Ref.TH.3854-CERN

EXCEPTIONAL ULTRAVIOLET FINITE YANG-MILLS THEORIES

J.-P. Derendinger, S. Ferrara and A. Masiero
CERN - Geneva

A B S T R A C T

We show that a new class of ultraviolet finite N = 2 supersymmetric Yang-Mills theories exists in which some hypermultiplets belong to irreducible, pseudoreal representations of the gauge group G. Some of these theories contain fermions which are strictly massless before the spontaneous breaking of G. A classification of these theories and of those which admit at least three generations of quarks and leptons is given.

Ref.TH.3854-CERN
March 1984

It is known that N = 2 supersymmetric Yang-Mills theories[1] can be made ultraviolet finite provided the one-loop beta function vanishes[2]. It has been shown that ultraviolet finiteness even persists if soft breaking terms of a particular form are introduced[3]. This opens up the possibility of constructing realistic models for unification of electroweak and strong interactions based on a softly broken global N = 2 supersymmetry[4]. A classification of ultraviolet finite N = 2 Yang-Mills theories has already been given[5] under the assumption that the matter spin 1/2-spin 0 hypermultiplets belong to pair-conjugate representations $R + \bar{R}$ of the gauge group G[1-5].

Under this circumstance, the condition of ultraviolet finiteness is simply given by[2]

$$C_2(G) = \sum_i n_i \, T(R_i) , \tag{1}$$

where $C_2(G)$ is the quadratic Casimir of the adjoint representation of the gauge groupe G, and

$$T(R) = C_2(R) \cdot \frac{\dim R}{\dim G} ; \tag{2}$$

T(R), multiplied by the rank of the group, is known as the second-order Dynkin index. It can be obtained from a realization of the generators T^A for the representation R with proper normalization through

$$T(R)\, \delta^{AB} = \mathrm{Tr}\,(T^A T^B) . \tag{3}$$

The sum in Eq. (1) goes over all hypermultiplet species, and n_i is the multiplicity of each hypermultiplet.

However, in deriving Eq. (1), it has not been taken into account that hypermultiplets of N = 2 supersymmetry can belong to irreducible representations of G, provided the representation is pseudoreal.

The reason why this can occur is that for pseudoreal representations of G, the hypermultiplet is PCT self-conjugate, and the doubling of states usually due to PCT-conjugation is not needed[6]. Consider, in fact, massless one-particle states given as follows:

$$\Omega^a\,,\quad Q_i^+\Omega^a\,,\quad Q_i^+Q_j^+\Omega^a\,;\quad i=1,2 \tag{4}$$

in which the Clifford vacuum has +1/2 helicity, and is defined by

$$Q_i\,\Omega^a = 0\,,$$

and a is a gauge group index. The four states given by Eq. (4) and their antiparticles correspond in general to fields

$$\psi_\alpha^a\,,\quad A_i^a\,,\quad \chi^{a\dot\alpha}\,, \tag{5}$$

in which Ψ and χ are Weyl spinors of opposite handedness and A_i^α are two scalar fields. However, if the representation of the gauge group G is pseudoreal, the four states in Eq. (4) are PCT self-conjugate and we can impose the conditions[7]

$$\begin{aligned} (A_i^a)^* &= \varepsilon^{ij} C_{ab} A_j^b\,,\\ \chi^{a\dot\alpha} &= \varepsilon^{\dot\alpha\dot\beta} C^{ab} (\psi_\beta^b)^*\,, \end{aligned} \tag{6}$$

where C_{ab} is the skew-symmetric metric of the pseudoreal representation of G.

Therefore for pseudoreal representations R_p, the hypermultiplet contains only 4 . dim R_p degrees of freedom instead of 8 . dim R for non-pseudoreal R of G.

Self-conjugate irreducible hypermultiplets appear in N = 8 supergravity if this theory is analysed in terms of N = 2 representations[8]. The fields of N = 8 supergravity transform in this case under SU(6) x SU(2). The 70 scalar fields belong to $(\underset{\sim}{15} + \overline{\underset{\sim}{15}}, \underset{\sim}{1}) + (\underset{\sim}{20}, \underset{\sim}{2})$, corresponding to 15 complex scalar fields belonging to 15 N = 2 vector multiplets, and 20 complex scalar fields belonging to a self-conjugate N = 2 irreducible hypermultiplet where Eq. (6) reads

$$\left(A_i^{[a_1 a_2 a_3]}\right)^* = \varepsilon^{ij}\varepsilon_{[a_1 \dots a_6]} A_j^{[a_4 a_5 a_6]} \quad ; \qquad (7)$$

$\varepsilon_{[a_1 \dots a_6]}$ is the invariant tensor of SU(6), and the threefold antisymmetric tensor $\underset{\sim}{20}$ of SU(6) is pseudoreal.

If we come back to the finiteness condition, it is clear that if a group G admits pseudoreal representations R_{p_i}, then Eq. (1) has to be modified according to

$$C_2(G) = \sum_i n_i T(R_i) + \frac{1}{2}\sum_j n_{P_j} T(R_{P_j}) . \qquad (8)$$

Our task is to give a complete list of solutions to Eq. (8) with some non-zero n_{p_i}.

It is obvious that cases where all n_{p_i} are even are particular solutions of Eq. (1). New theories will be obtained when some n_{p_i} will be odd.

Before discussing the solutions of Eq. (8) with non-zero n_{p_i}, we have first to recall the precise definition of a pseudoreal representation.

A representation of the Lie algebra of a group G is called pseudoreal (or symplectic) if the generators T^A for this representation satisfy the property

$$C T^{A*} C = T^A , \tag{9}$$

with C real, skew-symmetric, and verifying $C^2 = -1$. The simplest example is the $\underset{\sim}{2}$ of SU(2) for which the generators are the Pauli matrices and $C = i\sigma_2$. All pseudoreal representations are even-dimensional. The simple (compact) Lie groups which possess pseudoreal representations are[9-10] SU(4n+2), Sp(2n), SO(n) with n = 3, 4, 5 modulo 8, E7. In order to satisfy Eq. (8), we have first to enumerate all pseudoreal representations R_p of these groups for which

$$T(R_p) \leq 2 C_2(G) . \tag{10}$$

The result is given in Table 1. The invariant $T(R_p)$ has been calculated using the standard normalization for the generators of the lowest dimensional representation: for SU(N), $T(\underset{\sim}{N}) = 1/2$; for SO(N), $T(\underset{\sim}{N}) = 2$; and for Sp(2N), $T(\underset{\sim}{2N}) = 1$. It is then simple to calculate T(R) for all other representations using either

$$T(R_1 \times R_2) = \dim R_1 \cdot T(R_2) + (1 \leftrightarrow 2)$$
$$T(R_1 + R_2) = T(R_1) + T(R_2) \tag{11}$$

or the eigenvalues of a specific generator of the Cartan subalgebra. Tables of T(R) already exist in the literature[10-11].

The pseudoreal representations of SU(4n+2), satisfying Eq. (10), are only the (2n+1)-fold antisymmetric tensors of SU(2) [$\sim$SO(3) $\sim$ Sp(2)] ($\underset{\sim}{2}$) and of SU(6) ($\underset{\sim}{20}$). For orthogonal groups, only spinorial representations can be pseudoreal. The invariant T(R) for the fundamental spinor is

$$2^{n-2} \quad \text{for} \quad SO(2n+1),$$
$$2^{n-3} \quad \text{for} \quad SO(2n). \tag{12}$$

Then, since SO(3) $\sim$ SU(2), SO(4) $\sim$ SU(2) x SU(2), and SO(5) $\sim$ Sp(4) will be considered below, only SO(11), SO(12) and SO(13) have pseudoreal spinors with $T(R_p) \leq 2C_2(G)$.

The pseudoreal $\underset{\sim}{2N}$ representations of Sp(2N) obviously always satisfy Eq. (10). The other solutions are only the threefold antisymmetric traceless tensor of Sp(6) ($\underset{\sim}{14'}$) and of Sp(8) ($\underset{\sim}{48}$) and the $\underset{\sim}{16}$ of Sp(4) $\sim$ SO(5). This latter representation is an SO(5)-spinor obtained from

$$\underset{\sim}{4} \times \underset{\sim}{5} = \underset{\sim}{4} + \underset{\sim}{16}\,. \tag{13}$$

For exceptional groups, only the lowest dimensional representation $\underset{\sim}{56}$ of E7 fulfils condition (10).

We must now construct all finite theories, solving Eq. (8), with some pseudoreal matter multiplets. Table 2 shows all such solutions. In order to cancel the one-loop beta function, some of these theories contain, in addition to the pseudoreal part, hypermultiplets with fermions belonging to $R + \bar{R}$ of G (R can be real or complex). This is the case for all SU(6) models, except for the one containing $4(\underset{\sim}{20})$. Since SU(6) contains SU(5), one can look for the embedding of quark and lepton generations $\bar{\underset{\sim}{5}} + \underset{\sim}{10}$, with the mirror fermions $\underset{\sim}{5} + \overline{\underset{\sim}{10}}$. The SU(6) models containing at least three generations are

$$
\begin{aligned}
&3(\underset{\sim}{20}) + 3(\underset{\sim}{6} + \bar{\underset{\sim}{6}}) \\
&\qquad = 3(\underset{\sim}{10} + \bar{\underset{\sim}{5}} + \overline{\underset{\sim}{10}} + \underset{\sim}{5}) + 6(\underset{\sim}{1}), \\
&2(\underset{\sim}{20}) + (\underset{\sim}{15} + \overline{\underset{\sim}{15}}) + 2(\underset{\sim}{6} + \bar{\underset{\sim}{6}}) \\
&\qquad = 3(\underset{\sim}{10} + \bar{\underset{\sim}{5}} + \overline{\underset{\sim}{10}} + \underset{\sim}{5}) + 4(\underset{\sim}{1}), \\
&\underset{\sim}{20} + 2(\underset{\sim}{15} + \overline{\underset{\sim}{15}}) + (\underset{\sim}{6} + \bar{\underset{\sim}{6}}) \\
&\qquad = 3(\underset{\sim}{10} + \bar{\underset{\sim}{5}} + \overline{\underset{\sim}{10}} + \underset{\sim}{5}) + 2(\underset{\sim}{1}).
\end{aligned}
\tag{14}
$$

There is no place for additional Higgs multiplets.

The pseudoreal spinors of orthogonal groups which appear in Table 2 decompose under the SO(10) subgroup according to

$$
\begin{aligned}
&SO(11): \quad \underset{\sim}{32} &&= \underset{\sim}{16} + \overline{\underset{\sim}{16}}, \\
&SO(12): \quad \underset{\sim}{32}, \underset{\sim}{32}' &&= \underset{\sim}{16} + \overline{\underset{\sim}{16}}, \\
&SO(13): \quad \underset{\sim}{64} &&= 2(\underset{\sim}{16} + \overline{\underset{\sim}{16}}).
\end{aligned}
\tag{15}
$$

Each $\underset{\sim}{16} + \overline{\underset{\sim}{16}}$ contains one generation of quarks and leptons and the mirror states. Thus, finite theories based on SO(N) groups can accommodate up to five generations, or four generations if additional Higgs multiplets are required.

Models based on symplectic groups cannot be used for realistic models except if very peculiar embeddings of SU(3) x SU(2) x U(1) into a very large group are accepted.

The only exceptional model, with gauge group E7, contains six generations of quarks and leptons, with the mirror states. In fact all embeddings of SU(5) into E7 are equivalent and give

$$\underset{\sim}{56} = \underset{\sim}{10} + \overline{\underset{\sim}{10}} + 3(\underset{\sim}{5} + \overline{\underset{\sim}{5}}) + 6(\underset{\sim}{1}) \tag{16}$$

The pseudoreal representations have an interesting property: their quadratic invariant is antisymmetric. This means that a supersymmetric mass term is only allowed for two different matter multiplets. If a theory contains an odd number of some pseudoreal representation R_p, then automatically one multiplet R_p will remain massless as long as the gauge group remains unbroken. Since soft terms breaking supersymmetry while preserving finiteness are not able to give masses to fermions in a matter multiplet, finite theories with pseudoreal representations can give rise to massless fermions. Some interesting cases can be found in Table 2. Consider, for instance, SO(13) with matter multiplets transforming according to $\underset{\sim}{64}$ + $7(\underset{\sim}{13}+\underset{\sim}{13})$. The $\underset{\sim}{64}$ will remain massless, giving rise to two massless generations of quarks and leptons with mirror fermions, as is clear from the embedding of SO(10) into SO(13) [Eq. (15)].

The same result will hold for SO(12) with matter multiplets

$$3(\underset{\sim}{32}) + \underset{\sim}{32}' + 2(\underset{\sim}{12} + \underset{\sim}{12}),$$

the two massless generations being embedded in $\underset{\sim}{32} + \underset{\sim}{32}'$. It is however impossible to find more than two massless generations (with mirror fermions) without enforcing the absence of an allowed mass term. Another interesting case is Sp(4) with matter multiplets $\underset{\sim}{16}$. This finite model has no mass scale at all. The presence of pseudoreal matter multiplets appears then as a generalization to finite N = 2 Yang-Mills theories of the so-called "survival hypothesis" of unified theories.

It is worth mentioning that pseudoreal representations R_p possess a symmetric coupling to the adjoint representation

$$(R_p \times R_p)_{SYM.} = \text{Adjoint} + \dots , \qquad (17)$$

which allows to construct the N = 2 coupling to the gauge N = 2 multiplet. Also we notice that hypermultiplets belonging to pseudoreal representations of G remain irreducible when restricted to N = 1 supersymmetry. In fact, being PCT self-conjugate, they have the same number of degrees of freedom as a N = 1 Wess-Zumino multiplet.

To summarize, in the present paper we have shown that an additional class of finite N = 2 Yang-Mills theories exists in which some hypermultiplets belong to PCT self-conjugate, irreducible representations of the gauge group. These representations allow naturally massless matter fields in N = 2 supersymmetry. We have given a complete classification of all finite N = 2 theories of this sort, including those which contain particle states with the quantum numbers of three generations of quarks and leptons. Finally, it should be stressed that our analysis is purely group-theoretical, and the construction of any realistic N = 2 SUSY GUT model is still an open problem, even after the inclusion of soft-breaking terms which preserve finiteness.

REFERENCES

1) P. Fayet, Nucl. Phys. B113 (1976) 135.
M. Sohnius, Nucl. Phys. B165 (1980) 483.

2) P. Howe, K. Stelle and P. West, Phys. Lett. 124B (1983) 55.
P. West, Proc. of La Jolla Conference, January 1983. To be published by APS.

3) A. Parkes and P. West, Phys. Lett. 127B (1983) 353.
P. West, Supersymmetry and finiteness, Lecture given at Shelter Island II, June 1983.

4) F. del Aguila, M. Dugan, B. Grinstein, L. Hall, G.G. Ross and P. West, Harvard preprint HUTP-84/A001 (1984).

5) I.G. Koh and S. Rajpoot, Phys. Lett. 135B (1984) 397.
Fang-Xiao Dong, Pei-You Xue, Tung-Sheng Tu and Xian-Jian Zhou, Beijing preprint BIHEP-TH-84-3 (1984).

6) S. Ferrara and C.A. Savoy, Supergravity '81, eds S. Ferrara and J.G. Taylor (Cambridge University Press, New York, p. 47).

7) B. de Wit, P.G. Lauwers, R. Philippe, S.Q. Su and A. van Proeyen, Phys. Lett. 134B (1984) 37.

8) J.P. Derendinger, S. Ferrara, A. Masiero and A. van Proeyen, CERN preprint TH-3813 (1984), to appear in Phys. Lett. B.

9) R. Slansky, Phys. Rep. 79 (1981) 1.

10) W.G. McKay and J. Patera, Tables of dimensions, indices, and branching rules for representations of simple Lie algebras. Lecture notes in Pure and Applied Mathematics (Marcel Dekker, Inc., New York and Basel, 1981), Vol. 69.

11) M.T. Vaughn, Z. Phys. C2 (1979) 111.

TABLE 1

All pseudoreal representations with $T(R_p) < 2C_2(G)$

G	$C_2(G)$	R_p	$T(R_p)$
SU(2)	2	**2**	1/2
SU(6)	6	**20**: $\varphi_{[ijk]}$	3
SO(11)	18	**32**: spinor	8
SO(12)	20	**32**: , **32'**: spinors	8
SO(13)	22	**64**: spinor	16
Sp(2n)	2(n+1)	**2n**: fundamental	1
Sp(4)	6	**16**: (1,1)	12
Sp(6)	8	**14'**: (0,0,1)	5
Sp(8)	10	**48**:(0,0,1,0)	14
E7	18	**56**	6

TABLE 2

Finite N = 2 theories with pseudoreal matter multiplets

G	Pseudoreal part	Real Part
SU(2)	8 (2)	
SU(6)	n(20)	+3(4-n)(6+$\overline{6}$) n= 1, 2, 3, 4
	2(20)	+(15+$\overline{15}$) + 2(6+$\overline{6}$)
	20	+2(15+$\overline{15}$) + (6+$\overline{6}$)
	20	+(15+$\overline{15}$) + 5(6+$\overline{6}$)
	20	+(21 + $\overline{21}$) + (6+$\overline{6}$)
SO(12)	n("32")	+2(5-n)(12+12) n = 1, 2, 3, 4, 5
	"32" means: 32 or 32'	
SO(11)	n(32)	+(9-2n)(11+11) n = 1, 2, 3, 4
SO(13)	2(64)	+3(13+13)
	64	+7(13+13)
Sp(2N)	4(N+1)2N	$\left.\right\}$ N > 2
Sp(2N)	8(2N)	+([2]$_{2N}$ + [2]$_{2N}$)
Sp(4)	16	
	4(4)	+2(5+5)
Sp(6)	14' + 3(6)	+(14+14)
	n(14') + (16-3n)(6)	n = 1, 2, 3
Sp(8)	48 + 6(8)	
E7	6(56)	