1$  dimensions with bulk euclidean action

$$S^{(bulk)}[t] = \int d^{d+1}x \sqrt{\det g_{ab}} \left( \frac{1}{2} g^{ab} \partial_a t \partial_b t + V(t) \right) \quad (1)$$

in a non-dynamical  $AdS_{d+1}$  background

$$g = \frac{1}{z^2} (dz^2 + \delta_{\mu\nu} dx^\mu dx^\nu) \quad (2)$$

where  $(x^\mu)$  are the QFT coordinates with  $x^d \equiv i x^0$  the euclidean time and the AdS scale has been set to 1. The boundary is located at  $z = 0$  (UV region) while the horizon is at  $z = \infty$  (IR region).

The dimensionless field variable  $t$  is normalized to have extrema of the potential at  $t = 0, 1$ . More precisely, we will consider potentials (throughout the paper we will indicate with a dot the derivative w.r.t. the bulk coordinate  $z$  and with a prime a field derivative)

$$V(0) = 0 \quad , \quad V'(0) = 0 \quad ; \quad V(1) < 0 \quad , \quad V'(1) = 0 \quad , \quad V''(1) > 0 \quad (3)$$

i.e.  $t = 1$  will be the true minimum, while at the origin the potential can have a minimum (being a false vacuum thus) or even a maximum, provided that it is in the Breitenlöhner-Freedman conformal window  $-d^2/4 < V''(0) < 0$ .

We will be interested in regular, Poincarè invariant solutions  $t = t(z)$  that interpolate between the UV (extremum at  $t = 0$ ) and IR (true minimum at  $t = 1$ ) regions. They obey the equation of motion

$$z^2 \ddot{t}(z) - (d-1) z \dot{t}(z) = V'(t) \quad (4)$$

and necessary behave in the UV and IR as

$$t(z) \xrightarrow{z \rightarrow 0} a_{UV} z^{\Delta^{UV}} \quad ; \quad t(z) \xrightarrow{z \rightarrow \infty} 1 + a_{IR} z^{d-\Delta^{IR}} \quad (5)$$

respectively, where

$$\Delta^{UV/IR} \equiv \frac{d}{2} + \nu_{UV/IR} \quad ; \quad \nu_{UV/IR} \equiv \sqrt{\frac{d^2}{4} + m_{UV/IR}^2} \quad (6)$$

with  $m_{UV}^2 \equiv V''(0)$  and  $m_{IR}^2 \equiv V''(1) > 0$  ( $t = 1$  is a minimum according to (3))<sup>1</sup>.

We recall as a last remark that the symmetries of  $AdS$  space translate in the scale invariance of equation (4), i.e. if  $t(z)$  is a solution so it is  $t(\lambda z)$ , a fact of great relevance.

<sup>1</sup>In the window  $-\frac{d^2}{4} < V''(0) < 0$  the term  $z^{d-\Delta^{UV}}$  could also be present in the small  $z$  power expansion of  $t(z)$ . From the AdS/CFT point of view this term is interpreted as a source that breaks explicitly the scale invariance of the boundary QFT; then we should not expect a Goldstone mode to appear, situation we are not interested in. These domain walls are interpreted as dual to renormalization group flows generated by deformation of the UV CFT by a relevant operator, i.e. one of dimension less than  $d$  [15].

### 3 Analytic solutions for approximated bulk potentials

In this section we shortly summarize the results presented in [16].

The interesting region for  $t$  is between the local minimum at 0 and the global minimum at 1. We will divide this region into a number of sections, and in each of them the potential can be locally approximated by a quadratic form:

$$V(t) = \frac{A}{2} t^2 + B t + C \quad (7)$$

The minimum number of such sections is three: (1)  $0 < t < t_1$ , (2)  $t_1 < t < t_2$ , (3)  $t_2 < t < 1$ . The coefficients in (7) are parameterized in each region as

$$A = \begin{cases} A_1 > 0 \\ A_2 < 0 \\ A_3 > 0 \end{cases} ; \quad B = \begin{cases} 0 \\ -A_2 t_M \\ -A_3 \end{cases} ;$$

$$C = \begin{cases} 0 \\ (A_1 - A_2) t_1^2/2 + A_2 t_M t_1 \\ (A_2 - A_3) t_2^2/2 + A_3 t_2 - A_2 t_M t_2 + (A_1 - A_2) t_1^2/2 + A_2 t_M t_1 \end{cases} \quad (8)$$

respectively. The strange choice of  $C$ 's is required by the continuity of the potential. Furthermore, we will require the continuity of the first derivatives of the potential which yields to

$$t_1 = \frac{-A_2 t_M}{A_1 - A_2} ; \quad t_2 = \frac{A_3 - A_2 t_M}{A_3 - A_2} \quad (9)$$

relations that automatically satisfy  $0 < t_1 < t_M < t_2 < 1$  for any  $0 < t_M < 1$ . In this way we remain with four relevant parameters, the  $A_i$ 's and  $t_M$ .

Similarly as in (6) we introduce

$$\Delta_i^\pm = d/2 \pm \nu_i ; \quad \nu_i^2 \equiv \frac{d^2}{4} + A_i \quad (10)$$

We will consider the case of real  $\nu_{1,3} > \frac{1}{2} (A_{1,3} > 0)$  and pure imaginary  $\nu_2 \equiv i \bar{\nu}_2 (A_2 < -\frac{d^2}{4})$  with  $\bar{\nu}_2 > 0$ .

The solution to (4) with the piece-wise quadratic potential (7) is known

$$t_b(z) = \begin{cases} t_1 (z/z_1)^{\Delta_1^+} & , \quad 0 < z < z_1 \\ t_M + D_+ (z/z_2)^{\Delta_2^+} + D_- (z/z_2)^{\Delta_2^-} & , \quad z_1 < z < z_2 \\ 1 - (1 - t_2) (z/z_2)^{\Delta_3^-} & , \quad z_2 < z < \infty \end{cases} \quad (11)$$

Continuity of the solution and its derivative at  $z_{1,2}$  requires

$$D_+ = \frac{(1 - t_M) \Delta_2^- \Delta_3^- (\nu_3 - i \bar{\nu}_2)}{2i \bar{\nu}_2 (A_3 - A_2)} , \quad D_- = D_+^* \quad (12)$$

$$t_M = \left( 1 - \frac{\Delta_1^+}{\Delta_3^-} \left( \frac{\nu_3^2 + \bar{\nu}_2^2}{\nu_1^2 + \bar{\nu}_2^2} \right)^{1/2} \left( \frac{z_2}{z_1} \right)^{d/2} \right)^{-1} \quad (13)$$

$$\bar{\nu}_2 \log(z_2/z_1) = (2k + 1)\pi - \alpha_1 - \alpha_3 \quad (14)$$

with

$$\alpha_i \equiv \arctan(\bar{\nu}_2/\nu_i) , \quad i = 1, 3 \quad (15)$$

Notice here two things:

- there is one relation (fine-tuning) among the potential parameters  $A_i, t_M$ , see eqs. (13) and (14),
- $\nu_2$  needs to be purely imaginary.

The whole procedure can be repeated with more intervals, but these two conclusions still remain: A non-trivial fine-tuning among parameters is needed, and at least in one interval  $\nu^2 = d^2/4 + V''$  needs to be negative. In the next two sections we will try to understand better these two issues.

#### 4 Why the potential cannot be generic

Eqs. (13) and (14) represent the quantization condition on the potential for the solution to exist at all. As it has been noted in [16] and remarked before, this follows from the invariance of the equation of motion under dilatations  $z \rightarrow \lambda z$  for any positive real  $\lambda$ . There is thus an infinite family of solutions: the location of the domain wall is not determined. In our previous example this is seen explicitly by the fact that the coordinates  $z_1$  and  $z_2$  cannot be determined both, but due to dilatation invariance of the original equation of motion only their ratio. The four equations (functions and derivatives at  $z_{1,2}$ ) cannot be satisfied by only three parameters  $D_{+,-}, z_2/z_1$ , so a non-trivial relation among potential parameters follow. This simple counting can be easily generalized to an arbitrary number of intervals.

What happens if a fine-tuned potential changes a bit, i.e. if we relax the constraint among the potential parameters? The numerical output will make  $t(z)$  diverge, so that for  $z \rightarrow \infty$  limit it will not reach the unit value. In other words, the transition is not from the extremum in the origin to the minimum at  $t = 1$ , but it escapes to infinity. In order to make the field land to the minimum, one needs a constrained value for the model parameters.

There are two simple ways to see why there must be some constraint among the model parameters, if we are looking for a solution of (4).

First of all, we have a second order differential equation. In the limit  $z \rightarrow 0$  this non-linear equation can be linearized, call the two independent solutions of this linearized version  $t_+(z)$  and  $t_-(z)$ . Let them be defined so that for  $z \rightarrow 0$ ,  $t_+(z) \propto z^{\Delta^{UV}}$  with  $\Delta^{UV}$  given in (6) and  $t_-(z) \propto z^{d-\Delta^{UV}}$ . This second  $t_-(z)$  is interpreted in the AdS-CFT dictionary as a source. All solutions to the original full non-linear equations have to evolve only towards  $t_+(z)$  for  $z \rightarrow 0$  in order for the source to vanish. There is however no guarantee that these solutions are finite for  $z \rightarrow \infty$ . In general it will not be the case, only solutions which evolve to some linear combination  $a t_+(z) + b t_-(z)$  for  $z \rightarrow 0$  will be finite in the opposite limit at  $z \rightarrow \infty$ . We can enforce  $b = 0$  and thus have a  $t(z)$  sourceless at  $z \rightarrow 0$  and finite at  $z \rightarrow \infty$  only by carefully choosing the parameters of the original Lagrangian, i.e. the potential. From here the fine-tuning among parameters.

Another way perhaps more familiar of setting the problem is through the linearized perturbation equation around the assumed solution  $t(z)$ . If we write the perturbation as  $\xi(z; q) e^{iq \cdot \frac{x}{L}}$ , such equation results (82). We can rewrite this linearized equation for perturbations in a Schrödinger-like form. Taking  $\xi(z; q) = z^{\frac{d-1}{2}} f(z; q)$  we get,

$$\ddot{f}(z; q) - \left[ q^2 + \frac{1}{z^2} \left( \frac{d^2 - 1}{4} + V''(t(z)) \right) \right] f(z; q) = 0 \quad (16)$$

Now, well-known symmetry arguments (in this case related to dilatation invariance) show that  $\xi(z; 0) \sim z \dot{t}(z)$  solves equation (82) with  $q^2 = 0$ . But (16) is a second order linear differential equation with two independent solutions and then standard quantum mechanics arguments work. By definition, necessary  $f(z; 0) \sim z^{\frac{1}{2}-\nu_{IR}}$  for  $z \rightarrow \infty$  and the solution that goes as  $z^{\frac{1}{2}+\nu_{IR}}$  must be discarded. Similarly,  $f(z; 0) \sim z^{\frac{1}{2}+\nu_{UV}}$  for  $z \rightarrow 0$  and the solution that goes as  $z^{\frac{1}{2}-\nu_{UV}}$  must be discarded too. The only way for this solution of (16) to exist is that in both cases we remain with the same function. As the “energy” is zero it cannot be quantized as it is usually the case in QM, so  $z \dot{t}(z)$  can exist only when a fine-tuned relation among parameters in the potential holds, and so also the solution  $t(z)$  of (4) exists only in this case.

#### 4.1 Fine-tuning the cosmological constant on the boundary

As it has been explained in [24] the fine-tuning needed for the potential parameters is nothing else than the requirement for a vanishing cosmological constant on the boundary. To see it more explicitly we have of course to reintroduce gravity, i.e. a non-zero  $\kappa$ .

Let us thus consider the gravity-scalar system defined by the action,

$$S = \int d^{d+1}x \sqrt{|g|} \left( \frac{1}{2\kappa^2} (R + d(d-1)) - \frac{1}{2} D^M t D_{Mt} - V(t) \right) \quad (17)$$

The following equations of motion follow,

$$\begin{aligned} R_{MN} &= -d g_{MN} + \kappa^2 \left( T_{MN} - \frac{T^P_P}{d-1} g_{MN} \right) \\ D^K D_{Kt} &= V'(t) \end{aligned} \quad (18)$$

where the energy momentum-tensor for the scalar field is,

$$T_{MN} = D_{Mt} D_{Nt} - \left( \frac{1}{2} D^K t D_{Kt} + V(t) \right) g_{MN} \quad (19)$$

We are going to consider the ansatz,

$$\begin{aligned} g &= d\rho^2 + A^2(\rho) \hat{g} \\ t &= t(\rho) \end{aligned} \quad (20)$$

where  $\hat{g} \equiv \eta_{mn} \hat{\omega}^m \hat{\omega}^n$  is the metric ( $\{\hat{\omega}^m\}$  is a vielbein) on a  $d$ -dimensional space-time with generic coordinates  $\Omega$ . In the obvious local basis,

$$\omega^m \equiv A(\rho) \hat{\omega}^m \quad , \quad m = 0, 1, \dots, d-1 \quad ; \quad \omega^d \equiv d\rho \quad (21)$$

the connections are,

$$\omega^m_n = \hat{\omega}^m_n \quad ; \quad \omega^m_d = \frac{A'(\rho)}{A(\rho)} \omega^m \quad (22)$$

where only in this subsection a prime means  $d/d\rho$ .

The two-forms defining the curvature tensor result,

$$\mathcal{R}_{mn} = \hat{\mathcal{R}}_{mn} - \frac{A'(\rho)^2}{A(\rho)^2} \omega_m \wedge \omega_n \quad ; \quad \mathcal{R}_{md} = -\frac{A''(\rho)}{A(\rho)} \omega_m \wedge \omega^d \quad (23)$$

Finally the Ricci tensor components are,

$$\begin{aligned} R_{mn} &= \frac{1}{A(\rho)^2} \hat{R}_{mn} - \left( \frac{A''(\rho)}{A(\rho)} + (d-1) \frac{A'(\rho)^2}{A(\rho)^2} \right) \eta_{mn} \\ R_{dd} &= -d \frac{A''(\rho)}{A(\rho)} \\ R_{md} &= 0 \end{aligned} \quad (24)$$

and the Ricci scalar,

$$R = \frac{1}{A(\rho)^2} \hat{R} - 2d \frac{A''(\rho)}{A(\rho)} - d(d-1) \frac{A'(\rho)^2}{A(\rho)^2} \quad (25)$$

With (20) and (24) the equations (18) become,

$$\hat{R}_{mn} - A^2 \left( \frac{A''(\rho)}{A(\rho)} + (d-1) \frac{A'(\rho)^2}{A(\rho)^2} - d + \frac{2\kappa^2}{d-1} V(t) \right) \eta_{mn} = 0$$

$$\begin{aligned} \frac{A''(\rho)}{A(\rho)} - 1 + \frac{\kappa^2}{d} \left( t'(\rho)^2 + \frac{2}{d-1} V(t) \right) &= 0 \\ t''(\rho) + d \frac{A'(\rho)}{A(\rho)} t'(\rho) - \frac{dV}{dt}(t) &= 0 \end{aligned} \quad (26)$$

We have now two possible cases.

### Case I: Vacuum solutions

Let us consider  $t(\rho) = t_v$  an extremum of the potential,  $V'(t_v) = 0$ , and let us take  $V(t_v) = 0$ . Then there exist three non equivalent, exact solutions to the gravity equations in (26),

$$\begin{aligned} g &= d\rho^2 + e^{2\rho} \hat{g} & ; & \quad \hat{R}_{mn} = 0 \\ g &= d\rho^2 + \cosh^2(\rho) \hat{g} & ; & \quad \hat{R}_{mn} = -(d-1) \eta_{mn} \\ g &= d\rho^2 + \sinh^2(\rho) \hat{g} & ; & \quad \hat{R}_{mn} = +(d-1) \eta_{mn} \end{aligned} \quad (27)$$

We recognize the first case as plane  $AdS_{1,d}$  if  $\hat{g}$  is identified with the flat Minkowski metric, the maximally symmetric case. On the other hand, the second/third solutions correspond to Einstein space-times of negative/positive curvature, being the most symmetric choices for  $\hat{g}$  the spaces  $AdS_{1,d-1}/dS_{1,d-1}$  with scale  $L = 1$ . However from (18) we see that in any case the equation for the bulk metric  $g$  is just  $R_{MN} = -d g_{MN}$ ; so the maximally symmetric choices should lead to the same space, i.e. the three cases in (27) must correspond to  $AdS_{1,d}$  sliced differently.<sup>2</sup> An observation:  $z \equiv e^{-\rho}$  is the usual coordinate with  $z = 0$  the boundary and  $z = \infty$  the horizon iff the  $\rho$ -coordinate is the one defined in the patch of the first solution, i.e.  $\rho$  represents different coordinates in each line of (27).

### Case II: Domain wall solutions

In this case the profile of the scalar must be non trivial; in particular we are interested in interpolating solutions like the ones considered in the papers. We can however always take the weak gravity, decoupling limit  $\kappa \rightarrow 0$ , and we must solve the scalar equation in the background (27). Now, if we consider the flat slicing, we found the need of fine-tuning the potential in order to get a solution. The question is: if we interpret the other two slicings as leading to  $AdS$  and  $dS$  space-time geometries of the boundary theory instead of Minkowski, is it necessary to fine-tuning the potential to get a domain wall solution also in these cases?

With the new variable  $z = e^{-\rho}$  (and for simplicity keeping the same notation for  $t = t(z)$ ) the equation to solve is,

$$z^2 \ddot{t}(z) - \frac{(d-1) + k(d+1)z^2}{1 - kz^2} z \dot{t}(z) - \frac{dV}{dt}(t) = 0 \quad (28)$$

where  $k = +1, 0, -1$  in the  $dS$ , Minkowski,  $AdS$  slicing. There is no dilatation symmetry anymore, so no need for fine-tuning. In the language of the piece-wise-quadratic potential, all the coordinates of different intervals can be determined, and not only ratios. No relations among parameters is needed for the solution to exist. It is now clear the physical meaning of it: it is just the fine-tuning of the boundary cosmological constant.

## 4.2 The BPS solutions

A solution that spontaneously breaks conformal invariance makes the on-shell action vanish (see for example [23]). This is a hint that the solution may be of the BPS type, i.e. it solves a first order equation [16]. Instead of proving this statement, we will show how one can define the superpotential that

<sup>2</sup>In fact the third form can be found in equation (3.1) of [25].

allows a smooth  $\kappa \rightarrow 0$  limit. Let's go back to (17). We will search for solutions to the equations of motion of the form

$$\begin{aligned} g &= \frac{1}{z^2} \left( d\vec{x}^2 + L^2 \frac{dz^2}{F(z)} \right) \\ t &= t(z) \end{aligned} \quad (29)$$

The b.c. at the boundary  $z = 0$  are,

$$t(z) \rightarrow 0 \quad ; \quad F(z) \rightarrow 1 \quad (30)$$

where,

$$V(0) = 0 \quad ; \quad V'(t)|_{t=0} = 0 \quad (31)$$

This assures for the solution to be asymptotically AdS with fixed radius  $L = 1$ .

At the horizon  $z = \infty$  we impose,

$$t(\infty) < \infty \quad ; \quad F(z) = F_h + \mathcal{O}\left(\frac{1}{z}\right) \quad (32)$$

The equations of motion result

$$\begin{aligned} z F'(z) &= \kappa^2 z^2 F(z) \dot{t}^2(z) \\ z^{d+1} F^{\frac{1}{2}}(z) \frac{d}{dz} \left( \frac{F^{\frac{1}{2}}(z)}{z^{d-1}} \frac{dt}{dz}(z) \right) &= V'(t) \end{aligned} \quad (33)$$

With no back-reaction ( $\kappa = 0$ ),  $F(z) = 1$  and it is the second equation to solve, just the scalar fields in the AdS background. When back-reaction is taken into account ( $\kappa > 0$ ) we can use the superpotential trick. The usual choice is consider potentials which can be written as

$$V(t) = \frac{1}{2} W'^2(t) - \frac{d \kappa^2}{4} W^2(t) + \frac{d}{\kappa^2} \quad (34)$$

Then it is possible to show that a solution of,

$$\begin{aligned} F(z) &= \frac{\kappa^4}{4} W^2(t)|_{t=t(z)} \\ t'(z) &= \frac{2}{\kappa^2 z} \frac{W'(t)}{W(t)} \Big|_{t=t(z)} \end{aligned} \quad (35)$$

solves (33).

This ansatz implicitly assume that  $\kappa \neq 0$ . On the other side, if we want eventually to get the no-backreaction limit  $\kappa \rightarrow 0$ , we choose a potential of the form

$$V(t) = \frac{1}{2} W'^2(t) - d W(t) - \frac{\kappa^2 d}{4} W^2(t) \quad (36)$$

It is then possible to show that a solution of,

$$\begin{aligned} F(z) &= H^2(W(t)) \Big|_{t=t(z)} \\ z \dot{t}(z) &= \frac{W'(t)}{H(W(t))} \Big|_{t=t(z)} \end{aligned} \quad (37)$$

where

$$H(W) \equiv 1 + \frac{\kappa^2}{2} W \quad (38)$$

is a solution of (33). The  $\kappa \rightarrow 0$  limit is now small and points toward the potential

$$V(t) = \frac{1}{2} W'^2(t) - dW(t) \quad (39)$$

and the following BPS like equation,

$$z \dot{t}(z) = W'(t(z)) \quad (40)$$

whose solutions satisfy also the full second order equation of motion (4) and for which the action (17) vanishes.

At least for polynomial superpotentials and potentials the fine-tuning for vanishing boundary cosmological constant is simply the special form (39). With this we mean that all coefficients of the polynomial in the potential are not independent and thus the potential itself is not generic.

Before ending this section, let us see some examples of superpotentials  $W(t)$  (in [18] we already showed another choice).

#### 4.2.1 The $Z_2$ symmetric case

An interesting case consists of the sixth order potential with the  $Z_2$  symmetry  $t \rightarrow -t$ . The ansatz for the superpotential

$$W(t) = \Delta \left( \frac{1}{2} t^2 - \frac{1}{4} t^4 \right) \quad (41)$$

leads to the solution,

$$t(z) = \frac{z^\Delta}{(1 + z^{2\Delta})^{1/2}} \quad (42)$$

From here we see that

$$\Delta^{UV} = \Delta \quad , \quad \Delta^{IR} = d + 2\Delta \quad (43)$$

#### 4.2.2 A case with $\Delta^{UV}$ and $\Delta^{IR}$ independent

In the examples of [18] and above a correlation between the UV and IR  $\Delta$ 's was present. This is however not a generic feature of the system. In fact, choosing for example

$$W(t) = -\frac{1}{4} (\Delta^{IR} - (d + \Delta^{UV})) t^4 + \frac{1}{3} (\Delta^{IR} - (d + \Delta^{UV}) - \Delta^{UV}) t^3 + \frac{\Delta^{UV}}{2} t^2 \quad (44)$$

we get the solution

$$z(t) = \left[ \frac{\Delta^{UV} + (\Delta^{IR} - (d + \Delta^{UV})) t}{1 - t} \right]^{\frac{1}{\Delta^{IR} - d}} \left[ \frac{t}{\Delta^{UV} + (\Delta^{IR} - (d + \Delta^{UV})) t} \right]^{\frac{1}{\Delta^{UV}}} \quad (45)$$

which has the limits (5) with

$$a_{UV} = (\Delta^{UV})^{\frac{\Delta^{IR} - (d + \Delta^{UV})}{\Delta^{IR} - d}} \quad ; \quad a_{IR} = -(\Delta^{IR} - d)^{-\frac{\Delta^{IR} - (d + \Delta^{UV})}{\Delta^{UV}}} \quad (46)$$

The parameters  $\Delta^{UV} > d/2$  (corresponding to the maximum or minimum in the UV) and  $\Delta^{IR} > d + \Delta^{UV}$  (minimum in the IR) can be otherwise arbitrary.



## 5 $V''(t) < -d^2/4$

As we said before, in a piece-wise quadratic potential at least in some interval the second derivative of the potential must be smaller than  $-d^2/4$  for the solution to exist. Let us here show this statement for a general potential  $V(t)$  characterized by (39). Let us define

$$F \equiv \int d\mu W'(t)^2 \left( V''(t) + \frac{d^2}{4} \right) \Big|_{t=t(z)} \quad (47)$$

where  $t(z)$  is the solution of the BPS equation (40) and to simplify the notation we will use in this subsection the abbreviation

$$\int d\mu \dots \equiv \int_0^\infty dz z^{-d-1} \dots \quad (48)$$

and omit the field dependence. Our aim is to show that the quantity  $F$  is non-positive, so that  $V'' < -d^2/4$  at least in some region.

First we rewrite (47) using (39)

$$F = \int d\mu \left( W'^2 W''^2 + W'^3 W''' - d W'^2 W'' + \frac{d^2}{4} W'^2 \right) \quad (49)$$

Now we use (assuming vanishing boundary terms, which is easily verified)

$$\int d\mu W'^2 = \frac{2}{d} \int d\mu W'^2 W'' \quad (50)$$

$$\int d\mu W'^3 W''' = \int d\mu (d W'^2 W'' - 2 W'^2 W''^2) \quad (51)$$

to rewrite (49) as

$$F = \frac{d}{2} \int d\mu W'^2 W'' - \int d\mu W'^2 W''^2 \quad (52)$$

Finally we use the Schwartz inequality

$$\int d\mu f g \leq \left( \int d\mu f^2 \right)^{\frac{1}{2}} \left( \int d\mu g^2 \right)^{\frac{1}{2}} \quad (53)$$

to derive from (50)

$$\int d\mu W'^2 \leq \frac{4}{d^2} \int d\mu W'^2 W''^2 \quad (54)$$

Using then (53) we get first

$$\int d\mu W'^2 W'' \leq \left( \int d\mu W'^2 \right)^{\frac{1}{2}} \left( \int d\mu W'^2 W''^2 \right)^{\frac{1}{2}} \quad (55)$$

from which finally it follows

$$F \leq 0 \quad (56)$$

This proves our statement: the inequality  $V'' < -d^2/4$  is valid at least in some region of  $z$  for any potential  $V$  of the form (39).

Notice that since at the horizon ( $z \rightarrow \infty$ ) the potential has a minimum and at the boundary ( $z = 0$ ) a minimum or a maximum in the conformal window (i.e.  $V'' + d^2/4 > 0$ ), there are always an even number of times that  $V''$  crosses the particular value  $-d^2/4$ .

## 6 The WKB approximation method

Here we shall try to apply the WKB method in order to compute the two-point correlation function of operators dual through the AdS/CFT correspondence to a bulk scalar field. The recipe to get it is to consider the solution to the perturbation equation (82), and identify the propagator by looking at the behavior near the boundary  $z \rightarrow 0$ ,

$$\xi(z; q) \sim z^{d-\Delta^{UV}} + G_2(q) z^{\Delta^{UV}} \quad (57)$$

The straightest way of doing it is to consider the Schrödinger-type equation (16) with “potential”

$$Q(z; q) \equiv q^2 + \frac{1}{z^2} \left( \frac{d^2 - 1}{4} + V''(t(z)) \right) \quad (58)$$

where we remember that  $t(z)$  is the solution of (4). For simplicity we consider the case  $V''(0) \equiv m_{UV}^2 > 0$ , although it is not necessary for the argument.

The WKB approximation results a good one if the slowly varying “Compton length” condition holds,

$$\left| \frac{d|Q(z; q)|^{-\frac{1}{2}}}{dz} \right| = \left| \frac{\dot{Q}(z; q)}{2|Q(z; q)|^{\frac{3}{2}}} \right| \ll 1 \quad (59)$$

This condition applied to (58) reads,

$$\frac{\left| \frac{d^2-1}{4} + V''(t(z)) - \frac{1}{2} V'''(t(z)) z \dot{t}(z) \right|}{\left| \frac{d^2-1}{4} + V''(t(z)) + q^2 z^2 \right|^{\frac{3}{2}}} \ll 1 \quad (60)$$

From here is straightforward to see that the WKB solution is trustable for any  $q^2$  around  $z = 0$  and  $z = \infty$  if,

$$\nu_{UV} \gg \frac{1}{2} \quad ; \quad \nu_{IR} \gg \frac{1}{2} \quad (61)$$

respectively, with  $\nu_{UV/IR}$  as in (6). Furthermore,  $Q(z; q)$  is positive near  $z = 0$  (and diverges quadratically there), but it is also positive for large  $z$  (going to  $q^2$  from above). What happens in the middle? From section 5 we know that for  $q$  small enough  $Q(z; q)$  must become negative; then for some  $z_M$  where  $t(z_M) = t_M$  it should have a local minimum. Then there must exist  $z_i = z_i(q)$ ,  $z_1(q) < z_M < z_2(q)$  such that,

$$z_i^2 Q(z_i; q) = \frac{d^2 - 1}{4} + V''(t(z_i)) + q^2 z_i^2 = 0 \quad ; \quad i = 1, 2 \quad (62)$$

Near these zeroes of  $Q(z; q)$  the WKB approximation breaks down.

If we admit that  $V''(t_M)$  is large enough then it is seen from (60) that in the region near  $z_M$  the WKB solution is trustable too. Therefore, calling *I*, *II*, *III* the regions near  $z = 0$ ,  $z_M$  and  $z \gg 1$  respectively, we can write the approximate WKB solution in each region as,

$$\xi_I(z; q) = C_I^+ z^{\frac{d}{2}} \frac{\exp\left(\int_{z_1}^z \frac{dz}{z} \sqrt{z^2 Q(z; q)}\right)}{(z^2 Q(z; q))^{\frac{1}{4}}} + C_I^- z^{\frac{d}{2}} \frac{\exp\left(-\int_{z_1}^z \frac{dz}{z} \sqrt{z^2 Q(z; q)}\right)}{(z^2 Q(z; q))^{\frac{1}{4}}} \quad (63)$$

$$\xi_{II}(z; q) = C_{II} z^{\frac{d}{2}} \frac{\exp\left(i \int_{z_1}^z \frac{dz}{z} \sqrt{-z^2 Q(z; q)}\right)}{(-z^2 Q(z; q))^{\frac{1}{4}}} + C_{II}^* z^{\frac{d}{2}} \frac{\exp\left(-i \int_{z_1}^z \frac{dz}{z} \sqrt{-z^2 Q(z; q)}\right)}{(-z^2 Q(z; q))^{\frac{1}{4}}} \quad (64)$$

$$\xi_{III}(z; q) = C_{III}^+ z^{\frac{d}{2}} \frac{\exp\left(\int_{z_2}^z \frac{dz}{z} \sqrt{z^2 Q(z; q)}\right)}{(z^2 Q(z; q))^{\frac{1}{4}}} + C_{III}^- z^{\frac{d}{2}} \frac{\exp\left(-\int_{z_2}^z \frac{dz}{z} \sqrt{z^2 Q(z; q)}\right)}{(z^2 Q(z; q))^{\frac{1}{4}}} \quad (65)$$

where the coefficients are related by,

$$C_I^\pm = \frac{1 \pm 3}{2} \text{Im}\left(C_{II} e^{\pm i\frac{\pi}{4}}\right) \leftrightarrow C_{II} = \frac{1}{2} e^{+i\frac{\pi}{4}} C_I^+ + e^{-i\frac{\pi}{4}} C_I^- = (C_{II}^*)^* \quad (66)$$

$$C_{III}^\pm = \frac{3 \pm 1}{2} \text{Im}\left(C_{II} e^{i(\varphi(q) \mp \frac{\pi}{4})}\right) \leftrightarrow$$

$$C_{II} = e^{-i\varphi(q)} \left(-\frac{1}{2} e^{-i\frac{\pi}{4}} C_{III}^+ + e^{+i\frac{\pi}{4}} C_{III}^-\right) = (C_{II}^*)^* \quad (67)$$

and,

$$\varphi(q) \equiv \int_{z_1(q)}^{z_2(q)} \frac{dz}{z} \sqrt{-z^2 Q(z; q)} \quad (68)$$

Now, imposing finiteness when  $z \rightarrow \infty$  implies  $C_{III}^+ = 0$ . By using the relations (66) and (67) we get all the constants in terms of  $C_{III}^-$ ; in particular for the solution near  $z = 0$  we get,

$$\xi_I(z; q) = C_{III}^- \left( 2 \cos \varphi(q) z^{\frac{d}{2}} \frac{\exp\left(\int_{z_1(q)}^z \frac{dz}{z} \sqrt{z^2 Q(z; q)}\right)}{(z^2 Q(z; q))^{\frac{1}{4}}} \right. \\ \left. + \sin \varphi(q) z^{\frac{d}{2}} \frac{\exp\left(-\int_{z_1(q)}^z \frac{dz}{z} \sqrt{z^2 Q(z; q)}\right)}{(z^2 Q(z; q))^{\frac{1}{4}}} \right) \quad (69)$$

From here we should be able to extract the propagator as a function of  $q^2$ , at least for  $q$  not so large. But we know from section 4 that for  $q = 0$  (69) must be equal to  $z t'(z)$  and thus going only as  $z^{\Delta_{UV}}$  for  $z \rightarrow 0$ . We will show now that this implies the constraint  $\varphi(0) = k \pi$  with  $k$  an integer. First we rewrite

$$\exp\left(\pm \int_{z_1(q)}^z \frac{dz}{z} \sqrt{z^2 Q(z; q)}\right) = \left(\frac{z}{z_1(q)}\right)^{\pm \sqrt{\nu_{UV}^2 - 1/4}} \\ \times \exp\left(\pm \int_{z_1(q)}^z \frac{dz}{z} \left(\sqrt{z^2 Q(z; q)} - \sqrt{\nu_{UV}^2 - 1/4}\right)\right) \quad (70)$$

Since we are interested only in  $\nu_{UV} \gg 1/2$  and leading behavior at  $z \rightarrow 0$ , we can see with the help of (70) that the first term on the r.h.s. of (69) goes like  $z^{\Delta_{UV}}$ , while the second goes like  $z^{d-\Delta_{UV}}$ . Since this last one should not be present in the solution  $z t'(z)$  of the  $q = 0$  perturbation, we have to impose (otherwise no solution with the right asymptotic behavior exists)

$$\varphi(0) \equiv \int_{z_1(0)}^{z_2(0)} \frac{dz}{z} \sqrt{-z^2 Q(z; 0)} = k \pi \quad (71)$$

This means that only potentials which satisfy this constraint are acceptable. This is the WKB analog of the fine-tuning mentioned before.

This simple conclusion is the reason for the  $1/q^2$  behavior of the boundary propagator. In fact, it is easy to derive the form of the propagator in the WKB approximation; from (57) we get:

$$G_2(q) = \frac{2 \exp\left(-2 \int_0^{z_1(q)} \frac{dz}{z} \left(\sqrt{z^2 Q(z; q)} - \sqrt{\nu_{UV}^2 - 1/4}\right)\right)}{(z_1(q))^2 \sqrt{\nu_{UV}^2 - 1/4} \tan \varphi(q)} \quad (72)$$

Clearly, due to (71), we get for  $q \rightarrow 0$  the usual Goldstone pole

$$G_2(q) \approx \frac{2 \exp\left(-2 \int_0^{z_1(0)} \frac{dz}{z} \left(\sqrt{z^2 Q(z; 0)} - \sqrt{\nu_{UV}^2 - 1/4}\right)\right)}{(z_1(0))^2 \sqrt{\nu_{UV}^2 - 1/4} (d\varphi(q)/dq^2)_{q^2=0}} \times \frac{1}{q^2} \quad (73)$$

where

$$\left. \frac{d\varphi(q)}{dq^2} \right|_{q^2=0} = -\frac{1}{2} \int_{z_1(0)}^{z_2(0)} dz \frac{z}{\sqrt{-z^2 Q(z; 0)}} \quad (74)$$

Although the denominator vanishes at the integration boundaries, the integral itself is finite.

## 7 The matching method to all orders

Let  $t(z)$  be the solution of the equation of motion (4) that behaves for  $z \rightarrow 0$  (UV) and  $z \rightarrow \infty$  (IR) as,

$$t(z) \xrightarrow{z \rightarrow 0} a_{UV} z^{\Delta_+^{UV}} (1 + b_{UV} z^{\alpha_{UV}} + \dots) \quad ; \quad t(z) \xrightarrow{z \rightarrow \infty} 1 + a_{IR} z^{\Delta_-^{IR}} (1 + b_{IR} z^{\alpha_{IR}} + \dots) \quad (75)$$

respectively. Here  $\alpha_{UV} > 0$  and  $\alpha_{IR} < 0$ , while that

$$\Delta_{\pm}^{UV/IR} \equiv \frac{d}{2} \pm \nu_{UV/IR} \quad ; \quad \nu_{UV/IR} \equiv \sqrt{\frac{d^2}{4} + m_{UV/IR}^2} \quad (76)$$

with  $m_{UV}^2 \equiv V''(0)$  and  $m_{IR}^2 \equiv V''(1) > 0$ . Note that in order for  $t(z)$  to be finite in the asymptotic expansions (75) neither  $\Delta_-^{UV}$  appears in the UV nor  $\Delta_+^{IR}$  in the IR.

Let us introduce for further use the following expansions of the functions  $\xi_{\pm}(z)$

$$\xi_{\pm}(z) = a_{\pm}^{UV/IR} z^{\Delta_{\pm/\mp}^{UV/IR}} \bar{\xi}_{\pm}^{UV/IR}(z) \quad ; \quad \bar{\xi}_{\pm}^{UV/IR}(z) \xrightarrow{z \rightarrow 0/\infty} 1 \quad (77)$$

that follow by plugging (75) in the definitions

$$\xi_+(z) \equiv z \dot{t}(z) \quad (78)$$

$$\xi_-(z) \equiv \xi_+(z) \left( \int_{z_i}^z dy \frac{y^{d-1}}{\xi_+^2(y)} + \frac{\xi_-(z_i)}{\xi_+(z_i)} \right) \quad (79)$$

where  $z_i$  and  $\xi_-(z_i)$  are integration constants. We find

$$a_+^{UV/IR} \equiv a_{UV/IR} \Delta_{+/-}^{UV/IR} \quad ; \quad a_-^{UV/IR} \equiv \left( a_+^{UV/IR} (d - 2 \Delta_{+/-}^{UV/IR}) \right)^{-1} \quad (80)$$

Clearly the UV/IR expansion of  $\xi_+(z)$  can not contain the  $z^{\Delta_{-/+}^{UV/IR}}$ -power, but  $\xi_-(z)$  could contain the  $z^{\Delta_{+/-}^{UV/IR}}$ -power.

Our aim is to solve the equation for perturbations around the solution  $t(z)$ , i.e. if we write (for a general treatment see for example the appendix of [16])

$$t(z; q) \equiv t(z) + \xi(z; q) e^{iq \frac{x}{L}} \quad (81)$$

then the second equation in (18) gives to first order in  $\xi(z; q)$

$$z^2 \ddot{\xi}(z; q) - (d-1) z \dot{\xi}(z; q) - (q^2 z^2 + V''(t(z))) \xi(z; q) = 0 \quad (82)$$

We will do it in two different approximations.

### 7.1 The large $z$ expansion.

We write (82) as

$$z^2 \ddot{\xi}(z; q) - (d-1) z \dot{\xi}(z; q) - (q^2 z^2 + m_{IR}^2) \xi(z; q) = \delta(z) \xi(z; q) \quad (83)$$

and consider  $\delta(z) \equiv V''(t(z)) - V''(1)$  small in the sense,

$$|\delta(z)| = |V''(t(z)) - V''(1)| \ll V''(1) \quad \longrightarrow \quad z > z_\infty \equiv \left| \frac{V''(1)}{V'''(1) a_{IR}} \right|^{\frac{1}{\Delta_{IR}}} \quad (84)$$

independently of the value of  $q$ . Then the solution for  $z > z_\infty$  can be hopefully expanded in orders of  $\delta(z)$ ,

$$\delta(z) = V'''(1) a_{IR} z^{\Delta_{IR}} (1 + b_{IR} z^{\alpha_{IR}} + \dots) \quad (85)$$

The order zero term is the solution to the l.h.s. of (83) equal to zero, which is given by,

$$\xi_\infty(z; q) \equiv \frac{2}{\Gamma(\nu_{IR})} \left(\frac{q}{2}\right)^{\nu_{IR}} z^{\frac{d}{2}} K_{\nu_{IR}}(qz) \quad (86)$$

where we have dropped the solution that diverges in the IR and fixed the normalization in such a way that  $\xi_\infty(z; 0) = z^{\Delta_{IR}^-}$ . It is not difficult to see that the expansion for large  $z > z_\infty$  is of the form,

$$\xi(z; q) = \xi_\infty(z; q) \left(1 + \frac{f_0(qz)}{z^{-\Delta_{IR}^-}} + \dots\right) \quad (87)$$

where for completeness we quote the first correction,

$$f_0(u) = V'''(1) a_{IR} x^{-\Delta_{IR}^-} \int_\infty^u \frac{dx}{x K_{\nu_{IR}}^2(x)} \int_\infty^x \frac{dy}{y^{1-\Delta_{IR}^-}} K_{\nu_{IR}}^2(y) \quad (88)$$

However corrections to the leading term of  $\xi(z; q)$  in negative powers of  $z$  will not be relevant in the matching procedure, at least not to compute the leading order behavior of the two-point function.

### 7.2 The small $q$ expansion.

This time we write (82) as

$$z^{d-1} \frac{d}{dz} \left( z^{1-d} \frac{d\xi(z; q)}{dz} \right) - \frac{V''(t(z))}{z^2} \xi(z; q) = q^2 \xi(z; q) \quad (89)$$

and consider  $q$  small in the sense,

$$q \ll \frac{|V''(t(z))|^{\frac{1}{2}}}{z} \quad (90)$$

This condition certainly holds in the UV region near  $z = 0$ , but also in the IR region if

$$q^2 z^2 \ll |V''(t(z))| \sim m_{IR}^2 \quad \longrightarrow \quad qz \ll m_{IR} \quad (91)$$

that is, when  $z$  is large and  $q$  small but  $qz$  fixed and small enough.

Under this condition we can try a solution for small  $q$  as a power series in  $q^2$ ,

$$\xi(z; q) = \sum_{m \geq 0} q^{2m} \xi^{(m)}(z; q) \quad (92)$$

Plugging this expansion in (89) we get,

$$z^{d-1} \frac{d}{dz} \left( z^{1-d} \frac{d\xi^{(0)}(z; q)}{dz} \right) - \frac{V''(t(z))}{z^2} \xi^{(0)}(z; q) = 0 \quad (93)$$

$$z^{d-1} \frac{d}{dz} \left( z^{1-d} \frac{d\xi^{(m)}(z; q)}{dz} \right) - \frac{V''(t(z))}{z^2} \xi^{(m)}(z; q) = \xi^{(m-1)}(z; q) \quad ; \quad m = 1, 2, \dots \quad (94)$$

The solution to lowest order is,

$$\xi^{(0)}(z; q) = C_+^{(0)}(q) \xi_+(z) + C_-^{(0)}(q) \xi_-(z) \quad (95)$$

where  $C_{\pm}^{(0)}(q)$  are integration constants. With  $\xi^{(0)}(z; q)$  we can determine  $\xi^{(1)}(z; q)$  from (94), and so on.

This iterative procedure yields the solution in the following form. First we introduce the set of functions,

$$\begin{aligned} f_{ij}^{(k)}(z) &\equiv \int_{z_i}^z \frac{dw}{w^{d-1}} \xi_i(w) \xi_j^{(k)}(w) \quad ; \quad i, j = +, - \quad , \quad k = 0, 1, \dots \\ \xi_{\pm}^{(0)}(z) &\equiv \xi_{\pm}(z) \end{aligned} \quad (96)$$

where

$$\xi_{\pm}^{(k)}(z) \equiv -f_{-\pm}^{(k-1)}(z) \xi_+(z) + f_{+\pm}^{(k-1)}(z) \xi_-(z) \quad , \quad k = 1, 2, \dots \quad (97)$$

All of them are obtained iteratively: first, from (96) with  $k = 0$  we get  $f_{ij}^{(0)}(z)$ , then we go to (97) with  $k = 1$  and get  $\xi_{\pm}^{(1)}(z)$ , then we come back to (96) with  $k = 1$  and get  $f_{ij}^{(1)}(z)$  and so on. The functions  $\xi_m(z; q)$  can be expressed in terms of the  $\xi_{\pm}^{(k)}(z)$ 's yielding the full expansion (92) in the form,

$$\xi(z; q) = \sum_{m \geq 0} q^{2m} \sum_{k=0}^m \left( C_+^{(m-k)}(q) \xi_+^{(k)}(z) + C_-^{(m-k)}(q) \xi_-^{(k)}(z) \right) \quad (98)$$

where the  $C_{\pm}^{(k)}$ 's are, as in (95), the integration constants of the homogeneous solution in (94). After some rearrangement, we can write (98) as,

$$\xi(z; q) = C_+(q) \sum_{m \geq 0} q^{2m} \xi_+^{(m)}(z) + C_-(q) \sum_{m \geq 0} q^{2m} \xi_-^{(m)}(z) \quad (99)$$

where we have redefined the coefficients

$$C_{\pm}(q) \equiv \sum_{k \geq 0} C_{\pm}^{(k)}(q) q^{2k} \quad (100)$$

We should not be surprised of this expression; after all (89) is a second order linear differential equation and both sums in (99) are linearly independent solutions of it as it can be quickly checked. Note furthermore that they are holomorphic in  $q^2$ ; the reason behind this fact can be traced directly to the assumption (92).

### 7.3 The two-point function.

For  $qz \ll m_{IR}$  expansion (99) hopefully holds, and it can be used to compute the two-point correlation function at low momenta as follows. After adjusting the constant of integration in (75) to get rid of the  $z^{\Delta_{+}^{UV}}$  term in  $\xi_-(z)$ , we parametrize the  $z \rightarrow 0$  behavior as

$$\begin{aligned} \xi(z; q) &\rightarrow \left[ (1 - q^2 \epsilon_{-+}^{UV}(q)) C_+(q) - q^2 \epsilon_{--}^{UV}(q) C_-(q) \right] a_+^{UV} z^{\Delta_+^{UV}} + \dots \\ &+ \left[ (1 + q^2 \epsilon_{+-}^{UV}(q)) C_-(q) + q^2 \epsilon_{++}^{UV}(q) C_+(q) \right] a_-^{UV} z^{\Delta_-^{UV}} + \dots \end{aligned} \quad (101)$$

where

$$\epsilon_{+\pm}^{UV}(q) = - \sum_{m \geq 0} q^{2m} \int_0^{z_i} dw w^{1-d} \xi_+(w) \xi_{\pm}^{(m)}(w) \quad (102)$$

while we were unable to find a closed expression for  $\epsilon_{-\pm}^{UV}$  without specifying the potential.

Applying the holographic recipe (57) the two-point function results,

$$G_2(q) \xrightarrow{q \rightarrow 0} \frac{a_+^{UV}/a_-^{UV}}{\left(q^2 \epsilon_{++}^{UV}(q) + \frac{C_-(q)}{C_+(q)}\right)_{q \rightarrow 0}} \quad (103)$$

The knowledge of the leading order behavior of the quotient  $C_-(q)/C_+(q)$  for  $q \rightarrow 0$  will allow to compute the leading power in  $q$  of  $G_2(q)$ . The  $z_i$ -dependence of the coefficients  $\epsilon_{++}^{UV}$  (and the  $z_i$ -independence of the physics) gives a hint that this power is  $-2$ , as we will confirm below.

#### 7.4 The infrared expansion

Here we define the functions  $\bar{F}_{ij}^{(m)}(z)$  and the constants  $\bar{\varphi}_{ij}^{(m)}$  by means of the integrals,

$$\begin{aligned} & a_i^{IR} a_j^{IR} \sigma_j^{(m)} \int_{z_i}^z dw w^{1+2m+\Delta(i)+\Delta(j)-d} \bar{\xi}_i^{(0)}(w) \bar{\xi}_j^{(m)}(w) \\ \equiv & \bar{\varphi}_{ij}^{(m)} + \frac{a_i^{IR} a_j^{IR} \sigma_j^{(m)} z^{2+2m+\Delta(i)+\Delta(j)-d}}{2+2m+\Delta(i)+\Delta(j)-d} \bar{F}_{ij}^{(m)}(z) \quad ; \quad m = 0, 1, \dots \end{aligned} \quad (104)$$

where  $\bar{\varphi}_{ij}^{(m)}$  is defined to be the only  $z$ -independent part in the large  $z$  expansion, and

$$\sigma_{\pm}^{(m)} \equiv \frac{\Gamma(1 \mp \nu_{IR})}{2^{2m} m! \Gamma(1 \mp \nu_{IR} + m)} \quad ; \quad m = 0, 1, \dots \quad (105)$$

With them we can calculate ( $m = 1, 2, \dots$ ),

$$\varphi_{ij}^{(m)} \equiv \bar{\varphi}_{ij}^{(m)} + \sum_{k=0}^{m-1} \left( \bar{\varphi}_{i-}^{(k)} \varphi_{+j}^{(m-1-k)} - \bar{\varphi}_{i+}^{(k)} \varphi_{-j}^{(m-1-k)} \right) \quad (106)$$

$$\bar{\xi}_{\pm}^{(m)}(z) \equiv \frac{1}{\nu_{IR}} \left( (\nu_{IR} \mp m) \bar{F}_{\mp\pm}^{(m-1)}(z) \bar{\xi}_{\pm}^{(0)}(z) \pm m \bar{F}_{\pm\pm}^{(m-1)}(z) \bar{\xi}_{\mp}^{(0)}(z) \right) \quad (107)$$

The general form of  $\xi_{\pm}^{(m)}(z)$  for  $m = 1, 2, \dots$ , results,

$$\begin{aligned} \xi_{\pm}^{(m)}(z) &= a_{\pm}^{IR} \sigma_{\pm}^{(m)} z^{\Delta_{\mp}^{IR}+2m} \bar{\xi}_{\pm}^{(m)}(z) \\ &+ \sum_{k=0}^{m-1} \left( -a_+^{IR} \sigma_+^{(k)} \varphi_{-\pm}^{(m-1-k)} z^{\Delta_-^{IR}+2k} \bar{\xi}_+^{(k)}(z) + a_-^{IR} \sigma_-^{(k)} \varphi_{+\pm}^{(m-1-k)} z^{\Delta_+^{IR}+2k} \bar{\xi}_-^{(k)}(z) \right) \end{aligned} \quad (108)$$

where the ingredients to construct it are iteratively computed as described above.

#### 7.5 The matching procedure.

According to (84) and (91), in the region

$$z > z_{\infty} \quad ; \quad x \equiv qz \ll m_{IR} \quad (109)$$

both expansions (87) and (99) hold and therefore they should coincide *exactly*, i.e.

$$\xi_{\infty}(z; q) \left( 1 + \frac{f_0(qz)}{z^{-\Delta_-^{IR}}} + \dots \right) = C_+(q) \sum_{m \geq 0} q^{2m} \xi_+^{(m)}(z) + C_-(q) \sum_{m \geq 0} q^{2m} \xi_-^{(m)}(z) \quad (110)$$

This equation must be used to compute the unknown coefficients  $C^\pm(q)$ . As we will see shortly, this is not an easy task in general; fortunately the leading order behavior necessary to compute (103) is relatively simple to get. To proceed we need the IR behavior of the  $\xi_\pm^{(m)}(z)$ 's. By plugging (108) in (99) we get,

$$\begin{aligned} z^{-\Delta_-^{IR}} \xi(z; q) &= z^{-\Delta_-^{IR}} r.h.s. \quad (110) \\ &= \left( (1 - q^2 \epsilon_{-+}^{IR}(q)) C_+(q) - q^2 \epsilon_{--}^{IR}(q) C_-(q) \right) a_+^{IR} \sum_{m \geq 0} \sigma_+^{(m)} x^{2m} \bar{\xi}_+^{(m)} \left( \frac{x}{q} \right) \\ &+ \left( (1 + q^2 \epsilon_{+-}^{IR}(q)) C_-(q) + q^2 \epsilon_{++}^{IR}(q) C_+(q) \right) \frac{a_-^{IR}}{q^{2\nu_{IR}}} \sum_{m \geq 0} \sigma_-^{(m)} x^{2m+2\nu_{IR}} \bar{\xi}_-^{(m)} \left( \frac{x}{q} \right) \end{aligned} \quad (111)$$

where we have introduced the holomorphic functions,

$$\epsilon_{ij}^{IR}(q) \equiv \sum_{m \geq 0} \varphi_{ij}^{(m)} q^{2m} \quad (112)$$

On the other hand, by using the series expansion of  $\xi_\infty \left( \frac{x}{q}; q \right)$  valid for  $x < 1$  we have,

$$z^{-\Delta_-^{IR}} \xi_\infty(z; q) = \sum_{m \geq 0} \left( \sigma_+^{(m)} x^{2m} + \gamma \sigma_-^{(m)} x^{2m+2\nu_{IR}} \right) \quad ; \quad \gamma \equiv \frac{\Gamma(-\nu_{IR})}{2^{2\nu_{IR}} \Gamma(\nu_{IR})} \quad (113)$$

Now from (110) we have that at fixed  $x < \text{minimum}(m_{IR}, 1)$ , in the limit  $q \rightarrow 0$  equations (111) and (113) should coincide. More specifically, if we introduce  $\delta C_\pm(q)$  by,

$$\begin{aligned} C_+(q) &\equiv \frac{1}{D(q)} \left( \frac{1}{a_+^{IR}} (1 + q^2 \epsilon_{+-}^{IR}(q)) + \frac{\gamma}{a_-^{IR}} q^{2+2\nu_{IR}} \epsilon_{--}^{IR}(q) \right) + \delta C_+(q) \\ C_-(q) &\equiv \frac{1}{D(q)} \left( -\frac{1}{a_+^{IR}} q^2 \epsilon_{++}^{IR}(q) + \frac{\gamma}{a_-^{IR}} q^{2\nu_{IR}} (1 - q^2 \epsilon_{-+}^{IR}(q)) \right) + \delta C_-(q) \end{aligned} \quad (114)$$

where,

$$D(q) = 1 + q^2 (\epsilon_{+-}^{IR}(q) - \epsilon_{-+}^{IR}(q)) + q^4 (\epsilon_{++}^{IR}(q) \epsilon_{--}^{IR}(q) - \epsilon_{+-}^{IR}(q) \epsilon_{-+}^{IR}(q)) \quad (115)$$

then we should get,

$$\begin{aligned} &\lim_{q \rightarrow 0} \left\{ \sum_{m \geq 0} \sigma_+^{(m)} x^{2m} \left( \bar{\xi}_+^{(m)} \left( \frac{x}{q} \right) - 1 \right) + \gamma \sum_{m \geq 0} \sigma_-^{(m)} x^{2m+2\nu_{IR}} \left( \bar{\xi}_-^{(m)} \left( \frac{x}{q} \right) - 1 \right) \right. \\ &+ \left( (1 - q^2 \epsilon_{-+}^{IR}(q)) \delta C_+(q) - q^2 \epsilon_{--}^{IR}(q) \delta C_-(q) \right) a_+^{IR} \sum_{m \geq 0} \sigma_+^{(m)} x^{2m} \bar{\xi}_+^{(m)} \left( \frac{x}{q} \right) \\ &\left. + \left( (1 + q^2 \epsilon_{+-}^{IR}(q)) \delta C_-(q) + q^2 \epsilon_{++}^{IR}(q) \delta C_+(q) \right) \frac{a_-^{IR}}{q^{2\nu_{IR}}} \sum_{m \geq 0} \sigma_-^{(m)} x^{2m+2\nu_{IR}} \bar{\xi}_-^{(m)} \left( \frac{x}{q} \right) \right\} = 0 \end{aligned} \quad (116)$$

While the first line is automatically zero, the second and third lines should be zero separately because they present different power series<sup>3</sup>. From the third line we get,

$$\delta C_-(q) \xrightarrow{q \rightarrow 0} -\varphi_{++}^{(0)} q^2 \delta C_+(q) + q^{2\nu_{IR}} A(q) \quad (117)$$

<sup>3</sup>A subtlety (not present in the case considered in the text) arises if  $\bar{\xi}_-^{(m)}(z)$  contains powers of the form  $z^{-2\nu_{IR}-2n}$  with  $n \in \mathbb{N}$ ; in that case it can be easily showed that the effect is that the coefficients of  $\delta C^\pm(q)$  on the second line of (116) get modified by holomorphic functions; this fact does not modify the subsequent arguments.



where  $A(q) \xrightarrow{q \rightarrow 0} 0$ . Then the second line of (116) yields,

$$\delta C_+(q) \xrightarrow{q \rightarrow 0} \varphi_{--}^{(0)} q^{2\nu_{IR}+2} A(q) \Rightarrow \delta C_-(q) \xrightarrow{q \rightarrow 0} q^{2\nu_{IR}} A(q) \quad (118)$$

Going to (114) with (118) we get the leading behaviors,

$$C_+(0) = \frac{1}{a_+^{IR}} \quad ; \quad C_-(q)|_{q \rightarrow 0} = -\frac{\bar{\varphi}_{++}^{(0)}}{a_+^{IR}} q^2 \quad (119)$$

This yields for the two-point function (103) the Goldstone pole,

$$G_2(q) \xrightarrow{q \rightarrow 0} \frac{\alpha}{q^2} \quad (120)$$

where by using (102) and (104), i.e.

$$\bar{\varphi}_{++}^{(0)} = \int_{z_i}^{\infty} dw w^{1-d} \xi_+(w)^2 \quad (121)$$

we get for the residue,

$$\alpha = \frac{2\nu_{UV} (a_+^{UV})^2}{\int_0^{\infty} dw w^{1-d} \xi_+(w)^2} \quad (122)$$

The result is reassuring in the sense that both contributions in the denominator of (103) add to yield a  $z_i$ -independent result.

## 8 Global symmetries and AdS/CFT

Let us now use all this machinery for a simple  $d$ -dimensional strongly coupled system with a spontaneously broken global symmetry. We would like to see explicitly what makes Nambu-Goldstone bosons massless in the AdS/CFT picture: it is the square integrability of the solution  $\xi_+(z)$  of the perturbation equation. In other words, a normalizable perturbation is massless.

In general let the original symmetry group be  $G$ , spontaneously broken to  $H$ . The simplest example seems to be  $SU(3) \rightarrow SU(2) \times U(1)$ , i.e.  $G = SU(3)$  and  $H = SU(2) \times U(1)$ . A physically more appealing case could be the case of the unification group  $SU(5) \rightarrow SU(3) \times SU(2) \times U(1)$ , which will not consider here though. The hope is that eventually one could then weakly couple the system to gauge bosons, i.e. gauge it. Let's consider a real adjoint transforming under  $SU(3)$  as  $\Sigma \rightarrow U\Sigma U^\dagger$ , which we parametrize as

$$\Sigma = \frac{1}{\sqrt{2}} \begin{pmatrix} t/\sqrt{3} + t_3 & t_1 - it_2 & w_1 - iw_2 \\ t_1 + it_2 & t/\sqrt{3} - t_3 & w_3 - iw_4 \\ w_1 + iw_2 & w_3 + iw_4 & -2t/\sqrt{3} \end{pmatrix} \quad (123)$$

and a complex fundamental (under  $SU(3)$ ,  $F \rightarrow UF$ ):

$$F^T = (T_1 \quad T_2 \quad H)^T \quad (124)$$

The  $SU(2) \times U(1)$  final symmetry will be preserved by  $\langle \Sigma \rangle = t\lambda_8$  (with  $\lambda_8$  the Gell-Mann matrix) and  $\langle F \rangle = 0$ . To get it let the  $SU(3)$  invariant superpotential be

$$W = \Delta \left( \frac{1}{2} Tr \Sigma^2 + \frac{\sqrt{6}}{3} Tr \Sigma^3 \right) + F^\dagger (m - \sqrt{6}\alpha \Sigma) F \quad (125)$$

The strange relation between the  $\Sigma^2$  and  $\Sigma^3$  coefficients are chosen so that the straightforward generalization of the potential (39)

$$\begin{aligned}
 V &= \frac{1}{2} \left( \frac{\partial W}{\partial t} \right)^2 + \frac{1}{2} \sum_{i=1}^3 \left( \frac{\partial W}{\partial t_i} \right)^2 + \frac{1}{2} \sum_{a=1}^4 \left( \frac{\partial W}{\partial w_a} \right)^2 \\
 &+ \sum_{i=1}^2 \left( \frac{\partial W}{\partial T_i} \right) \left( \frac{\partial W}{\partial T_i^*} \right) + \left( \frac{\partial W}{\partial H} \right) \left( \frac{\partial W}{\partial H^*} \right) - d W
 \end{aligned} \tag{126}$$

has an extremum at  $t = 0$  and a minimum at  $t = 1$  with all other fields vanishing, and the potential for  $t$  is the same as in [18]:

$$W(t) = \Delta \left( \frac{t^2}{2} - \frac{t^3}{3} \right) \tag{127}$$

$$\begin{aligned}
 V(t) &= \frac{1}{2} W'^2(t) - d W(t) \\
 &= \Delta(\Delta - d) \frac{t^2}{2} - \Delta(3\Delta - d) \frac{t^3}{3} + 2\Delta^2 \frac{t^4}{4}
 \end{aligned} \tag{128}$$

The solution to the e.o.m. is

$$t(z) = \frac{z^\Delta}{1 + z^\Delta}, \quad t_i, w_a = 0 \tag{129}$$

One can calculate the mass matrix

$$\frac{\partial V}{\partial t^2} \equiv m^2(t) = \Delta(\Delta - d) - 2\Delta(3\Delta - d)t + 6\Delta^2 t^2 \tag{130}$$

$$\frac{\partial V}{\partial w_a \partial w_b} \equiv m_w^2(t) \delta^{ab} = (\Delta(\Delta - d) - \Delta(3\Delta - d)t + 2\Delta^2 t^2) \delta^{ab} \tag{131}$$

$$\frac{\partial V}{\partial t_i \partial t_j} \equiv m_t^2(t) \delta^{ij} = (\Delta(\Delta - d) + 2\Delta(3\Delta - d)t + 2\Delta^2 t^2) \delta^{ij} \tag{132}$$

$$\frac{\partial V}{\partial T_\alpha^* \partial T_\beta} \equiv m_T^2(t) \delta^{\alpha\beta} = (m(m - d) - \alpha(2m + \Delta - d)t + \alpha(\alpha + \Delta)t^2) \delta^{\alpha\beta} \tag{133}$$

$$\frac{\partial V}{\partial H^* \partial H} \equiv m_H^2(t) = m(m - d) + 2\alpha(m + \Delta - d)t + 2\alpha(2\alpha - \Delta)t^2 \tag{134}$$

with all other elements vanishing.

This means that it is easy to solve the perturbation equation since the different modes decouple. In an obvious notation:

$$z^2 \ddot{\xi}(z; q) - (d - 1) z \dot{\xi}(z; q) - (q^2 z^2 + m^2(t(z))) \xi(z; q) = 0 \tag{135}$$

$$z^2 \ddot{\xi}^w(z; q) - (d - 1) z \dot{\xi}^w(z; q) - (q^2 z^2 + m_w^2(t(z))) \xi^w(z; q) = 0 \tag{136}$$

$$z^2 \ddot{\xi}^t(z; q) - (d - 1) z \dot{\xi}^t(z; q) - (q^2 z^2 + m_t^2(t(z))) \xi^t(z; q) = 0 \tag{137}$$

$$z^2 \ddot{\xi}^T(z; q) - (d - 1) z \dot{\xi}^T(z; q) - (q^2 z^2 + m_T^2(t(z))) \xi^T(z; q) = 0 \tag{138}$$

$$z^2 \ddot{\xi}^H(z; q) - (d - 1) z \dot{\xi}^H(z; q) - (q^2 z^2 + m_H^2(t(z))) \xi^H(z; q) = 0 \tag{139}$$

The first equation (135) has a well known solution at  $q^2 = 0$  (corresponding to the dilaton)

$$\xi_+(z) = z \dot{t}(z) \quad (140)$$

The second one (136) is for the Goldstone-bosons of the global symmetry. Since

$$m_w^2(t) = \frac{1}{t} V'(t) \quad (141)$$

the well-behaved solution at  $q^2 = 0$  is simply

$$\xi_+^w(z) = t(z) \quad (142)$$

This is why it has a pole at  $q^2 = 0$ . We just need to do the usual expansion derived in general in the previous section, see also [18], with the result for the propagator

$$G_2^w(q) = \frac{\alpha}{q^2} \quad (143)$$

with the general expression

$$\alpha = \frac{2\nu_{UV} (a_+^{UV})^2}{(\int_0^\infty dx x^{1-d} (\xi_+^w(x))^2)} \quad (144)$$

In our specific case (136) we have

$$a_+^{UV} = 1 \quad (145)$$

with the integral in the denominator finite.

Then, what about the third equation (137), i.e. for  $\xi^t$ ? One can easily find the solution for  $q = 0$ :

$$\xi^t(z) = C_1 z^\Delta (1 + z^\Delta)^2 + C_2 \frac{z^{d-\Delta}}{1 + z^\Delta} {}_2F_1(1, -5 + d/\Delta, -1 + d/\Delta, -z^\Delta) \quad (146)$$

For  $z \rightarrow \infty$  we get

$$\xi^t(z) \rightarrow C_1 z^\Delta (1 + z^\Delta)^2 + C_2 \frac{z^{d-\Delta}}{1 + z^\Delta} z^{5\Delta-d} \frac{\Gamma(6 - d/\Delta)\Gamma(-1 + d/\Delta)(1 + z^\Delta)^3}{2z^{3\Delta}} \quad (147)$$

and so

$$C_1 = -\frac{C_2}{2} \Gamma(6 - d/\Delta)\Gamma(-1 + d/\Delta) \quad (148)$$

In the opposite limit  $z \rightarrow 0$  (137) becomes

$$\xi^t(z) \rightarrow C_1 (z^\Delta + \dots) + C_2 (z^{d-\Delta} + \dots) \quad (149)$$

so that due to (148) we get in the IR limit  $q \rightarrow 0$

$$G_2^t(0) = \frac{C_1}{C_2} = -\frac{1}{2} \Gamma(6 - d/\Delta)\Gamma(1 - d/\Delta) \quad (150)$$

Obviously there is no pole here at  $q = 0$ , a pole is expected at finite  $q$ . The reason for no pole at  $q = 0$  is thus due to the fact that there is no solution finite in the whole positive  $z$ -axis. This was true

for  $\xi(z) = z \dot{t}(z)$  and  $\xi^w(z) = t(z)$ , and this is why the next order in  $q$  was needed there. In other words, if the integral in (122) is finite, the propagator obeys (120), if it is not, then the leading term in this expansion is a constant.

Equations (138) and (139) for  $\xi^T$  and  $\xi^H$  seem to point to the same conclusion as for  $\xi^t$ : no pole, i.e. no light degree of freedom. So if we would like one of the two to be light, i.e. for example  $\xi^H$  (the analog of the light SM doublet in SU(5)), we would need to further fine-tune the system, similarly as one obtains the usual doublet-triplet splitting in a SU(5) grand unified theory.

## 9 Summary of results

We have studied some simple examples of the AdS/CFT correspondence, i.e. the domain wall solution in the AdS bulk of a scalar field in the limit of no gravity back-reaction. To obtain analytical results we rewrote the bulk scalar potential in terms of an appropriate superpotential. The correspondence between the two is a bit unusual, i.e. it is the one that survives the infinite Planck scale limit. Since the superpotential was written as a simple polynomial of the scalar field, the potential itself is obviously tuned, i.e. it is not the most general one of given order. We showed explicitly that such tuning of the parameters is necessary for the solution to exist at all, and that it is just a consequence of a vanishing boundary cosmological constant, in accord with a known result from the literature. We proved another general requirement for the existence of such domain walls: the second field derivative of the potential must be smaller than  $-d^2/4$  at least in some interval of the bulk coordinate, where  $d$  is the boundary space-time dimension. We then found the solution of the perturbation equation for the dilaton mode by the WKB approximation and by the matching method to all orders. We generalized in this way the known result of a massless mode corresponding to the spontaneously broken Nambu-Goldstone excitation (dilaton) due to the spontaneously broken translational symmetry in AdS. The existence of a massless mode is essentially due to the existence of a normalizable mode at  $q^2 = 0$ : this interpretation is confirmed in the case of spontaneously broken internal global symmetries. We gave an explicit example in terms of a SU(3) theory spontaneously broken by an adjoint vev to SU(2)→U(1).

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## References

- [1] J. M. Maldacena, “The Large  $N$  Limit of Superconformal Field Theories and Supergravity,” *Adv. Theor. Math. Phys.* **2** (1998) 231 [hep-th/9711200].
- [2] S. S. Gubser, I. R. Klebanov, A. M. Polyakov and , “Gauge Theory Correlators from Noncritical String Theory,” *Phys. Lett. B* **428** (1998) 105 [hep-th/9802109].
- [3] E. Witten, “Anti-de Sitter Space and Holography,” *Adv. Theor. Math. Phys.* **2** (1998) 253 [hep-th/9802150].
- [4] L. Girardello, M. Petrini, M. Porrati, A. Zaffaroni, “Novel Local CFT and Exact Results on Perturbations of  $\mathcal{N} = 4$  Super Yang Mills from AdS Dynamics,” *JHEP* **9812** (1998) 022 [hep-th/9810126].
- [5] D. Z. Freedman, S. S. Gubser, K. Pilch, N. P. Warner, “Continuous Distributions of D3-Branes and Gauged Supergravity,” *JHEP* **0007** (2000) 038 [hep-th/9906194].
- [6] G. Arutyunov, S. Frolov, S. Theisen, “A Note on Gravity Scalar Fluctuations in Holographic RG Flow Geometries,” *Phys. Lett. B* **484** (2000) 295 [hep-th/0003116].

- [7] W. Mück, “Correlation Functions in Holographic Renormalization Group Flows,” Nucl. Phys. B **620** (2002) 477 [hep-th/0105270].
- [8] D. Martelli, A. Miemiec, “CFT / CFT Interpolating RG Flows and the Holographic C Function,” JHEP **0204** (2002) 027 [hep-th/0112150].
- [9] M. Berg, H. Samtleben, “Holographic Correlators in a Flow to a Fixed Point,” JHEP **0212** (2002) 070 [hep-th/0209191].
- [10] D. Z. Freedman, C. Nunez, M. Schnabl, K. Skenderis, “Fake Supergravity and Domain Wall Stability,” Phys. Rev. D **69** (2004) 104027 [hep-th/0312055].
- [11] M. Cvetič, S. Griffies, S. -J. Rey, “Static Domain Walls in  $\mathcal{N} = 1$  Supergravity,” Nucl. Phys. B **381** (1992) 301 [hep-th/9201007].
- [12] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri, Y. Oz, “Large  $N$  Field Theories, String Theory and Gravity,” Phys. Rept. **323** (2000) 183 [hep-th/9905111].
- [13] O. DeWolfe, D. Z. Freedman, “Notes on Fluctuations and Correlation Functions in Holographic Renormalization Group Flows,” hep-th/0002226.
- [14] E. D’Hoker, D. Z. Freedman, “Supersymmetric Gauge Theories and the AdS / CFT Correspondence,” hep-th/0201253.
- [15] K. Skenderis, “Lecture Notes on Holographic Renormalization,” Class. Quant. Grav. **19** (2002) 5849 [hep-th/0209067].
- [16] B. Bajc, A. R. Lugo and M. B. Sturla, “Spontaneous Breaking of a Discrete Symmetry and Holography,” JHEP **1204** (2012) 119 [arXiv:1203.2636 [hep-th]].
- [17] M. Bianchi, D. Z. Freedman, K. Skenderis, “How to Go with an RG Flow,” JHEP **0108** (2001) 041 [hep-th/0105276].
- [18] B. Bajc and A. R. Lugo, “On the Matching Method and the Goldstone Theorem in Holography,” JHEP **1307** (2013) 056 [arXiv:1304.3051 [hep-th]].
- [19] C. Hoyos, U. Kol, J. Sonnenschein and S. Yankielowicz, “The Holographic Dilaton,” arXiv:1307.2572 [hep-th].
- [20] U. Kol, “On the Dual Flow of Slow-Roll Inflation,” arXiv:1309.7344 [hep-th].
- [21] C. Hoyos, U. Kol, J. Sonnenschein and S. Yankielowicz, “The A-Theorem and Conformal Symmetry Breaking in Holographic RG Flows,” arXiv:1207.0006 [hep-th].
- [22] S. A. Hartnoll, “Lectures on holographic methods for condensed matter physics,” Class. Quant. Grav. **26**, 224002 (2009) [arXiv:0903.3246 [hep-th]].
- [23] D. S. Berman and E. Rabinovici, hep-th/0210044.
- [24] F. Coradeschi, P. Lodone, D. Pappadopulo, R. Rattazzi and L. Vitale, “A Naturally Light Dilaton,” arXiv:1306.4601 [hep-th].
- [25] J. Maldacena, “Vacuum Decay into Anti De Sitter Space,” arXiv:1012.0274 [hep-th].