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GLUEBALL MASSES AND SYMMETRY RESTORATION IN SU(3) LATTICE GAUGE THEORY: A HIGH STATISTICS MONTE CARLO STUDY

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ABSTRACT

We calculate 0^{++}, 2^{++} and 1^{-+} correlation functions for a wide range of momenta in a high statistics SU(3) study on an 8^4 lattice: 28,000 sweeps at $\beta = 5.7$ and 18,000 at $\beta = 5.9$. We obtain an accurate confirmation of the restoration of the continuum relativistic dispersion relation, $E^2 = p^2+m^2$, and of rotational invariance. We obtain accurate 2^{++} mass estimates up to two lattice spacings, and confirm consistency with asymptotic scaling. For the 1^{-+} the results are much poorer and we can only present some very crude mass estimates. We compare our 0^{++} data to our previous calculations with a source, and make some statements about the relative efficiencies of source and variational calculations in this range of couplings.

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The physics interest of the SU(3) non-Abelian gauge theory arises from the fact that it provides the underlying dynamics of QCD. When the theory is regularized by replacing Euclidean space-time with a hypercubic lattice of points[1)], its non-perturbative physics becomes amenable to solution by numerical simulation[2)] In the last year there have appeared calculations of the scalar glueball mass[3)], the string tension[4)] and the deconfining temperature[5)], which are reliable enough in their control of systematic and statistical errors as to make it meaningful to ask the question of whether the spatial cut-off is small enough that one is addressing the physics of the continuum theory.

In the present paper we focus on calculations of the energies, over a wide range of momenta, of 0^{++}, 2^{++} and 1^{-+} glueball states. We would like to emphasize immediately that these calculations are not in the same class of reliability as, say, our recent 0^{++} mass calculation[3)], and in presenting our results we will try to make explicit their weak points. Nonetheless, these results are a significant improvement on previous calculations of the same quantities, and the physics one can extract is of some importance.

The first piece of physics concerns the glueball energy-momentum dispersion relation - does it have the continuum relativistic form, $E^2 = p^2+m^2$? In a previous work[6)] such a relation was confirmed for both SU(2) and SU(3): however, the statistical errors in the latter case were too large to give this confirmation a great significance. Here we shall confirm the continuum E(p) dependence with much greater precision. At the same time we shall confirm continuum spatial rotational invariance: E depends only on $|\underline{p}|$. Finding at which (bare) couplings such continuum symmetries are dynamically restored is of great importance because it provides unmistakeable evidence that the lattice spacing is becoming small enough to be invisible to typical non-perturbative physics. Thus it provides a criterion for where it is sensible to try to calculate continuum physics.

Secondly we shall obtain a high statistics estimate of the 2^{++} glueball mass on a reasonably large 8^4 lattice. We shall obtain an accurate confirmation of previous results that were obtained either with low statistics on a similar 8^4 lattice[7)], or on much smaller $4^3 8$ lattices[8)].

Our third piece of interesting physics should have been the (oddball) 1^{-+} glueball mass: it will not be because we were not able to simultaneously control statistical and systematic errors. However, assuming the latter errors to be small we can extract statistically accurate masses and this we shall do, albeit with considerable reservations.

Finally we shall compare our 0^{++} measurements with those we previously obtained[3)] using a source method: this will enable us to compare the relative efficiencies of source and variational methods. Although a rather technical point, this comparison should be very useful to anyone who wishes to pursue the calculation of such quantities.

Throughout the present calculation we use the standard Wilson plaquette action[1)] on an 8^4 hypercubic lattice with periodic boundary conditions. We use a ten hit Metropolis algorithm[2)] for updating the lattice. Our glueball measurements are performed every sweep and use all four directions for measuring correlation functions. We have a total of 28,000 sweeps at $\beta(\equiv 6/g^2) = 5.7$ and 18,000 at $\beta = 5.9$. The statistical errors were computed by splitting the data into bins of 500 sequential sweeps: our experience is that on lattices of the size being considered, and for these couplings, bins of this size give statistically independent sub-averages. The large number of bins (56 at $\beta = 5.7$, 38 at $\beta = 5.9$) ensures accurate error estimates.

In the next section we remind the reader what it takes to perform a "reliable" lattice Monte Carlo calculation. Later in the paper we shall continuously refer back to this section in evaluating the reliability of our results. Then we briefly remind the reader how to calculate energies of different J^{PC} glueballs for non-zero (as well as zero) momenta. We address the technical problem of how to do this on a lattice without mixing in 0^{++} operators. From there we go on to our studies of Lorentz and rotational invariance, the 2^{++} and 1^{-+} glueball masses, and a direct comparison of source and variational calculations for the 0^{++} glueball. In the conclusions we shall collect our results and point out how they could be improved (as they should be).

CRITERIA FOR A RELIABLE CALCULATION

To perform a calculation of the lowest mass m with some definite quantum numbers, Q, one calculates the correlation function, C(t), of some (vacuum-subtracted) operator ϕ_Q with these same quantum numbers, and extracts the mass from its asymptotic exponential decay in (Euclidean) time

$$C(t) \equiv \langle \phi_Q(t)\, \phi_Q(0) \rangle \underset{t \to \infty}{\sim} \alpha\, e^{-mt} \qquad (1)$$

At small t, C(t) will receive contributions from the exchange of all states with the appropriate quantum numbers: the first criterion for a reliable calculation is, therefore, that one unambiguously verifies the asymptotic exponential decay (1). This is difficult in a Monte Carlo calculation where the exponentially small signal in Eq. (1) rapidly disappears in statistical noise. Nonetheless the only reasonable way to be sure that one is extracting the mass m is to carry the calculation far enough so that at least three values of C(t) $[t=n_t a,\ (n_t+1)a,\ (n_t+2)a]$ lie on a single exponential, and that the errors on these points are small enough as to make this fit statistically compelling.

Having calculated the mass m in the above fashion on some $L_s^3\, L_t$ lattice with lattice spacing $a(\beta)$, the next step is to repeat the calculation on larger lattices, so as to obtain the desired infinite volume ($L_s \to \infty$) and zero temperature $[(L_t a)^{-1} \to 0]$ limit of m. This procedure can be considerably accelerated if one has an analytic expression for the leading large volume correction[9)].

Having followed the above steps we can at least be sure that we have calculated the correct mass, $m(\beta)$, for lattice spacing $a(\beta)$. Of course this mass will be calculated in terms of the lattice spacing itself (since this is the only explicit scale in the problem), that is to say as the dimensionless product $m(\beta).a(\beta)$. Assuming our ambition is to be calculating the continuum mass we need to redo the calculation for several values of β and test whether for some $\beta > \beta_c$ the calculated mass $m(\beta)$ becomes independent of β, when measured in physical units: if so it is reasonable to claim one is obtaining continuum physics for $\beta > \beta_c$. Expressing $m(\beta)$ in "physical units" is of course a procedure that needs to be defined. If we are simultaneously measuring two masses, and if

$$\frac{m_1(\beta).\,a(\beta)}{m_2(\beta).\,a(\beta)} \equiv \frac{m_1(\beta)}{m_2(\beta)} \underset{\beta \geq \beta_c}{=} \text{ind. of } \beta \qquad (2)$$

then it is reasonable to believe that we are seeing continuum values of m_1, m_2. An alternative procedure is available if $a(\beta)$ is small enough to be accurately represented, in terms of β, by the two-loop perturbative formula: in that case

$$m(\beta)\,.\,a(\beta) = \frac{m(\beta)}{\Lambda_L}\; e^{-\frac{4\pi^2}{33}\beta}\left[\frac{8\pi^2}{33}\beta\right]^{51/121}$$

$$\underset{\beta \geq \beta_c}{=} \frac{m}{\Lambda_L}\cdot e^{-\frac{4\pi^2}{33}\beta}\left[\frac{8\pi^2}{33}\beta\right]^{51/121} \qquad (3)$$

where the second line follows if $m(\beta)$ attains its continuum value for $\beta > \beta_c$. If our measurements of $m(\beta).a(\beta)$ have the β dependence in Eq. (3) then it is reasonable to assume that $m(\beta)$ is indeed attaining its continuum value for $\beta > \beta_c$, and also that $a(\beta)$ is indeed small enough to be given by its two-loop expression.

Of course in practice one at best tries to confirm (2) or (3) over a finite β range, say $\beta_c < \beta < \beta_c + \Delta\beta$. In addition the measured values of $m(\beta).a(\beta)$ will possess statistical errors. This raises the practical question of how to evaluate the significance of an apparent verification of (2) or (3): how small do the errors have to be for an apparent confirmation of (2) or (3) to be reasonably compelling? In the case where we are testing for continuum scaling, as in (3), a reasonable criterion is that the statistical errors on $ma(\beta)$ be very much less than the amount by which the two-loop expression for $a(\beta)$ changes in the $\Delta\beta$ interval being tested. For example in the present paper we calculate masses at $\beta = 5.7$ and $\beta = 5.9$. So we would want the statistical error δm, to satisfy

$$\left[\left(\frac{\delta m(5.7)}{m(5.7)}\right)^2 + \left(\frac{\delta m(5.9)}{m(5.9)}\right)^2\right]^{1/2} \ll \left.\frac{a(5.7)-a(5.9)}{a(5.7)}\right|_{2\text{-loop}} \approx 0.2 \qquad (4)$$

Calculations that begin to meet the above standards are now possible, and indeed some are available[3)-5)]. We have gone in some detail into the desirable standards not because we are going to meet them in the present work, but because they will enable us to put our results in an honest perspective.

CALCULATING GLUEBALL ENERGIES: $E(J^{PC};\underline{p})$

Glueball energies can be calculated by measuring the asymptotic exponential decay of the correlation function of two operators with the desired $(J^{PC},\underline{p})$ quantum numbers; as in Eq. (1).

The first step is to construct local colour singlet operators whose $\underline{p} = 0$ piece will have the desired J^{PC} properties. The trace of a closed loop is colour singlet: the real part has C = + and the imaginary part C = −. Suitable linear combinations of rotations and spatial inversions of such loops will then have the desired J^{P} quantum numbers. Consider for example the 2×2 plaquette centred on the site $\underline{n}_\mu$. There are three such spatial loops which we may label by the spatial direction orthogonal to the plane of the loop: so $\phi_i(n_\mu)$, i = x,y,z. We can form a 0^{++} operator using the maximally symmetric combination

$$\phi_{0^{++}}(n_\mu) = \mathrm{Re}\,\mathrm{tr}\left\{\phi_x(n_\mu) + \phi_y(n_\mu) + \phi_z(n_\mu)\right\} \tag{5}$$

and a 2^{++} operator, with spin projection 2 along the z axis, by

$$\phi_{2^{++}}(n_\mu : m_z = 2) = \mathrm{Re}\,\mathrm{tr}\left\{\phi_x(n_\mu) - \phi_y(n_\mu)\right\} \tag{6}$$

(In the continuum limit these operators also project onto 4^{++},... states.) There exists a systematic variational procedure[10] for choosing operators that have a large projection onto the state of interest.

If one wants to calculate the lowest <u>mass</u> m in some J^{PC} channel, one can take the correlation function of appropriate local operators; however, one will obviously do much better to project onto zero momentum if one wishes to see the asymptotic exponential decay at a t which is as small as possible:

$$\phi(\underline{p}=0\,;n_t) = L_s^{-3/2}\sum_{\underline{n}_i}\phi(\underline{n}_i\,,n_t)$$

$$C(\underline{p}=0\,;n_t) = \langle\phi(\underline{p}=0;n_t)\,\phi(\underline{p}=0\,;0)\rangle \underset{n_t\to\infty}{\sim} e^{-ma.n_t} \tag{7}$$

To calculate the lowest energy for a given momentum $\underline{p}$ in some J^{PC} channel, we need to construct an appropriate operator, $\phi_{J^{PC}}(\underline{p},n_t)$, with the required quantum numbers. The simplest choice[6),7),11)] (correct for the 0^{++}) might seem to be

$$\phi(\underline{p}, n_t) = L_s^{-3/2} \sum_{\underline{n}} e^{i\underline{p}a\cdot\underline{n}} \phi_{J^{PC}}(\underline{n}, n_t) \tag{8}$$

However, while ϕ certainly has momentum $\underline{p}$, it does not, in general, have the pure J^{PC} quantum numbers of the $\underline{p} = 0$ piece of the component local operators. [This mixing was noted in Ref. 11).] This is simple to see: the J^{PC} properties of the operator ϕ can be seen by boosting ϕ from momentum $\underline{p}$ to momentum zero (i.e., to the rest frame). Under such a boost the local rotational properties are mixed. A simple example is given by the "2^{++}" operator

$$\phi(\underline{p}, t) = \int d^3x \; e^{i\underline{p}\cdot\underline{x}} \left\{ F_{xy}^2(\underline{x}, t) - F_{yz}^2(\underline{x}, t) \right\} \tag{9}$$

which we write in continuous space-time to emphasize that the lattice discretisation is irrelevant to the argument. Suppose $\underline{p}$ is parallel to the x axis. Boosting to $\underline{p} = 0$ rotates

$$x \longrightarrow x' = x\cos\theta + t\sin\theta \tag{10}$$

and hence in (8) one will pick up pieces of the form $F_{xy}F_{yt}$ which project onto 0^{++} as well as 2^{++}.

In the continuum, curing this presents no problem; however, on the lattice one would in general have to include an appropriate linear combination of timelike loops, which is undesirable in the present region of couplings. One notes, however, that if our momentum is parallel to the spin (helicity ±2 for our example), the boost will leave the helicity unchanged so that the state does not mix with $J < 2$ states.

So to construct an operator with $(J^{PC}, \underline{p})$ quantum numbers one uses Eq. (8); with impunity if $J^{PC} = 0^{++}$, and with maximal helicity local operators otherwise. Then the lowest energy can be calculated as usual:

$$\langle \phi_{J^{PC}}(\underline{p}, n_t)\, \phi_{J^{PC}}(\underline{p}, 0)\rangle \underset{n_t \to \infty}{\sim} \propto e^{-E(J^{PC};\underline{p})\cdot a n_t} \tag{11}$$

We observe, finally, that the operator mixing under boosts disappears as $p/m \to 0$. Moreover the operators that are mixed in, such as $F_{xy}F_{yt}$ in our example, have in fact a very small projection onto the 0^{++}. Hence one may expect that correlation functions using operators defined as in Eq. (8) will always be dominated for moderate n_t and for p/m small by exchanges of the naive J^{PC}. We shall see later what actually happens in practice.

DYNAMICAL RESTORATION OF LORENTZ INVARIANCE

Using as our basic local loops 1×1 and 2×2 plaquettes, we construct 0^{++} operators with momenta, $\underline{p}$, covering the range

$$p_i = \frac{n_i \pi}{4a} \tag{12}$$

where

$$n_x, n_y = 0,1,2$$
$$n_z = 0,1,2,3,4$$

so that at least one momentum component takes all possible values. We then calculate correlation functions of these operators up to the maximum distance of t = 4a on our periodic 8^4 lattice.

Our statistics are not good enough and our lattice is not long enough for us to identify the asymptotic exponential decay, as in Eq. (1). However, our previous (source-method) calculation[3)] at $\beta = 5.7$, for the $\underline{p} = 0$ correlation function, showed that for a lattice of the present size the effective mass one extracts between t = a and t = 2a is already a good (though not perfect) approximation: that is to say the admixture of higher mass states is at this distance relatively small. Since our data ceases to be very accurate further than t = 2a, we shall focus on energy estimates obtained from a to 2a:

$$E(p)\cdot a \approx \ln\left[\frac{C(p;a)}{C(p;2a)}\right] \tag{13}$$

A second weakness of our calculation is that we have no explicit control of finite size effects: we calculate on only one lattice size. Once again we have recourse to our previous calculation[3)] where we demonstrated that at $\beta = 5.7$ an 8^3 lattice is (very probably) close to the infinite volume limit.

So, we extract E(p)·a from our $\beta = 5.7$ measurements using Eq. (13). In Fig. 1a we plot the extracted $[E(p)\cdot a]^2$ versus $(p\cdot a)^2$ for the lowest momenta. We find an excellent confirmation of the continuum relativistic dispersion relation:

$$(Ea)^2 = (pa)^2 + (ma)^2 \tag{14}$$

Indeed our data are completely inconsistent with the leading strong coupling result

$$E(p).a = ma \quad ; \ \forall p \tag{15}$$

[as well as with Monte Carlo measurements[6] of E(p).a taken at $\beta = 4.0$]. It is amusing to note that our data are accurate enough to rule out the rotationally invariant but non-relativistic continuum energy momentum relation

$$E.a = ma + (pa)^2/2ma \tag{16}$$

In Fig. 1b we plot $(E.a)^2$ versus $(p.a)^2$ for a different local operator and over our full available momentum range. One expects that as one increases p_x towards its maximum value while keeping p_y and p_z fixed and small, $[E(p).a]^2$ must bend over to a maximum at $p_x = p_{max} = \pi/a$. In Fig. 2 we show such plots for $(p_y, p_z) = (0,0)$ and $(0,\pi/4a)$ respectively. We observe a signal of the expected behaviour: moreover the transition from low momentum continuum physics, to the ultra-violet lattice physics appears smooth.

We now form non-zero momentum 2^{++} operators (as explained previously) and measure their correlation functions. In Fig. 3 we display $(Ea)^2$ versus $(pa)^2$, as extracted from $t = a$ to $t = 2a$, for both $\beta = 5.7$ and $\beta = 5.9$. The open points come from operators with helicity (spin projection along $\underline{p}$) two: as shown previously these will be operators with minimum J of 2. The solid points may suffer some mixing with the 0^{++} state. We shall discuss the quality of this 2^{++} data in more detail later on. For the moment we remark that it is consistent with the continuum dispersion relation and that there is, at small momenta, no sign of the kind of flattening of E(p).a which one would expect if there were a significant admixture of the 0^{++} state.

In summary: we find good evidence that by $\beta = 5.7$ continuum Lorentz invariance has been (dynamically) restored. Our control of systematic errors in this calculation is not direct: we need to appeal to our previous calculation[3] of the 0^{++} mass. Thus the systematic errors on the <u>energies</u> begin to slip out of our control with increasing momentum: the reader should bear this in mind particularly when interpreting the data with the very largest momenta. To do better, one should repeat the calculation we previously performed for the mass, using a source that projects onto the desired non-zero momenta (the construction of such a source is trivial).

RESTORATION OF CONTINUUM ROTATIONAL INVARIANCE

On a cubic lattice continuous rotational symmetry is broken down to that of $\pi/2$ rotations. As we approach the continuum limit, the long-distance physics should recover full rotational invariance and the cubic nature of the lattice should be felt only near the ultra-violet cut-off.

In practical calculations the restoration of rotational symmetry will manifest itself in the fact that the energy is a smooth function of $|\underline{p}|$ only (for p small compared to the maximum momentum). Equally the heavy quark-antiquark potential, $V(\underline{r})$, should be a smooth function of $|\underline{r}|$ only, for $|\underline{r}|$ much bigger than the lattice spacing. Indeed SU(2) rotational invariance restoration was first confirmed by the latter type of calculation[12]. Later evidence came from the smoothness of momentum-dependent correlation functions, this time for both SU(2) and SU(3)[6),13], and then, in the case of SU(3), also from the potential[14].

With our present calculation we are able to test the rotational invariance of momentum dependent correlation functions much more accurately. In Fig. 4 we plot C(p;a)/C(p;0) for the 0^{++} state at $\beta = 5.7$: we distinguish the values of $\underline{p}$ according to whether they lie along a cube axis, a cube face diagonal, a cube diagonal, or otherwise. The accurate smoothness of the correlation functions is apparent.

In addition we note that there are two momenta accessible to our measurements, which have the same $|\underline{p}|$: $\underline{p}_A = (1,2,2)(\pi/4a)$ and $\underline{p}_B = (0,0,3)(\pi/4a)$. In Fig. 5a,b we plot the values of $C(\underline{p}_A,a)/C(\underline{p}_A,0)$ and $C(\underline{p}_B,a)/C(\underline{p}_B,0)$ for the 2×2 local operator for $\beta = 5.7$ and 5.9, respectively. We also plot these ratios for the immediately neighbouring momenta so that one may judge the significance of the comparison. It is clear that again we have a remarkably accurate confirmation that continuum spatial rotational invariance has been restored by $\beta = 5.7$.

THE MASS OF THE 2^{++} GLUEBALL

Our 2^{++} correlation functions are of useful (statistical) accuracy only up to t = 2a. Accordingly we shall extract our energies using Eq. (13). How much one can justify this procedure is left to the following section: not because the problems are minor, but rather because they are difficult! For the moment we merely remark that because of the typical decomposition

$$\begin{aligned} C(p,t) &\equiv \langle \phi(p,t)\,\phi(p,0)\rangle \\ &= \langle \phi\, e^{-Ht}\,\phi \rangle \\ &= \sum_n e^{-E_n(p)\cdot t}\,|\langle n|\phi|\Omega\rangle|^2 \end{aligned} \tag{17}$$

we know that any energy we extract using Eq. (13) will be an <u>upper bound</u> on the true minimum energy in that channel.

The most direct way to get a mass is from the ratio of $\underline{p} = 0$ correlation functions at t = 2a and at t = a. This gives us

$$m \cdot a = \begin{cases} 2.65^{+0.37}_{-0.27} & \beta = 5.7 \\ 2.09^{+0.22}_{-0.19} & \beta = 5.9 \end{cases} \tag{18}$$

To reduce the statistical errors further one can use the measured energies E(p) with maximal helicity, and, assuming $E^2 = p^2 + m^2$, extract a mass. This gives

$$m \cdot a = \begin{cases} 2.39^{+0.13}_{-0.10} & \beta = 5.7 \\ 2.02^{+0.12}_{-0.10} & \beta = 5.9 \end{cases} \tag{19}$$

In practice we use $p \leq \pi/2a$ and the final numbers in (19) are dominated by $p = 0$ and $\pi/4a$. That is to say

$$\left(p/m\right)^2 \lesssim \begin{cases} 0.1 & \beta = 5.7 \\ 0.15 & \beta = 5.9 \end{cases} \tag{20}$$

and the uncertainty introduced by the use of $E^2 = p^2 + m^2$ is presumably minimal.

If we take our values in Eqs. (18) and (19) and express them in units of Λ_L [by using the two-loop perturbative formula for a(β)] we obtain

$$\underline{p} \parallel \underline{s} \quad \begin{cases} m(\beta = 5.7) = 726^{+39}_{-30} \cdot \Lambda_L \\ m(\beta = 5.9) = 770^{+46}_{-38} \cdot \Lambda_L \end{cases} \tag{21}$$

$$\underline{p} = 0 \quad \begin{cases} m(\beta = 5.7) = 805^{+112}_{-82} \cdot \Lambda_L \\ m(\beta = 5.9) = 797^{+84}_{-72} \cdot \Lambda_L \end{cases} \tag{22}$$

These values are plotted in Fig. 6 [with 0^{++} mass values from Ref. 3)]. We confirm asymptotic continuum scaling to ~10% over a region of couplings where the inverse two-loop lattice spacing changes by ~25%: that is to say, we have a reasonably significant confirmation of asymptotic scaling for the 2^{++} glueball state (as measured herein).

THE MASS OF THE 1^{-+} GLUEBALL

Our 1^{-+} correlation functions are only statistically accurate up to t = a: to obtain statistical accuracy up to t = 2a one must include $\underline{p} \nparallel \underline{s}$ contributions, and this incurs problems of mixing.

So from our zero momentum data at t = 0 and t = a we obtain the following upper bounds:

$$m.a \leq \begin{cases} 4.56(5) & \beta = 5.7 \\ 4.53(6) & \beta = 5.9 \end{cases} \tag{23}$$

However, we can hardly pretend to get realistic mass estimates from such short time intervals. To see what we can extract at t = 2a we calculate the energies, $E(\underline{p})$, for all momenta, using Eq. (13), and then extract a mass, ma, assuming $(ma)^2 = (Ea)^2 - (pa)^2$. The extracted masses are displayed in Fig. 7a,b for β = 5.7 and 5.9, respectively. The solid circles represent masses from maximal helicity states, i.e., where there should be no mixing of lower spin states. It is clear that the latter data contains no information at β = 5.9, and just barely some at β = 5.7. From the latter data we can extract the mass estimate

$$\left.\begin{matrix} \underline{p} // \underline{s} \\ \beta = 5.7 \end{matrix}\right\} \qquad ma = 2.8^{+0.7}_{-0.4} \tag{24}$$

which is not accurate enough to be useful.

To do better we take a slightly greater risk (of being wrong). Noting that the masses in Fig. 7 show no sign of decreasing with increasing $|\underline{p}|$ - as one might have expected from any intermixing of light J = 0 states - and noting in addition that the $|\underline{p}| = \pi/4a$ states should have the least of this mixing, we boldly go forth to extract from our data, at this momentum, the mass estimates

$$|\underline{p}| = \frac{\pi}{4a} \quad : \qquad ma = \begin{cases} 1.66\ (6) & \beta = 5.7 \\ 2.41\ (15) & \beta = 5.9 \end{cases}$$

$$= \begin{cases} 504\,(18) \cdot \Lambda_L & \beta = 5.7 \\ 920\,(57) \cdot \Lambda_L & \beta = 5.9 \end{cases} \tag{25}$$

Equation (25) represents, evidently, a dramatic violation of asymptotic scaling. We conclude that either our mass estimates, Eq. (25), are simply incorrect (due to our lack of control of systematic errors), or, if they are correct, we are not yet seeing any evidence for continuum 1^{-+} physics.

HOW GOOD ARE OUR MASS ESTIMATES?

It is now time to assess how reliable are our 2^{++} and 1^{-+} mass estimates. Since, in fact, we do not have any very serious 1^{-+} results, we focus on the case of the 2^{++} glueball (any reservations concerning the latter may be carried over, and even more so, to the 1^{-+} case).

There are three major causes for concern. Two have been alluded to previously: are we far enough along the correlation function to be extracting something close to the lowest 2^{++} glueball mass, and how important are finite volume corrections? The third problem (which does not arise for the lowest mass 0^{++} state) is that if the 2^{++} glueball mass is greater than twice the 0^{++} mass (and our estimates suggest it is) then the truly asymptotic exponential decay of a 2^{++} correlation function will be given not by the 2^{++} glueball but by the $L = 2$ 0^{++}-0^{++} cut. How should we deal with this ambiguity?

To address the first question we begin by presenting, in Fig. 8, the masses extracted from our data for $C(p;3a)/C(p;2a)$ (at low momenta) - the values represented by shaded circles should have no lower J mixing, while the open circles in principle may have. We simultaneously plot the values expected (shaded triangles) if our mass estimates in Eq. (21) are correct. The values are mutually consistent; but due to the large errors the significance of this consistency is weak. What is perhaps much more interesting is that we can see 2^{++} signals at $t = 3a$, and with an extra factor of ~10 in statistics we might expect to get useful numbers at three lattice spacings. This would require about 75 hours of one CPU of a Cray X-MP: an accessible amount of time.

In Fig. 9, we plot the masses extracted from $t = a$ and $t = 2a$ using Eqs. (13) and (14). The data suggest, simultaneously, that there is no significant 0^{++} mixing (which should lead to ma decreasing with increasing pa) and that Lorentz invariance is being respected. This would also suggest that one state is dominating the correlation function - otherwise there would be no reason for the "effective mass" to be independent of momentum. Of course, this apparent independence might be accidental.

Perhaps the best argument comes from comparing with 0^{++} correlation functions: at $\beta = 5.7$ the lowest mass glueball can be extracted from $t = a$ to $t = 2a$ with an error of only about 10%[3]. Of course, while this argument suggests that our $\beta = 5.7$ 2^{++} mass estimate may be reasonably accurate, the same argument would suggest that the $\beta = 5.9$ estimate is not: at this β the 0^{++} mass extracted from $t = a$ to $t = 2a$ is about 50% higher than the true mass! There is, however, some evidence that allows us to be optimistic even at $\beta = 5.9$. The argument goes as follows. In Fig. 10a, we plot $C(p=0;a)/C(p=0;0)$ for the 0^{++} 2×2 plaquette operator as a function of β (using the present as well as previous published[7),8)] and unpublished[15)] data). This quantity rises rapidly as β increases, reaches a maximum near $\beta = 5.7$, and then begins to drop rapidly. The reason for this behaviour is that at smaller β the lattice spacing is large, so that this ratio of correlation functions is dominated by $\exp(-ma)$ and hence increases as β is increased and $a(\beta)$ decreases; at sufficiently large β, however, the 2×2 operator becomes ultra-violet, projects rapidly less onto the lowest lying glueball, and its correlation function over short distances decreases, being dominated by large mass exchanges. Hence where this ratio begins to fall is where, at $t = a$, the correlation function loses knowledge of the lowest mass. In Fig. 10b we show the same ratio for the 2^{++}; the flattening occurs at higher β than for the 0^{++}, near $\beta = 5.9$. We may take this as providing evidence that a $t = 2a$ 2^{++} mass extraction at $\beta = 5.9$ will be roughly as reliable as a 0^{++} $t = 2a$ mass at $\beta = 5.7$: that is to say, that it will be within ~10% of the true mass.

Leaving this question with this (overly?) optimistic point of view, we now address the question of finite size corrections. The functional form of the leading finite (spatial) size corrections is known[9]:

$$m(L) = m(\infty)\left[1 - c\,\frac{\exp\{-\text{const.}\, m(\infty)\cdot aL\}}{m(\infty)\cdot aL}\right] \tag{26}$$

for a lattice of spatial length L. These corrections are the same as for the 0^{++} except for the value of the coupling contained in c. (They arise from the longest distance piece of the self-energy: a 0^{++} is emitted by the 2^{++} and is reabsorbed after winding through the periodic spatial boundary.) If c is not much greater for the $2^{++}.2^{++}.0^{++}$ vertex than for the 30^{++} vertex then we can make a statement as follows: at $\beta = 5.7$ the finite spatial corrections are small (~5%) but at $\beta = 5.9$ they will be large. In addition the finite length in time induces finite temperature corrections whose size is not known, but which could well be important at $\beta = 5.9$ where the temperature associated with eight lattice spacings is a large fraction of the critical deconfining temperature[5),16)]

$$T(\beta = 5.9\,;\, 8a) \approx 0.75\, T_c \tag{27}$$

(Note, however, that there is no reason why the mass of the 2^{++} resonance should change dramatically as we cross into the deconfined phase.) We conclude that finite size corrections are very probably small at $\beta = 5.7$: at $\beta = 5.9$ they could be large. [In this context, we are tempted to point out that previous 2^{++} calculations on a $4^3 8$ lattice[8] (extracting the mass from $t = a$ to $t = 2a$) found

$$ma = \begin{cases} 2.23\,^{+0.21}_{-0.17} & \beta = 5.7 \\ 1.81\,^{+0.26}_{-0.20} & \beta = 5.9 \end{cases} \tag{28}$$

which tells us that if indeed the lowest mass is dominating at $t = a$ to $t = 2a$, then finite size effects are not large at either β. However, this last observation should certainly not be overstressed - a similar point can be made about the 0^{++} glueball[7], yet, nonetheless, we now know that finite size effects over this range of lattice sizes are in fact large [3]!]

We now turn to the third problem: if indeed the 2^{++} glueball mass is greater than twice the 0^{++} mass, and our best estimate is

$$m_{2^{++}} \approx 2.5 \; m_{0^{++}} \qquad (29)$$

then the asymptotic exponential decay will come from the L = 2 two 0^{++} glueball cut. How can we measure the 2^{++} mass, and indeed are we certain that our present 2^{++} mass estimates are not already badly contaminated by 0^{++}-0^{++} cut contributions? This is of course not a new question, and it has been addressed in some detail in the literature[17)]. Generally these ambiguities can only be resolved with measurements which are sufficiently accurate to reliably isolate both types of contribution in several different correlation functions. A more practical observation is that the projection onto the cut will typically be spread over a large energy range, so that the projection onto energies close to the branch point E_b, say

$$E \leq 2.5 \, m_{0^{++}} = 1.25 \, E_b \qquad (30)$$

will be very small. In Ref. 17), operators which largely project onto the L = 2 two 0^{++} glueball cut were constructed for SU(2) and typically the correlation functions would fall much more steeply - over the range of time intervals being considered herein - than the 2^{++} correlation functions we have obtained in the present work. We refer to Ref. 17) for more details: it seems reasonable to expect that the cut contributions are very small for $t \leq 2a$:

0^{++}: COMPARING SOURCE AND VARIATIONAL METHODS

The high statistics of the present simulation gives us an opportunity to compare the relative efficiencies of variational (as herein) and source [as in Ref. 3)] methods. Although such a comparison is preliminary in the sense that both types of method are open to substantial improvement, it should be of a very practical interest to anyone who wishes to pursue calculations of this kind.

In a source calculation[18)] one places a source at t = 0, e.g., by fixing all t = 0 spacelike links to unity [as in Ref. 3)], and measures how the expectation value of a suitable colour singlet operator, ϕ, approaches its value at "$t = \infty$":

$$\langle \phi(t) \rangle \underset{t\to\infty}{\simeq} \langle \phi(\infty) \rangle + c\, e^{-mt} \qquad (31)$$

where m is the lowest mass state sharing the quantum numbers of ϕ and the source. The major advantage of this method is that the statistical errors come only from the measurements of ϕ: in the variational method the errors come from both ends of the correlation function, and so typically the error/signal ratio would be worse. A disadvantage of the source method is that if one extracts a mass at finite t, assuming Eq. (31) to be valid, then this mass can be larger or smaller than the true mass: with the variational method, by contrast, such a mass will always be greater than the true mass. Here we do not wish to enter into the relative merits of the two methods [the reader is referred to Ref. 3) for a slightly more detailed discussion]: we shall confine ourselves to comparing their relative statistical errors.

Such a comparison is inevitably ambiguous. To reduce these ambiguities it is best to use a lattice of the same spatial size with both methods. This is possible at $\beta = 5.7$ where we can compare the results of this paper obtained on an 8^4 lattice, with source method calculations on an $8^3 16$ lattice[3)]. At $\beta = 5.9$, however, we shall have to compare our 8^4 results with source calculations on a $10^3 20$ lattice.

The 0^{++} correlation functions in this paper have been constructed from both 1×1 and 2×2 plaquettes. Since the latter loop gives larger values of $\langle\phi(a)\phi(0)\rangle/\langle\phi(0)\phi(0)\rangle$ we shall only consider its (vacuum-subtracted) correlation function here (this being in the spirit of the variational approach). We now extract effective masses by making local cosh fits (the lattice is periodic) to the correlation functions between $t = (n_t-1)a$ and $t = n_t a$:

$$\frac{\langle \phi(n_t)\, \phi(0) \rangle}{\langle \phi(n_t-1)\, \phi(0) \rangle} = \frac{ch\left[\left(\frac{L_t}{2} - n_t\right) m_{eff} a\right]}{ch\left[\left(\frac{L_t}{2} - n_t + 1\right) m_{eff} a\right]} \qquad (32)$$

where $L_t = 8$ is the timelike extent of the lattice. In Fig. 11 we plot $m_{eff}\cdot a$ (open circles) as a function of $n_t a/(n_t-1)a$. We repeat the same procedure with the $pa = \pi/4$ correlation function to extract an effective energy, E_{eff}, and then, using

$$(E_{eff}\, a)^2 = (pa)^2 + (m_{eff}\, a)^2 \qquad (33)$$

and effective mass, $m_{eff}a$. This is also plotted in Fig. 11 (solid points). We now repeat the procedure for our source data on the $8^3 16$ lattice: i.e., we use Eq. (32) but replacing the left-hand side:

$$\frac{\langle \phi(n_t)\ \phi(0)\rangle}{\langle \phi(n_t-1)\ \phi(0)\rangle} \longrightarrow \frac{\langle \phi(n_t) - \phi(L_t/2)\rangle}{\langle \phi(n_t-1) - \phi(L_t/2)\rangle} \tag{34}$$

We obtain the crosses in Fig. 11. We now compare the errors for $n_t = 3$:

$$\frac{\text{error}\ (\text{var}\ ;\ p=0)}{\text{error}\ (\text{source}\ ;\ p=0)} \approx 8$$

$$\frac{\text{error}\ (\text{var}\ ;\ p=\pi/4)}{\text{error}\ (\text{source}\ ;\ p=0)} \approx 3.3 \tag{35}$$

The number of $8^3 16$ configurations used was comparable to half the number (28,000) of 8^4 configurations: that is, the computing time was comparable for both calculations. So the effective gain in CPU time through using the source is

$$\left.\begin{matrix}\text{CPU gain}\\ \text{factor}\end{matrix}\right)_{\beta=5.7} \approx \begin{cases} O(60) & p=0 \\ O(10) & p=\pi/4a \end{cases} \tag{36}$$

depending how we perform the comparison. Of course, it is perhaps misleading to compare the $p = \pi/4a$ variational results with a $p = 0$ source: perhaps a $p = \pi/4a$ source would do better. A second important point to note from Fig. 11 is the evidence that the source correlation function reaches its asymptotic exponential decay slightly later than the conventional method. In any case the gain in statistics obtained by using the source at this β is overwhelming, as we see from Eq. (36). (Note that at fixed β and for $p = 0$ the improvement the source brings relative to the variational method should grow linearly with the spatial volume: our results are consistent with this expectation.)

We now go to $\beta = 5.9$ where we compare the 8^4 results in this paper, with the 0^{++} correlation function on a $10^3 20$ lattice with source (mesurements with a 2×2 loop)[3]. Repeating the same procedure as above, but using only $p = 0$ correlation functions, we get the effective masses shown in Fig. 12. Comparing the errors for $n_t = 3$ we find:

$$\frac{\text{error}\ (\text{var}\ ;\ p=0)}{\text{error}\ (\text{source}\ ;\ p=0)} \approx 5 \tag{37}$$

However the CPU time spent on the $10^3 20$ lattice was ~6.5 times that spent on the 8^4 calculation. Hence

$$\left.\begin{matrix}\text{CPU gain}\\ \text{factor}\end{matrix}\right)_{\beta=5.9} \approx 4 \qquad (p=0) \tag{38}$$

One might argue that since the conventional calculation require a computing time that grows as the spatial volume to reach a given error/signal ratio, we should multiply the 4 in Eq. (38) by $(10/8)^3 \approx 2$. On the other hand, the information from the $p = \pi/4a$ correlation functions on a $10^3.L_t$ lattice might well push the comparison in favour of the conventional calculation. A more important point is that at this β the source effective mass appears to lag by almost a whole lattice spacing behind the variational m_{eff}: i.e., the asymptotic exponential decay sets in significantly earlier in the latter case. In summary: by $\beta = 5.9$ the enormous advantage of the source method over the variational method has almost completely evaporated. With increasing β the naive source calculation gets worse much more quickly than the naive variational method.

CONCLUSIONS

The high statistics of the present calculation enables us to extract much cleaner glueball correlation functions than in any previous SU(3) variational-type calculation. We show examples in Fig. 13. This has enabled us to present an accurate confirmation[6)] of the restoration of the continuum relativistic energy-momentum dispersion relation, $E^2 = p^2+m^2$, for the 0^{++} glueball at $\beta = 5.7$ (Fig. 1); and also to see the expected deviations from the continuum relation when the momenta approach the ultra-violet lattice cut-off (Fig. 2). In addition we have obtained accurate qualitative (Fig. 4) and quantitative (Fig. 5) demonstrations of spatial rotational symmetry restoration at $\beta = 5.7$ and 5.9, in the sense that we find our 0^{++} correlation functions to depend only on the modulus of the momentum vector, $\underline{p}$.

We have also calculated the energy of the 2^{++} glueball as a function of momentum (Fig. 3). There are extra problems here due to the possible intermixing of the 0^{++} glueball state for $\underline{p} \neq 0$: we point out that there is no such mixing for maximal helicity wave functions.

Our demonstrations of symmetry restoration at $\beta = 5.7$ are nonetheless open to improvement. We extract our energies from correlation functions between $\Delta t = a$ and $\Delta t = 2a$. From Ref. 3) we know that for $\underline{p} = 0$ the admixture of higher mass states will in fact be non-zero: however, it will be small. For $\underline{p} \neq 0$ we have no direct control over this particular systematic error, and this lends an uncertainty to our results that, obviously, increases with momentum. It would be useful to repeat the type of calculation performed in Ref. 3) for some non-zero momenta to remove this uncertainty.

Our high statistics study should also lay to rest a long-standing argument[19] about whether the 2^{++} glueball mass as extracted from $t = a$ to $2a$ along the correlation function is indeed very different from that extracted between $t = 0$ and $t = a$. It is: the latter mass is higher, and violates asymptotic scaling. The mass we extract in this paper (Fig. 6) is sufficiently accurate so that its consistency with asymptotic scaling is statistically significant, i.e., we have real evidence that we are seeing continuum 2^{++} glueball physics. The mass we extract

$$m_{2^{++}} \cdot a = \begin{cases} 2.39 \begin{smallmatrix} +0.13 \\ -0.10 \end{smallmatrix} & \beta = 5.7 \\ 2.02 \begin{smallmatrix} +0.12 \\ -0.10 \end{smallmatrix} & \beta = 5.9 \end{cases} \qquad (39)$$

can be expressed in physical units by using the ρ mass as extracted[20] in the valence quark approximation: if we do so, we find

$$m_{2^{++}} = \begin{cases} 1.84 \begin{smallmatrix} +0.10 \\ -0.08 \end{smallmatrix} \text{ GeV} & \beta = 5.7 \\ 1.99 \begin{smallmatrix} +0.12 \\ -0.10 \end{smallmatrix} \text{ GeV} & \beta = 5.9 \end{cases} \qquad (40)$$

(It goes without saying that such a translation into physical units is open to whatever uncertainties are involved in the ρ mass determination.)

Having congratulated ourselves on the extent to which our present 2^{++} calculation improves upon previous ones, it remains to confess that several important systematic errors are still largely out of our control. Three particularly important problems are: finite volume effects; have we gone far enough along the correlation function to isolate the 2^{++} ground state; if the 2^{++} is heavier than twice the 0^{++} mass, how do we untangle the resonance and cut contributions? In examining these problems (Figs. 8, 9 and 10) we concluded that the third problem should not be serious for $\Delta t \leq 2a$ (on the basis of previous calculations[17] of typical cut contributions to 2^{++} correlation functions) and that the other two should not be serious at $\beta = 5.7$ (on the basis of what we learn from the 0^{++} calculations[3]). At $\beta = 5.9$ there is every possibility that the last two errors are important. So we emphasize that our 2^{++} masses and their apparent continuum behaviour need considerable further work, and our results should be regarded as rather preliminary.

The situation with respect to our 1^{-+} glueball mass estimates is significantly worse. This is unfortunate because the 1^{-+} is an "oddball", i.e., its quantum numbers are not accessible to a pure $q\bar{q}$ system, and hence its experimental detection would be particularly significant. To get mass estimates at $\Delta t = 2a$, we were forced to use $\underline{p} \neq 0$ correlation functions with $\underline{p} \not\parallel \underline{s}$ so that intermixing with different quantum number states might have occurred. We minimized the risk of the latter by using only the lowest non-zero momentum. Our results were

$$m_{1^{-+}} = \begin{cases} 1.66\,(6)\cdot a^{-1} & \beta = 5.7 \\ 2.41\,(15)\cdot a^{-1} & \beta = 5.9 \end{cases}$$

$$= \begin{cases} 1.28\,(5)\ \text{GeV} & \beta = 5.7 \\ 2.38\,(15)\ \text{GeV} & \beta = 5.9 \end{cases}$$

where to express the mass in GeV units we once again use the ρ mass[20]. The fact that the mass changes rapidly with the (bare) coupling means that there is no evidence that the mass has yet attained its continuum value. Nonetheless we remark that the $\beta = 5.9$ value is similar to that obtained in a previous calculation[21] where the spacelike and timelike lattice spacings differed (and roughly corresponded to $\beta \approx 5.8$ and $\beta \approx 6.0$, respectively). The relatively light mass (a little above that of the 2^{++}) we find for the 1^{-+} at our highest β value is very motivating for improving the calculation of this important state.

The final part of our paper involved a comparison of the variational type[10] of mass calculation (as used herein) with the source method[18] for mass calculations [as used in ref. 3)] for the 0^{++} glueball state. At $\beta = 5.7$ the source method is more efficient (in CPU time) by a factor that lies somewhere between 10 and 100 depending on how one does the comparison. The source method results are so much more accurate that we have not bothered, in this paper, to extract masses from our 0^{++} data except insofar as we require them for this comparison (Figs. 11 and 12). The story at $\beta = 5.9$ is, however, very different: the methods seem to have become of comparable efficiency - the overwhelming advantage of the simple source method at $\beta = 5.7$ evaporates rapidly with increasing β as shown in Fig. 14. On the other hand the source method still has the very nice feature that the computer time to reach a given signal-to-error ratio (for $p = 0$ correlation functions) is naively independent of volume [the argument to the contrary in Ref. 3) is not correct] - while it increases linearly for the variational method. This emphasizes the desirability of constructing improved sources.

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FIGURE CAPTIONS

Fig. 1 The energy momentum dispersion relation at $\beta = 5.7$ (extracted from $\Delta t = a$ and $\Delta t = 2a$ correlation functions) for the 0^{++} glueball: (a) for the 2×2 loop operator, and the smallest momenta; (b) for the 1×1 loop operator and all measured momenta.

Fig. 2 The variation of the energy [extracted from $C(\underline{p},2a)/C(\underline{p},a)$] as a function of p_x at fixed p_y and p_z.

Fig. 3 As in Fig. 1a but for the 2^{++} glueball: (a) at $\beta = 5.7$; (b) at $\beta = 5.9$.

Fig. 4 Ratio of 0^{++} correlation functions, $C(\underline{p},a)/C(\underline{p},0)$, for 1×1 and 2×2 loop operators as a function of momentum, for momentum along axis ( ), a face diagonal (Δ), a cube diagonal (o) or otherwise (•). Errors within points unless shown otherwise.

Fig. 5 Comparison of 0^{++} correlation functions (based on the 2×2 local operator) for two different momenta of the same magnitude and for (a) $\beta = 5.7$, (b) $\beta = 5.9$.

Fig. 6 2^{++} glueball masses extracted from $t = a$ and $t = 2a$ correlation functions. Also shown are the 0^{++} masses of Ref. 3).

Fig. 7 The 1^{-+} mass as extracted from the energy, $E(p)$, using $E^2 = p^2+m^2$: (a) at $\beta = 5.7$; (b) at $\beta = 5.9$.

Fig. 8 2^{++} glueball masses extracted from $t = 2a$ to $t = 3a$ and compared with the masses as extracted from $t = a$ to $t = 2a$ along the correlation functions.

Fig. 9 As Fig. 7 but for the 2^{++}.

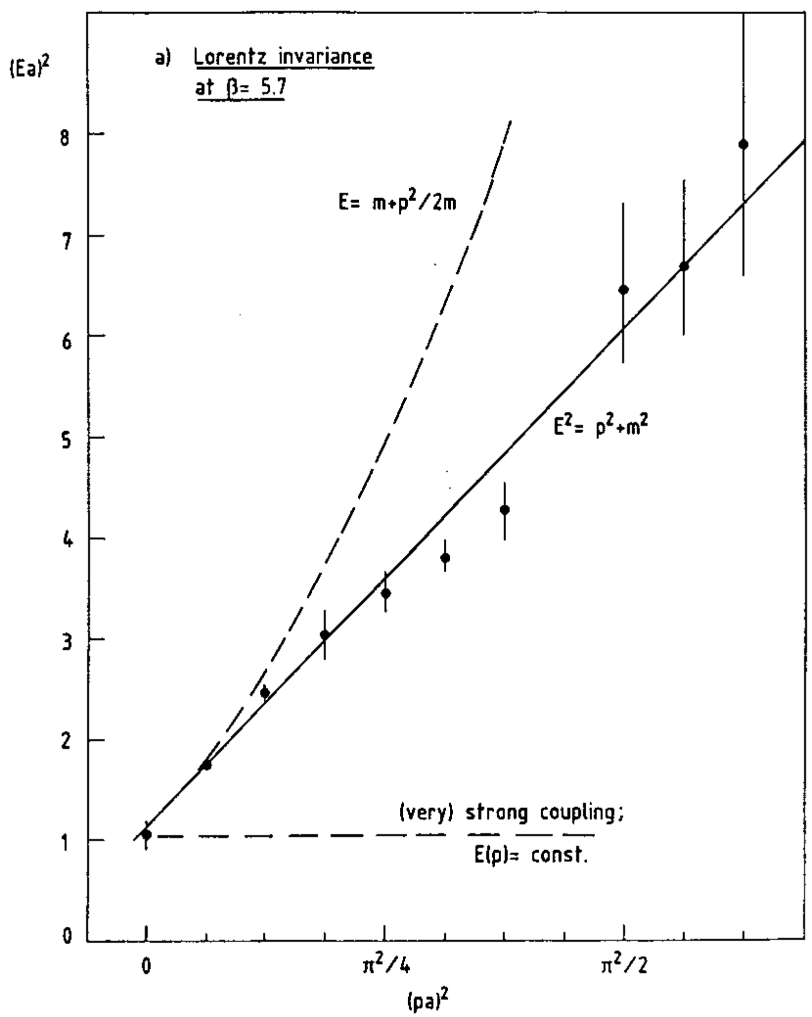
Fig. 10 The ratio of (a) 0^{++}, (b) 2^{++} correlation functions $C(p=0;a)/C(p=0;0)$ as a function of β.

Fig. 11 0^{++} masses extracted at increasing distances along correlation functions obtained with source on an $8^3 16$ lattice [Ref. 3)] and by the conventional method of this paper on an 8^4 lattice. All at $\beta = 5.7$.

Fig. 12 As in Fig. 11, but at $\beta = 5.9$ and the source results [Ref. 3)] come from a $10^3 20$ lattice.

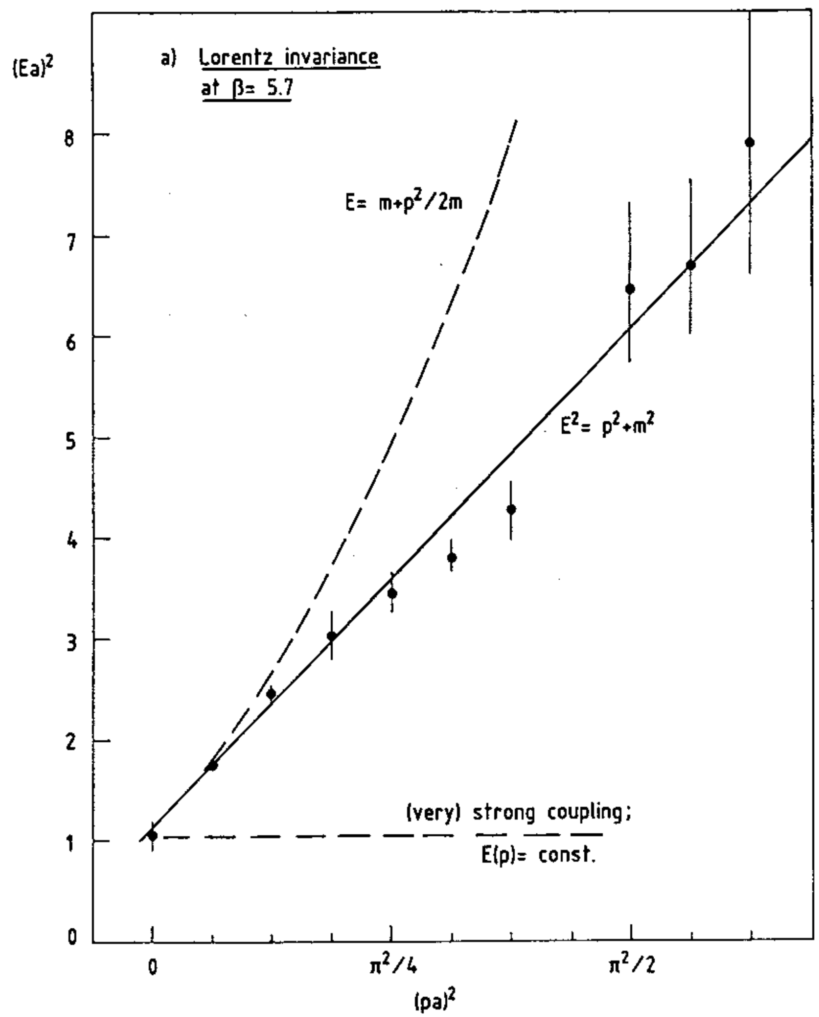
Fig. 13 0^{++} correlation functions from the present work.

Fig. 14 Ratio of CPU times required by variational and source methods [as used herein and in Ref. 3), respectively] to achieve a given error/signal ratio, at distances where the 0^{++} correlation function begins to be dominated by the lowest mass: a range of values is given at each β (see text). The data comes from this work, Refs. 3) and 7), and unpublished work[13].



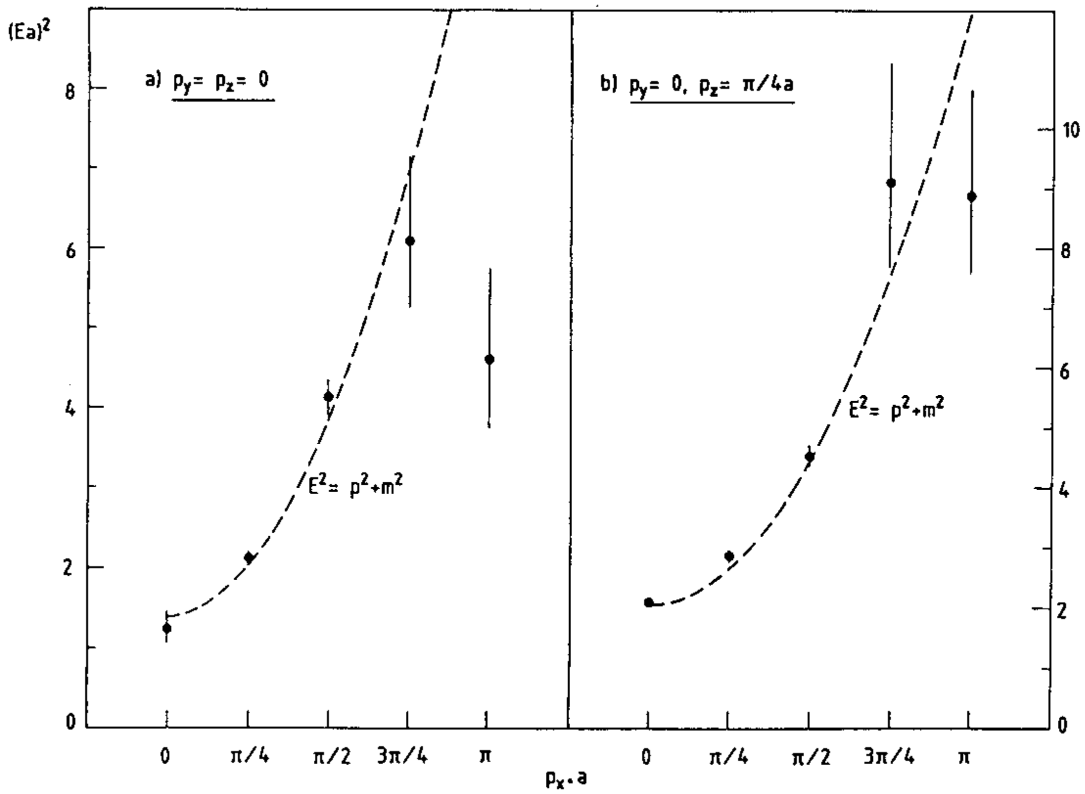
(Ea)²
a) Lorentz invariance
at β= 5.7
E= m+p²/2m
E²= p²+m²
(very) strong coupling;
E(p)= const.
8
7
6
5
4
3
2
1
0
0
π²/4
π²/2
(pa)²

- Fig. 1a -



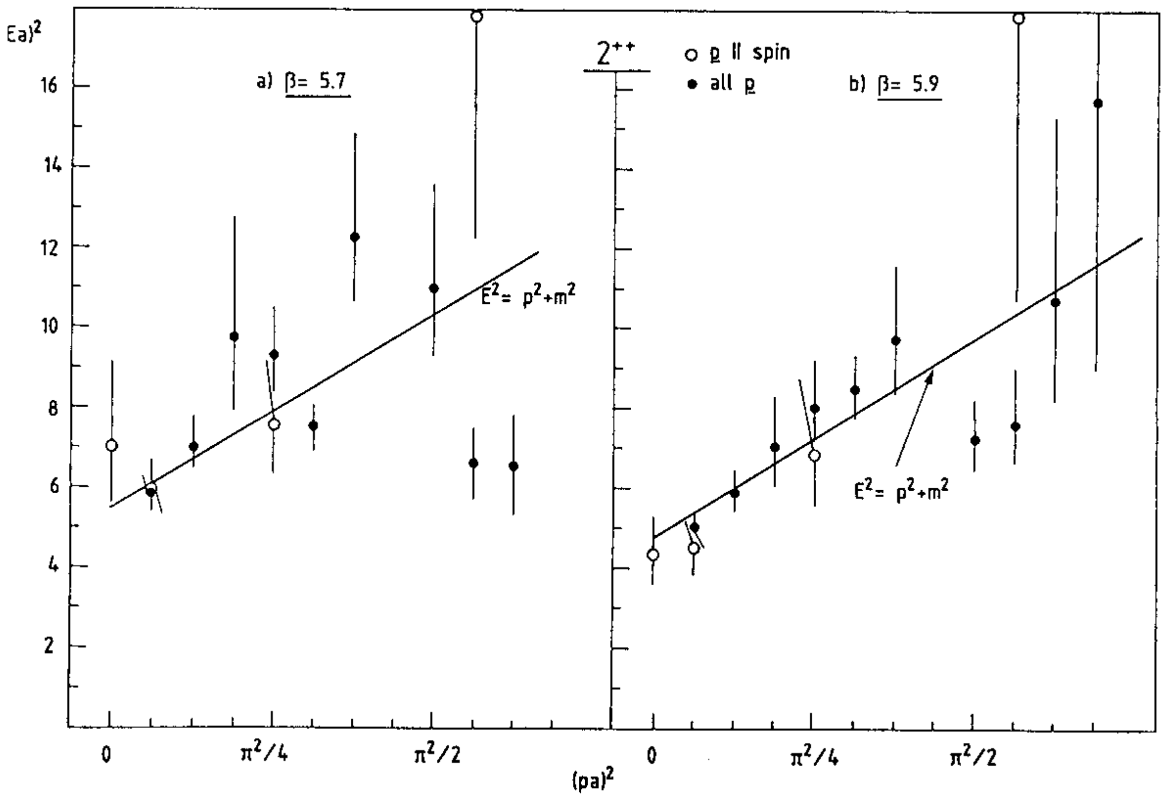
(Ea)²
a) Lorentz invariance
at β= 5.7
E= m+p²/2m
E²= p²+m²
(very) strong coupling;
E(p)= const.
8
7
6
5
4
3
2
1
0
0
π²/4
π²/2
(pa)²

- Fig. 1b -



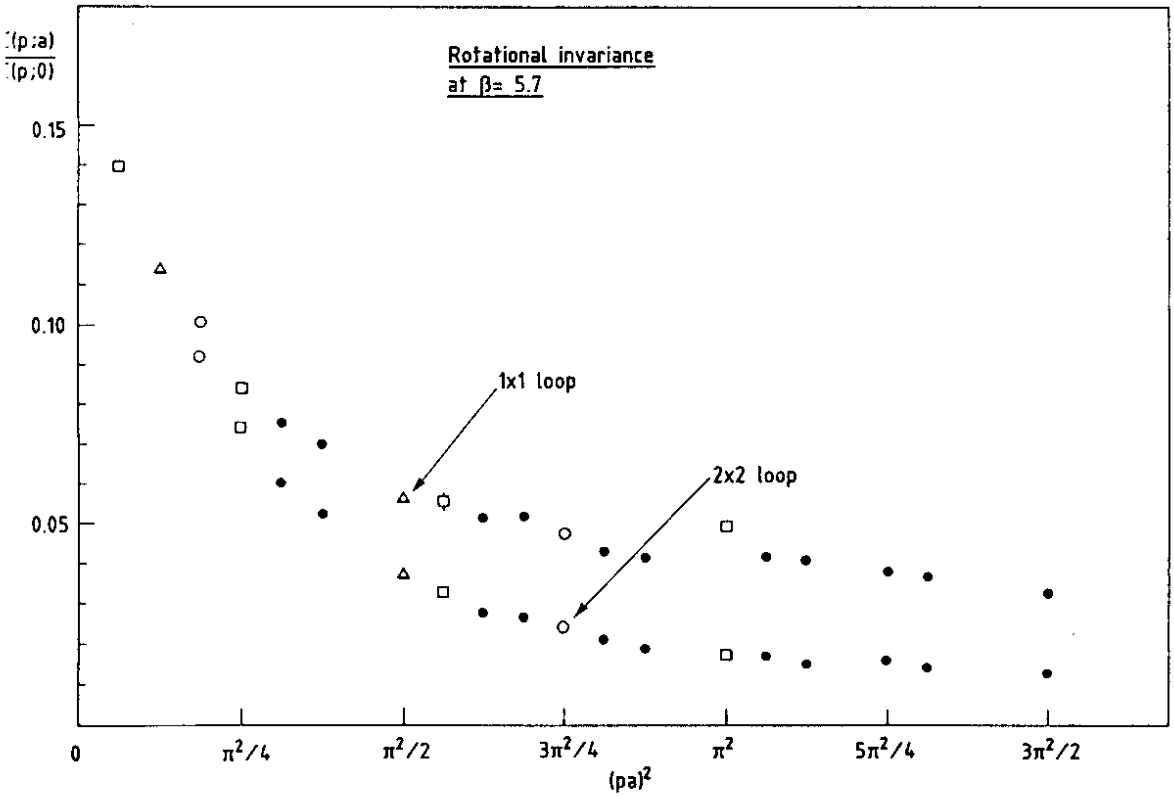
(Ea)^2
a) $p_y = p_z = 0$
b) $p_y = 0,\ p_z = \pi/4a$
$E^2 = p^2 + m^2$
$E^2 = p^2 + m^2$
0
2
4
6
8
10
0 π/4 π/2 3π/4 π
0 π/4 π/2 3π/4 π
$p_x \cdot a$

– Fig. 2 –



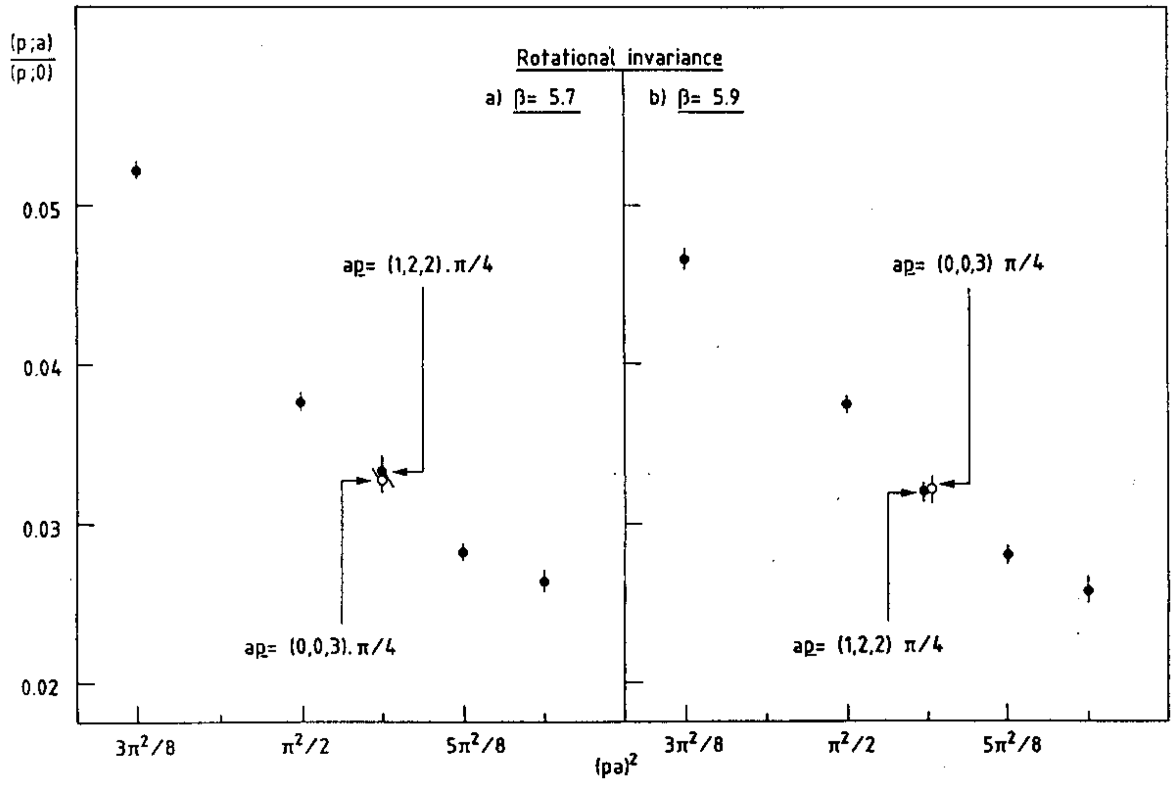
Ea)^2
2^{++}
o p ll spin
• all p
a) β= 5.7
b) β= 5.9
$E^2 = p^2 + m^2$
$E^2 = p^2 + m^2$
2
4
6
8
10
12
14
16
0 π²/4 π²/2
0 π²/4 π²/2
(pa)^2

– Fig. 3 –



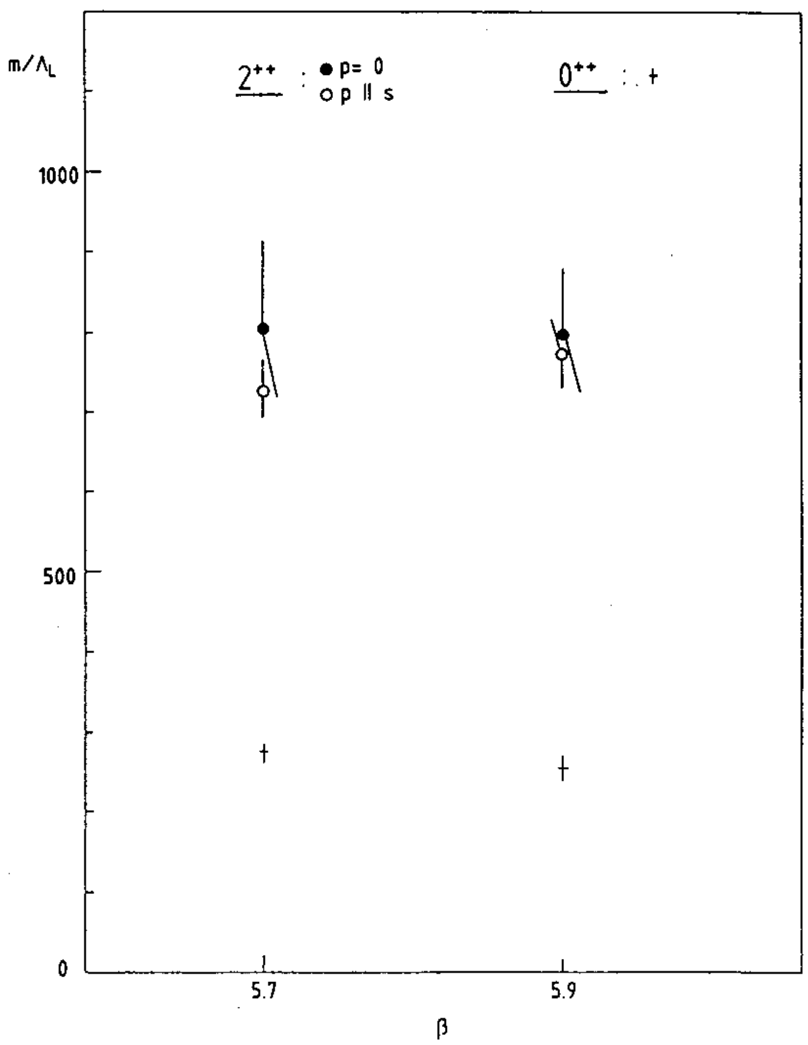
Rotational invariance
at β= 5.7
1x1 loop
2x2 loop
0.15
0.10
0.05
0
π²/4
π²/2
3π²/4
π²
5π²/4
3π²/2
(pa)²

- Fig. 4 -



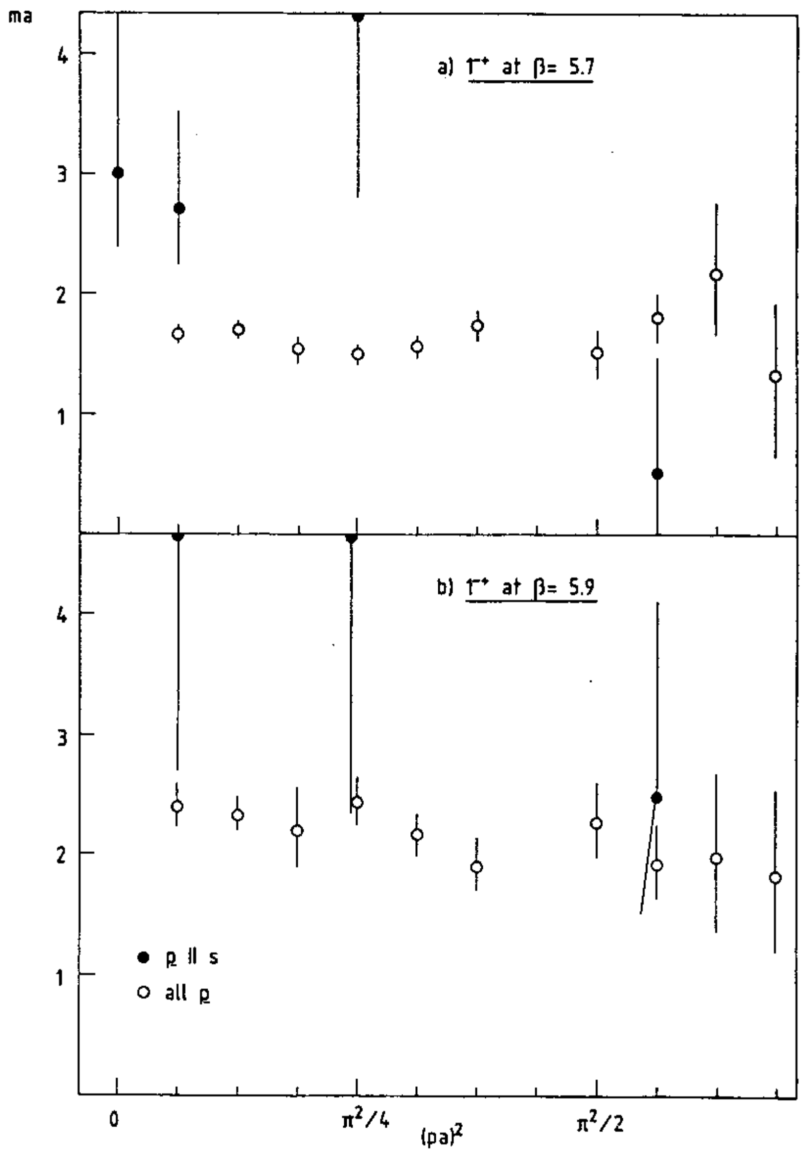
(p;a)
(p;0)
Rotational invariance
a) β= 5.7
b) β= 5.9
ap= (1,2,2).π/4
ap= (0,0,3).π/4
ap= (0,0,3) π/4
ap= (1,2,2) π/4
0.05
0.04
0.03
0.02
3π²/8
π²/2
5π²/8
(pa)²

- Fig. 5 -



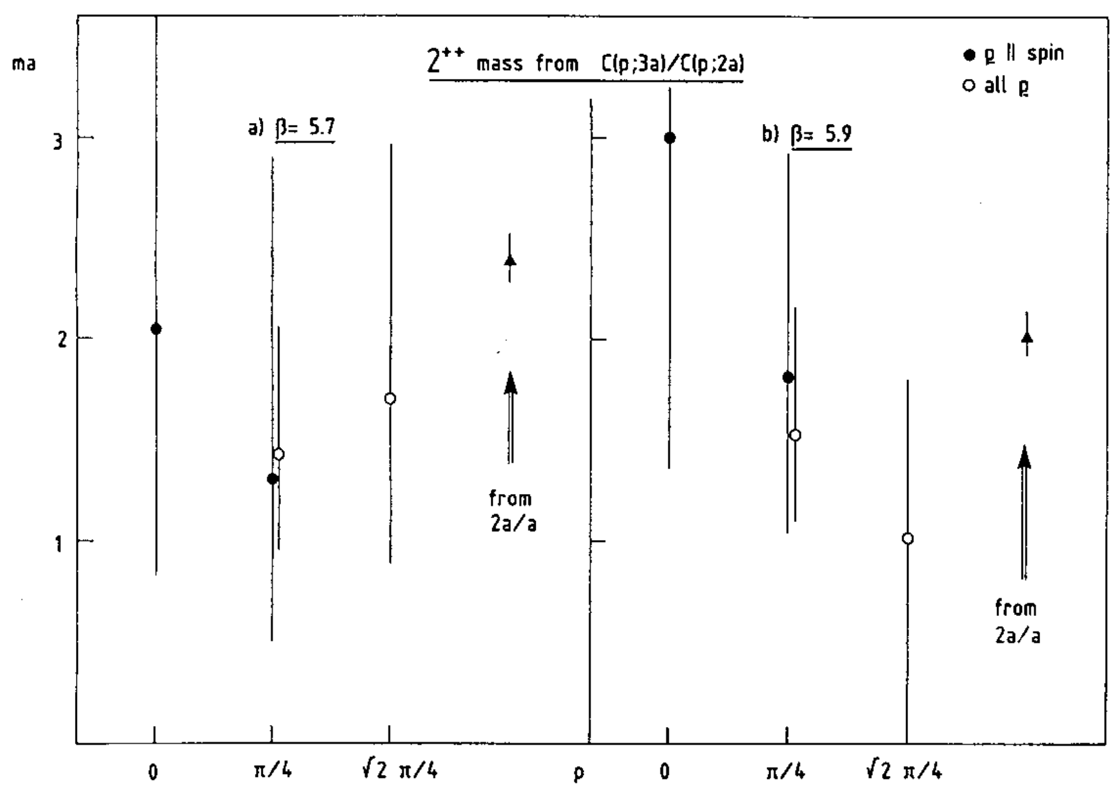
m/Λ_L
2++ : ● p= 0
○ p ∥ s
0++ : +
1000
500
0
5.7
5.9
β

- Fig. 6 -

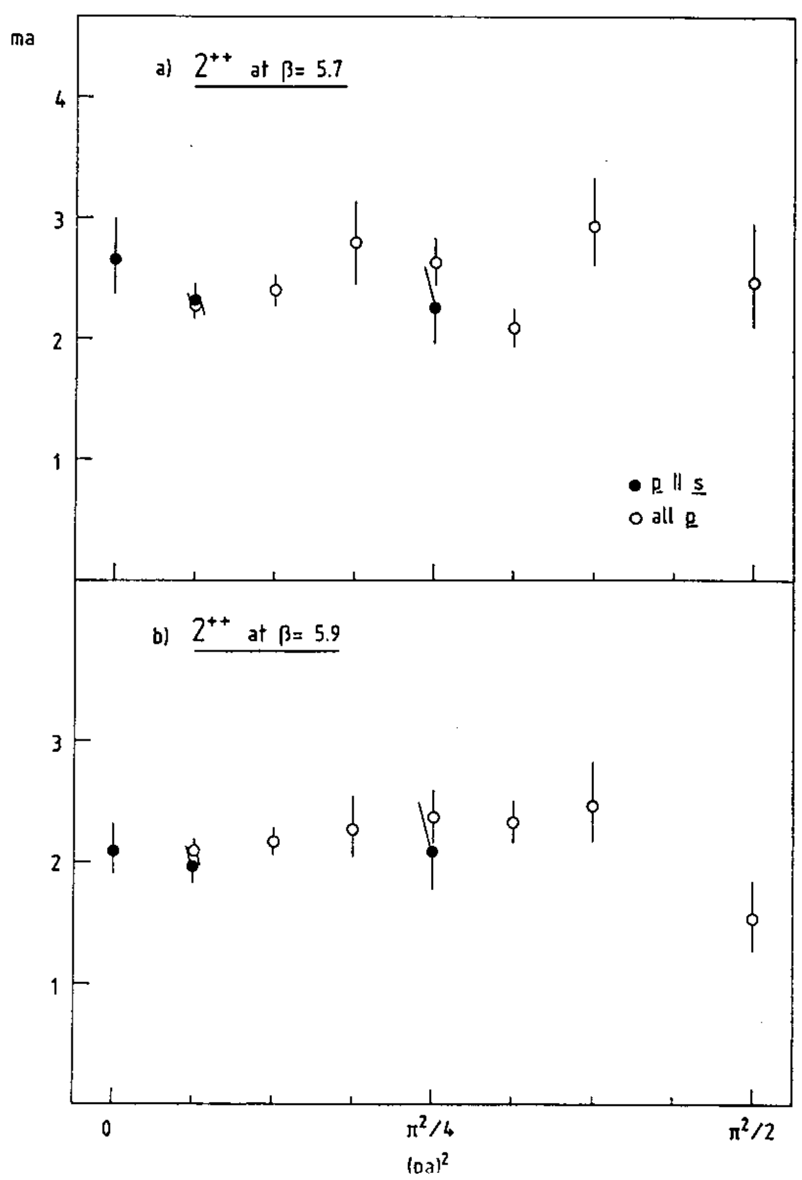


ma
a) 1-+ at β= 5.7
b) 1-+ at β= 5.9
4
3
2
1
● p ∥ s
○ all p
0
π²/4
(pa)²
π²/2

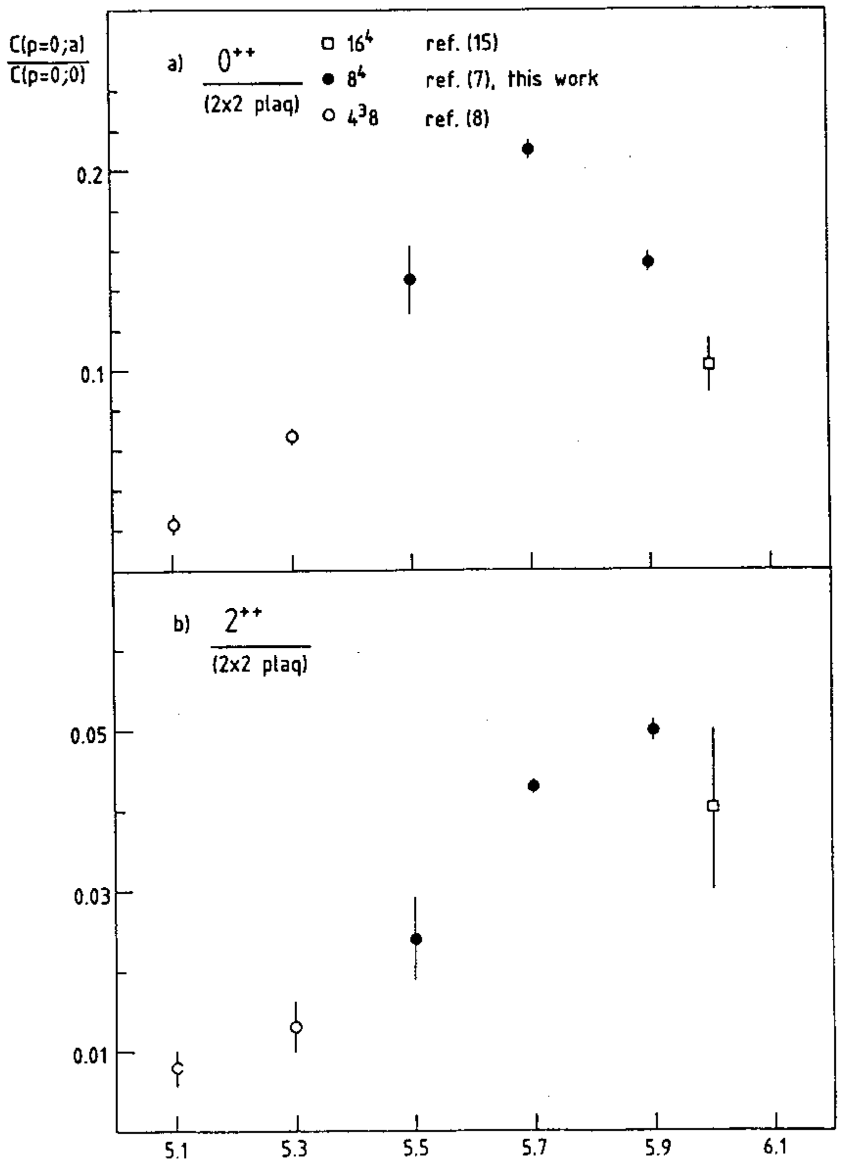
- Fig. 7 -


ma
2⁺⁺ mass from C(p;3a)/C(p;2a)
● p ‖ spin
○ all p
a) β= 5.7
b) β= 5.9
from
2a/a
from
2a/a
0
π/4
√2 π/4
p
0
π/4
√2 π/4
3
2
1

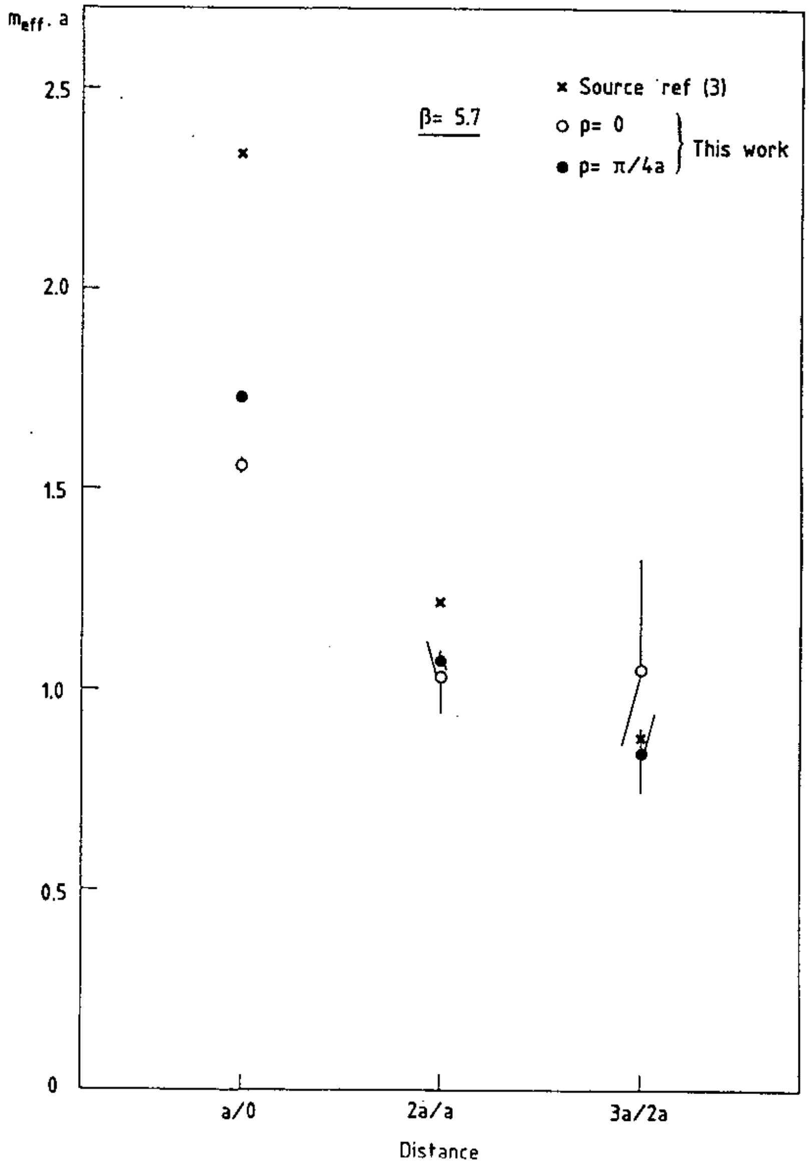
- Fig. 8 -


ma
a) 2⁺⁺ at β= 5.7
b) 2⁺⁺ at β= 5.9
● p ‖ s
○ all p
4
3
2
1
3
2
1
0
π²/4
π²/2
(pa)²

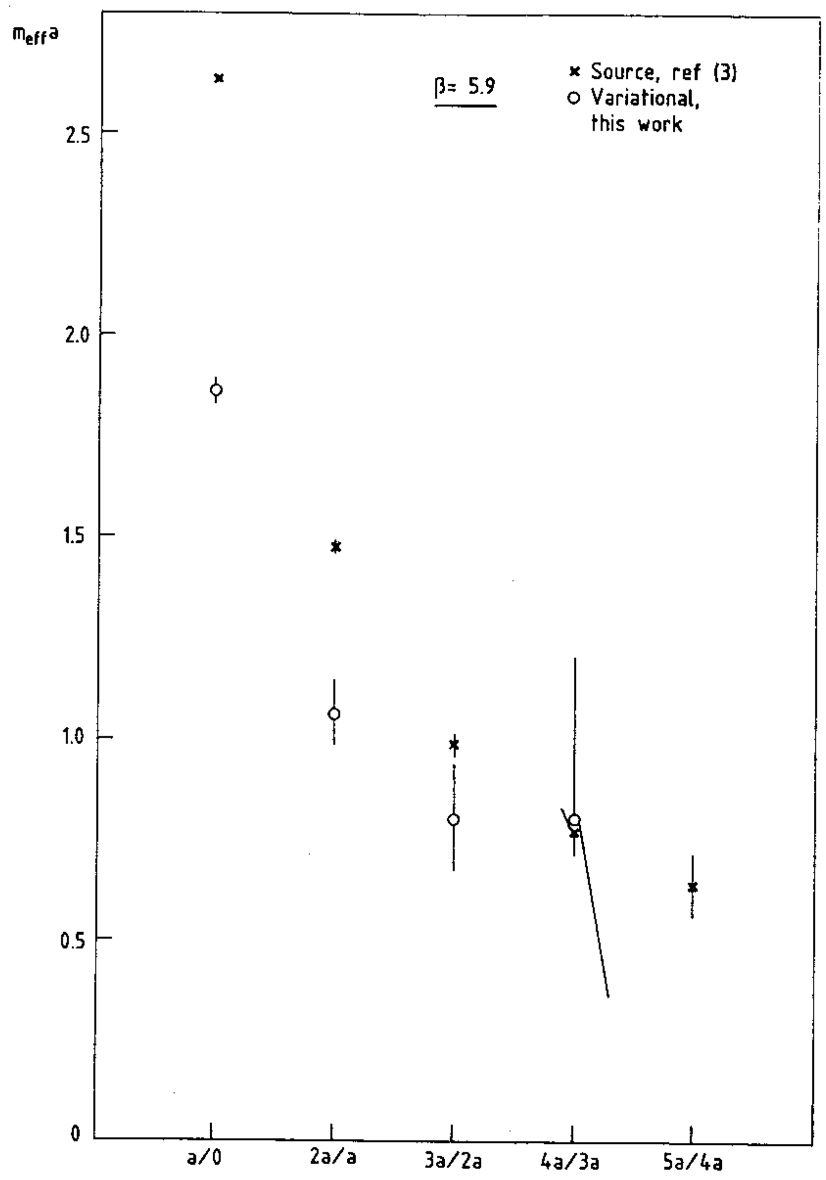
- Fig. 9 -


C(p=0;a) / C(p=0;0)
a) 0++ / (2x2 plaq)
b) 2++ / (2x2 plaq)
16⁴ ref. (15)
8⁴ ref. (7), this work
4³8 ref. (8)
0.2
0.1
0.05
0.03
0.01
5.1
5.3
5.5
5.7
5.9
6.1

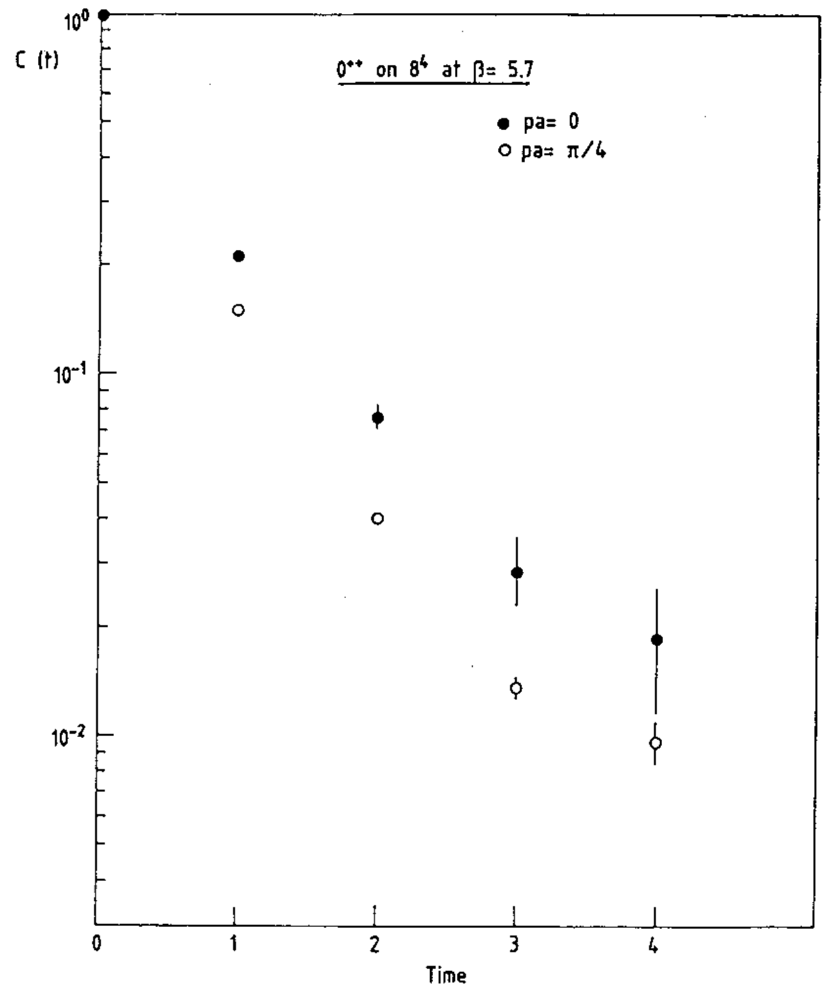
- Fig. 10 -


m_eff . a
β= 5.7
Source ref (3)
p= 0
p= π/4a
This work
2.5
2.0
1.5
1.0
0.5
0
a/0
2a/a
3a/2a
Distance

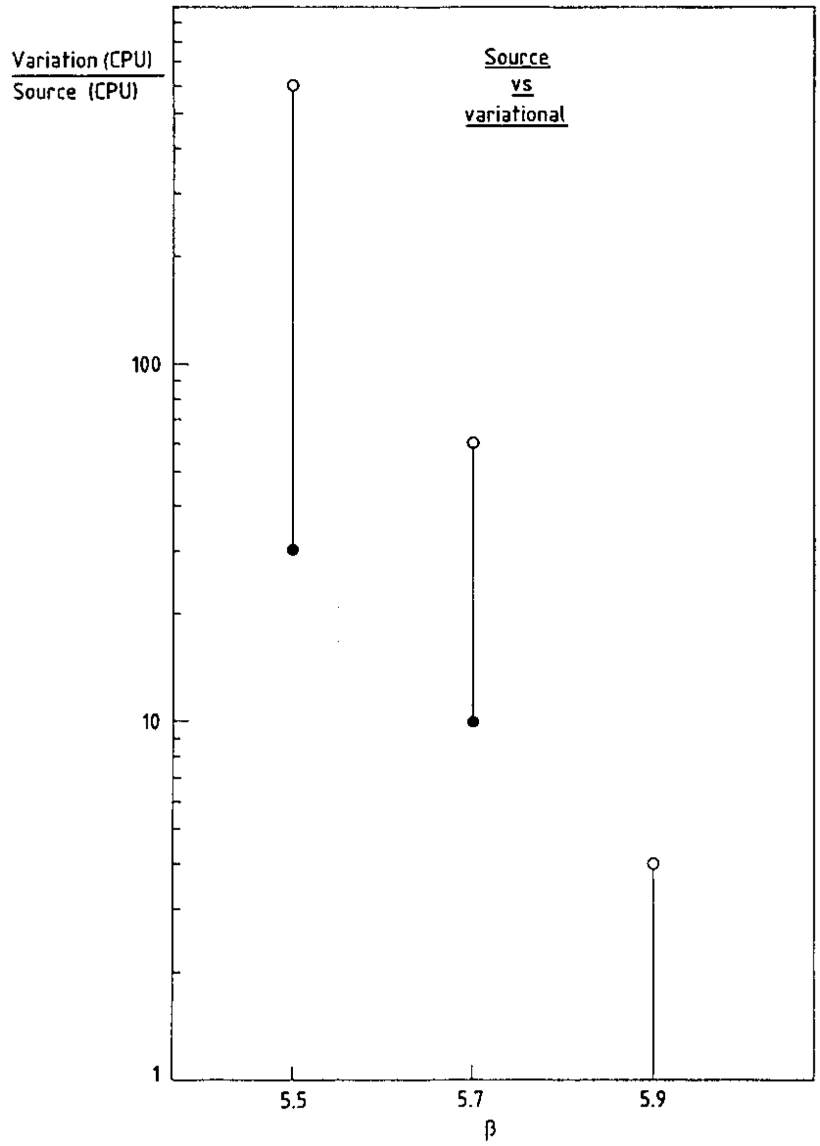
- Fig. 11 -


$m_{eff}a$
2.5
2.0
1.5
1.0
0.5
0
a/0
2a/a
3a/2a
4a/3a
5a/4a
β= 5.9
× Source, ref (3)
○ Variational, this work

– Fig. 12 –


C (t)
10^0
10^{-1}
10^{-2}
0^{++} on 8^4 at β= 5.7
● pa= 0
○ pa= π/4
0
1
2
3
4
Time

– Fig. 13 –


Variation (CPU)
Source (CPU)
Source
vs
variational
100
10
1
5.5
5.7
5.9
β

- Fig. 14 -