Published for SISSA by Springer

Received: November 8, 2012
Accepted: December 14, 2012
Published: February 11, 2013

d-geometries revisited

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Abstract: We analyze some properties of the four dimensional supergravity theories which originate from five dimensions upon reduction. They generalize to $N > 2$ extended supersymmetries the d-geometries with cubic prepotentials, familiar from N=2 special Kähler geometry. We emphasize the role of a suitable parametrization of the scalar fields and the corresponding triangular symplectic basis. We also consider applications to the first order flow equations for non-BPS extremal black holes.

Keywords: Supersymmetry and Duality, Black Holes in String Theory, Supergravity Models

ArXiv ePrint: 1210.5983v2

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doi:10.1007/JHEP02(2013)059

Contents

1 Introduction

The allowed scalar manifolds for the $N = 2$ five-dimensional supergravity coupled to $n_V - 1$ Abelian vector multiplets, parametrized by scalar fields φ^x ($x = 1, \ldots, n_V - 1$), can be described as the $(n_V - 1)$-dimensional cubic hypersurface $\frac{1}{3!} d_{ijk} \hat{\lambda}^i \hat{\lambda}^j \hat{\lambda}^k = 1$ of an ambient space spanned by n_V coordinates $\hat{\lambda}^i = \hat{\lambda}^i(\varphi^x)$ ($i = 1, \ldots, n_V$) [1]. The cubic nature of this polynomial constraint is related to the presence of the Chern-Simons term $d_{ijk} F^i F^j A^k$ in the Lagrangian for the n_V vector fields A^i_μ ($\mu = 0, 1, 2, 3, 4$), with n_V denoting the total number of $D = 5$ vector potentials (including the $D = 5$ graviphoton). A complete classification of the allowed homogeneous scalar manifolds was given in [2, 3], and many interesting properties, especially when they are restricted to be a symmetric coset of the Jordan family, were already analyzed in [1]. When this theory is dimensionally reduced to four dimensions, it yields a particular class of $N = 2$ four-dimensional matter coupled models with special Kähler target space geometries, which were studied in [3] under the name "d-spaces". There, the uplift between four and five dimensions was called "r-map", since it associates real scalars to the $N = 2$ four dimensional complex scalar fields belonging to the n_V $D = 4$ vector multiplets: $z^i = X^i / X^0 = a^i - i\ \lambda^i$, with a^i, λ^i real and with the index 0 pertaining to the $D = 4$ graviphoton. The axions a^i originate by Kaluza-Klein (KK) reduction from the vector components A^i_4, and the $\lambda^i = \hat{\lambda}^i e^{2\phi}$ are n_V real scalars parametrizing the $D = 5$ scalars ϕ^x and the KK scalar $\phi = g_{44}$. In this sense, the r-map is similar to the c-map, relating the moduli spaces of special Kähler vector multiplets to the quaternionic hypermultiplets scalar manifolds in $N = 2$ theories [3, 4]. In superstring theories, the c-map relates IIA and IIB string theories compactified on the same $(2, 2)$ superconformal field theory at $c = 9$, while in a purely supergravity context, it can simply be viewed as a consequence of dimensional reduction from 4 to 3 dimensions [4]. Actually, these $N = 2$ matter coupled theories, where the holomorphic prepotential takes the cubic form

$$F(X) \equiv \frac{1}{3!} d_{ijk} \frac{X^i X^j X^k}{X^0}\,, \tag{1.1}$$

were first studied in [5], where they were shown to lead to supergravity couplings with flat potentials characterized by the completely symmetric rank-3 tensor d_{ijk}. They are particularly relevant in connection with the large volume limit of Calabi-Yau compactifications of type IIA superstrings where the d-tensors are related to intersection forms of the Calabi-Yau manifold.

Formally, the d-tensor appears in the expression for the curvature of any special Kähler manifold [6]

$$R_{i\bar{j}k\bar{l}} = -g_{i\bar{j}} g_{k\bar{l}} - g_{i\bar{l}} g_{k\bar{j}} + C_{ikp} \overline{C}_{\bar{j}\bar{l}\bar{p}} g^{p\bar{p}} \tag{1.2}$$

since in "special coordinates" the covariantly holomorphic quantity C_{ijk} is given by $C_{ijk} = e^{K(z,\bar{z})} d_{ijk}$, with $K(z, \bar{z})$ denoting the Kähler potential.

Notice that a generic d-geometry of complex dimension n_V is not necessarily a coset space, but nevertheless it admits $n_V + 1$ real isometries, corresponding to Peccei-Quinn shifts of the n_V axions, and to an overall rescaling of the prepotential [3].

This paper aims to study d-geometries in a framework broader than $N = 2$, considering the r-map for $N \geq 2$ extended supergravities along the lines of previous work on this 4D/5D relation in the context of black hole supergravity solutions and their attractors [7–9]. Due to the structure of 5D spinors, these generalized d-geometries encompass all extended supergravities with a number of supercharges multiple of 8, and thus an even number of supersymmetries $N = 2, 4, 6, 8$.

d_{ijk} is an invariant tensor of the underlying classical duality group G_5 of the $D = 5$ action [10], corresponding to the continuous version of the non-perturbative string symmetries $G_5(\mathbb{Z})$ of [11]. The dimensional reduction yields interesting relations between the scalar manifolds and the isometries of the 5D and 4D theories: G_5 is embedded into the $D = 4$ electric-magnetic duality group G_4, whose isometries are included in $\mathrm{Sp}(2n_V + 2, \mathbb{R})$ (for generic $N > 1$, one has $\mathrm{Sp}(2n, \mathbb{R})$ for a theory with n vector potentials; for $N = 2$, $n = n_V + 1$). More precisely, one always has the chain of embeddings

$$G_5 \times \mathrm{SO}(1,1) \subset G_4 \subset \mathrm{Sp}(2n_V + 2, \mathbb{R}). \tag{1.3}$$

Our main point is that the five-dimensional origin of all generalized d-geometries naturally selects a particular branching of the $D = 4$ scalars, given by the axions a^I, the Kaluza-Klein scalar ϕ and the 5D scalars λ^x:

$$\Phi = \left\{a^I, \phi,\ \lambda^x\right\}. \tag{1.4}$$

When $N > 2$ these latter transform in a suitable representation of H_5, the maximal compact subgroup of G_5, which depends on N: for instance, in $N = 8$ there are 42 of them, sitting in the rank-4 antisymmetric skew-traceless representation **42** of USp(8), and there are 27 axions.

Remarkably, only in $N = 2$ the number of axions exactly matches the number of scalars plus 1, so that the two sets can be combined to give complex scalars. For this case we will use a small index i rather than I, to emphasize its complex nature. We will illustrate that the a^I and ϕ give rise to a universal sector which is present in any $N = 2, 4, 6, 8$ -extended supergravity in $D = 4$ endowed with generalized d-geometry for the vector multiplet sigma model.

In the study and classification of BPS and non-BPS extremal black hole supergravity solutions, the relation between 4D and 5D for cubic holomorphic prepotentials $F(X)$ (1.1) was used in [7] to relate the two $N = 2$ effective black hole potentials and to derive the 4D attractors and Bekenstein-Hawking classical entropies from the 5D ones. The key idea was to reformulate the 4D effective black hole potential in terms of 5D real special geometry data, implementing the natural splitting (1.4) of the 4D scalar fields.

Some extra features arise in *symmetric* special geometries, where the d-symbols satisfy the relation [1]

$$d_{r(pq}d_{ij)k}d^{rkl} = \frac{4}{3}\delta^l_{(p}d_{qij)}, \tag{1.5}$$

and one can define cubic , G_5-invariant, and quartic, G_4-invariant polynomials of electric (q_0, q_i) and magnetic charges (p^0, p^i) by [12]:

$$I_4\left(p^0, p^i, q_0, q_i\right) = -\left(p^0 q_0 + p^i q_i\right)^2 + 4\left[q_0 I_3\left(p\right) - p^0 I_3\left(q\right) + \frac{\partial I_3\left(q\right)}{\partial q_i}\frac{\partial I_3\left(p\right)}{\partial p^i}\right], \tag{1.6}$$

$$I_3(p) \equiv \frac{1}{3!}d_{ijk}p^i p^j p^k,\ \ I_3(q) \equiv \frac{1}{3!}d^{ijk}q_i q_j q_k\,. \tag{1.7}$$

The simplest example of rank-3 symmetric d-geometry is provided in $N = 2$ by the *stu* model [13, 14], with 3 complex scalar fields spanning the coset $(\mathrm{SU}(1,1)/\mathrm{U}(1))^3$, which serves as the ubiquitous toy model in the context of black holes arising from superstring and M-theory.

The generalization of $N = 2$ special geometry is achieved in terms of a generalized symplectic formalism, established in [15], which enlarges the rich geometric structure of special Kähler manifolds [3] to the other extended supergravities. In fact, an important difference between $N = 2$ and $N > 2$ extended theories is that for $N > 2$ the scalar sigma model is always given by a symmetric space G/H.

The formalism of [15] hinges on the definition of generalized sections $(\mathbf{f}, \mathbf{h})$ of a flat symplectic bundle [16], which relates to $N > 2$ the flat bundle underlying special Kähler geometry [17]. Even in $N = 2$ the sections are fundamental, since they allow to describe also theories where the holomorphic prepotential $F(X^\Lambda)$ does not exist [18, 19]. More precisely, the sections $V_A = (f^\Lambda_A, h_{\Lambda A})$, with $\Lambda = 0, \ldots, n_V$ and $A = 0, a$, are square complex matrices defined in $N = 2$ supergravity by

$$(\mathbf{f}, \mathbf{h}) = (L^\Lambda, \overline{D}_{\bar{a}} \overline{L}^\Lambda; M_\Lambda, \overline{D}_{\bar{a}} \overline{M}_\Lambda) \,, \tag{1.8}$$

with $(L^\Lambda, M_\Lambda) = e^{K/2}(X^\Lambda, F_\Lambda)$, D_a denoting the flat covariant derivative in the scalar manifold: $D_a = e^a_i D_i$, $g_{i\bar{\jmath}} = e^a_i e^b_{\bar{\jmath}} \delta_{ab}$ and $D_i = \partial_i + \frac{1}{2}\partial_i K$. They satisfy

$$h_{\Lambda A} = \mathcal{N}_{\Lambda\Sigma} f^\Sigma{}_A \tag{1.9}$$

where $\mathcal{N}_{\Lambda\Sigma}(z)$ is the 4D complex vector kinetic matrix. The sections encode a generic element $\mathbf{L}$ of the flat $\mathrm{Sp}(2n_V + 2, \mathbb{R})$-bundle over the $D = 4$ scalar manifold as [15]

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \longrightarrow \begin{pmatrix} \mathbf{f} \\ \mathbf{h} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} A & -\mathrm{i}B \\ C & -\mathrm{i}D \end{pmatrix} , \tag{1.10}$$

or the inverse transformation

$$\mathbf{L} \equiv \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \sqrt{2} \begin{pmatrix} \mathrm{Re}\,\mathbf{f} & -\mathrm{Im}\,\mathbf{f} \\ \mathrm{Re}\,\mathbf{h} & -\mathrm{Im}\,\mathbf{h} \end{pmatrix} , \tag{1.11}$$

with the symplectic property $\mathbf{L}^T \Omega \mathbf{L} = \Omega = \left(\begin{smallmatrix} 0 & -\mathbf{1} \\ \mathbf{1} & 0 \end{smallmatrix}\right)$ yielding the conditions

$$i(\mathbf{f}^\dagger \mathbf{h} - \mathbf{h}^\dagger \mathbf{f}) = \mathbb{1} \,, \qquad \mathbf{f}^T \mathbf{h} - \mathbf{h}^T \mathbf{f} = 0 \,. \tag{1.12}$$

This paper studies in detail the properties of a certain parametrization (2.2), (2.13) of four-dimensional generalized d-geometries, which reflects their five-dimensional origin, yielding a lower-triangular structure (2.13) for the matrix $\mathbf{L}$ characterizing the flat symplectic bundle sigma model which generalizes the one of $N = 2$ special Kähler d-geometry to any for any $N = 2, 4, 6, 8$. This parametrization exploits nilpotent (of degree 4) translations [17, 20, 21] parametrized by axion scalars a^I, and it acts on the same space where the d-tensor is defined. The sigma model is parametrized by additional block diagonal elements in the matrix $\mathbf{L}$, one of them being a dilatation in terms of the KK radius ϕ, and by a symmetric matrix, which depends on the $5D$ data and is related to the kinetic term of the $5D$ vector fields.

It should be stressed that the proposed basis turns out to be different from the standard parametrization of $N = 2$ d-geometry (1.8), although it leads to the same $4D$ vector kinetic matrix. We will emphasize that the two symplectic frames are in fact related by a unitary transformation M that was introduced in [9], which only depends on the $5D$ data. The unitary transformation M, that rotates the usual $N = 2$ complex basis of special geometry into the basis where $\mathbf{f}$ is real and $\mathbf{L}$ is lower triangular, allows to make a precise connection with the $N = 2$ *stu* model, viewed as a sub sector of the full $N = 8$ theory [15, 22, 23]. In the t^3 model, this unitary transformation is numerical (cfr. appendix B), because the relevant $5D$ uplifted theory is the pure $N = 2$, $D = 5$ supergravity.

Symmetric d-geometries can be related to Euclidean *Jordan algebras* of rank 3 [1, 24], which were classified in [25]; in this case, the nilpotent axionic translations fit into a Jordan algebra irreducible representation. The reduction to $D = 4$ yields a *Freudenthal triple system* (see e.g. [12]).

Our results have interesting applications to non-BPS extremal black holes, that we illustrate by making a precise and non trivial comparison between the methods of [22] and [26] in the computation of the *fake superpotential* [27] for non-BPS solutions and (p^0, q_0) charge configuration in the stu-truncation of $N = 8$ supergravity.

Beyond their interest in relation to supergravity structure and solutions, one may hope that these general properties of $N \geq 2$ d-geometries and the corresponding triangular symplectic frame (with degree-4 nilpotent axionic translations) could play a role in understanding the symmetry structure of supergravity counterterms, in order to clarify the issue of ultraviolet finiteness of $N = 8$ and other extended supergravity theories in $D = 4$ space-time dimensions [28].

The paper starts in section 2 with the universal decomposition for the $D = 4$ symplectic element $\mathbf{L}$ in the proposed basis 1.4, where axion are singled out. Then, the relation between $\mathbf{L}$ and the matrix $\mathcal{M}$ entering the black hole effective potential is elucidated in section 3. Other geometrical identities in a 5-dimensionally covariant formalism are presented in section 4. The simpler case of $N = 4$, $D = 4$ pure supergravity (with no matter coupling) is discussed in section 5. For d-geometries based on symmetric spaces G/H, the computation of the *Vielbein* and of the H-connection is carried out in section 6, in particular focusing on $N = 8$ supergravity. Next, in section 7 the $N = 2$ axion basis is related to the reformulation of special Kähler geometry as flatness condition of a symplectic connection [17].

A detailed treatment of $N = 2$ d-geometries is then given in section 8, where we elaborate on the results of [9] on the unitary matrix M rotating the axion basis to the usual special coordinates one. Geometrical identities for M and the related matrix $\widehat{M}$ are derived in section 9.

An application of the axion basis to the first order formalism for extremal black holes is considered in section 10. After a preliminary analysis for the *stu* model in sections 10.1.1 and 10.1.2 , explicit computations for the t^3 limit in the $\left(p^0, q_0\right)$ $(D0 - D6)$ charge configuration are performed in sections 10.1.3, and the known fake non-BPS superpotential is retrieved in section 10.2. In table 1 we list the allowed Rank-3 Euclidean Jordan algebras J_3 and corresponding symmetric generalized d-geometries, characterized by a parameter q related to the number of vector and scalar fields for each $N = 2, 4, 6, 8$.

Some appendices conclude the paper. In appendix A useful results on exponential matrices are collected, while appendix B contains some explicit computations in the t^3 model, displaying the matrix M. The purely imaginary nature of the *Vielbein* of the *stu* model and its consistent embedding into the $N = 8$ theory are discussed in appendix C. Finally, appendix D deals with the duality-invariant polynomial and the first order fake superpotential in the $D0 - D6$ configuration of the *stu* model with $i_3 = 0$.

2 Universal decomposition for the $D = 4$ symplectic element in the axion basis

We are interested in general features of all $D = 4$ Maxwell-Einstein (super)gravity theories admitting an uplift to $D = 5$. The classification of the tensors d_{IJK} associated to homogeneous Riemannian d-spaces was performed in [3]. For symmetric geometries, d_{IJK} can be characterized as the cubic norm of an associated rank-3 Jordan algebra[1] [1, 25]. In this case, the general properties are given in terms of a parameter q reported in table 1.

The number of $D = 5$ vectors is $n_V = 3q + 3$, while the number of $D = 4$ 2-form field strengths and their duals is $6q + 8$. Only in $N = 2$ theories, the number of 5D real scalars is $3q + 2$, while the number of 4D complex scalars is $3q + 3$ (one for each 4D Abelian vector multiplet). Quite generally, the relation between the number of vector and scalar fields in theories derived from five dimensions is such that

$$\begin{aligned} \#\ 4D \text{ scalars } &= \#\ 5D \text{ scalars} + \#\ 5D \text{ vectors} + 1 \\ \#\ 4D \text{ vectors } &= \#\ 5D \text{ vectors} + 1 = n_V + 1\,, \end{aligned} \tag{2.1}$$

where the n_V axions arise from the total number of $5D$ vectors.

We will show that in these generalized d-geometries, the representation of the $D = 4$ axions a^I is nilpotent of degree four and that, together with the Kaluza-Klein $\mathrm{SO}(1,1)$ radius parametrized by the real scalar ϕ, it provides a universal sector of the scalar manifold of the $D = 4$ theory, regardless of its specific geometry. This reflects the property of special Kähler d-geometries [3], of always having as minimal isometry of the scalar manifold the n_V axionic Peccei-Quinn translations and the $\mathrm{SO}(1,1)$ overall rescaling.

To prove the above statement, we split the symplectic element $\mathbf{L}$ according to the decomposition of the $D = 4$ scalars (1.4), and we demonstrate that[2]

$$\mathbf{L}\left(a^I, \phi, E\left(\lambda\right)\right) = \mathcal{A}(a^I)\mathcal{D}(\phi)\mathcal{G}(E)\,. \tag{2.2}$$

In order to identify the various factors in (2.2), one must consider the definition (1.11) and complement it with the results of [9], where the 4D/5D connection was used for $N = 8$

[1]With the exception of the *non-Jordan symmetric sequence* [29] of $N = 2$, $D = 5$ vector multiplets' scalar manifolds $\frac{\mathrm{SO}(1,n_V)}{\mathrm{SO}(n_V)}$.

[2]In the following we will switch the axion index from i into I, whenever our analysis holds for generic $N \geqslant 2$ d-geometries.

to determine the 28×28 symplectic sections $(f^{\Lambda}_{A}, h_{\Lambda A})$ in a five-dimensionally covariant symplectic frame, where the indices split as $\Lambda = (0, I)$ and $A = (0, a)$. They take the form:

$$f^{\Lambda}{}_{A} = \frac{1}{\sqrt{2}} \left(\begin{array}{c|c} e^{-3\phi} & 0 \\ \hline e^{-3\phi} a^I & e^{-\phi} (a^{-1/2})^I{}_a \end{array} \right) ; \tag{2.3}$$

$$h_{\Lambda\, A} = \frac{1}{\sqrt{2}} \left(\begin{array}{c|c} -e^{-3\phi} \frac{d}{6} - i e^{3\phi} & -\frac{1}{2} e^{-\phi} d_K (a^{-1/2})^K{}_a + i e^{\phi} a^K (a^{1/2})_K{}^a \\ \hline \frac{1}{2} e^{-3\phi} d_I & e^{-\phi} d_{IJ} (a^{-1/2})^J{}_a - i e^{\phi} (a^{1/2})_I{}^a \end{array} \right) , \tag{2.4}$$

with

$$d \equiv d_{IJK} a^I a^J a^k , \qquad d_I \equiv d_{IJK} a^J a^k , \qquad d_{IJ} \equiv d_{IJK} a^K , \tag{2.5}$$

and where

$$E(\lambda) \equiv (a^{-1/2})_a{}^J = E_a{}^J \tag{2.6}$$

is the coset representative of the 5D scalar manifold G_5/H_5. Notice that in this basis the section $\mathbf{f}$ is real and it takes a lower triangular form, and that the 5D scalars enter the sections only through $E(\lambda)$.

By generalizing this 5D/4D approach to the class of theories under consideration and interpreting the indices Λ, A on the appropriate representations, we determine the generic expression for each factor in (2.2).

The axionic generators

$$\mathcal{A}(a) \equiv e^{T(a)} , \tag{2.7}$$

also appeared in [30] in the context of gauging of flat groups in 4D supergravity, and they are given by the $2(n_V + 1) \times 2(n_V + 1)$ block-matrix

$$T(a) = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ a^J & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -a^I \\ 0 & d_{IJ} & 0 & 0 \end{array} \right) . \tag{2.8}$$

It is easily checked that $T(a)$ is nilpotent of order four:

$$T^4(a) = 0 \Rightarrow \mathcal{A}(a) = \mathbb{1} + T(a) + \frac{1}{2} T^2(a) + \frac{1}{3!} T^3(a), \tag{2.9}$$

which, by definition (2.7), yields

$$\mathcal{A}(a) = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ a^J & 1 & 0 & 0 \\ \hline -\frac{1}{6} d & -\frac{1}{2} d_I & 1 & -a^I \\ \frac{1}{2} d_J & d_{IJ} & 0 & 1 \end{array} \right) . \tag{2.10}$$

As we will discuss in section 8, this is in agreement with the $N = 2$ interpretation of [21]. The 1-dimensional Abelian SO(1, 1) factor in (2.2) is given by

$$\mathcal{D}(\phi) = \left(\begin{array}{cc|cc} e^{-3\phi} & 0 & 0 & 0 \\ 0 & e^{-\phi} & 0 & 0 \\ \hline 0 & 0 & e^{3\phi} & 0 \\ 0 & 0 & 0 & e^{\phi} \end{array} \right) , \tag{2.11}$$

JHEP02(2013)059

whereas the $(2n_V + 2) \times (2n_V + 2)$ matrix $\mathcal{G}$ is

$$\mathcal{G}(\lambda) = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & E & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & E^{-1} \end{array} \right) . \tag{2.12}$$

By matrix multiplication of (2.10)–(2.12) according to (2.2), one finds that the symplectic matrix $\mathbf{L}$ (1.11) acquires the triangular form:

$$\mathbf{L}(a^I, \phi, E(\lambda)) = \left(\begin{array}{cc|cc} e^{-3\phi} & 0 & 0 & 0 \\ a^I e^{-3\phi} & E_a{}^I e^{-\phi} & 0 & 0 \\ \hline -\frac{1}{6} d e^{-3\phi} & -\frac{1}{2} d_K E^K{}_a e^{-\phi} & e^{3\phi} & -a^K (E^{-1})^a{}_K e^{\phi} \\ \frac{1}{2} d_I e^{-3\phi} & d_{IK} E_a{}^K e^{-\phi} & 0 & (E^{-1})^a{}_I e^{\phi} \end{array} \right) . \tag{2.13}$$

We see that, in this particular basis, $B = \mathrm{Im}\,\mathbf{f} = 0$, since the $\mathbf{f}$ section is purely real:

$$\mathbf{f} = \mathrm{Re}\mathbf{f} = \frac{1}{\sqrt{2}} A(a^I, \phi, E(\lambda)). \tag{2.14}$$

On the other hand, one has

$$\mathbf{h} = \frac{1}{\sqrt{2}}(C - iD) \quad \Rightarrow \quad \begin{array}{l} \mathrm{Re}\mathbf{h} = \frac{1}{\sqrt{2}} C(a^I, \phi, E(\lambda), d_{IJK}) \\ \mathrm{Im}\mathbf{h} = -\frac{1}{\sqrt{2}} D(a^I, \phi, E(\lambda)), \end{array}$$

along with the normalization

$$\mathbf{f}^T \mathrm{Im}\mathbf{h} = \frac{1}{2}. \tag{2.15}$$

Notice that the C sub-block is the only one depending on d_{IJK}.

Conversely, one can say that the formula (2.13) for the symplectic representative yields an explicit expressions for the symplectic sections $\mathbf{f}$ and $\mathbf{h}$ which match eqs. (2.3) and (2.4).

To make the discussion concrete, let us consider $N = 8$ supergravity [31, 32], based on the rank-3 Euclidean Jordan algebra $J_3^{\mathbb{O}_s}$ over the split octonions; the $D = 5$ U-duality group is $G_5 = E_{6(6)}$ and d_{IJK} is the invariant tensor of the fundamental irrep. **27** ($I, J, K = 1, \ldots, 27 = n_V - 1$, $x = 1, \ldots, 42$, $i = 1, \ldots 70$). The $\mathrm{Sp}(56, \mathbb{R})$ matrix $\mathbf{L}$ (1.11) is the coset representative of the rank-7 symmetric $D = 4$ scalar manifold

$$\frac{G_4}{H_4} = \frac{E_{7(7)}}{\mathrm{SU}(8)}, \ \dim_{\mathbb{R}} = 70, \tag{2.16}$$

where H_4 is the maximal compact subgroup of $E_{7(7)}$. The 70 real $D = 4$ scalars z^i sit in the rank-4 self-dual antisymmetric irrep. **70** of $\mathrm{SU}(8)$.

The symplectic sections (2.3) and (2.4) are given in the particular symplectic frame defined by the *partial* decomposition of $\mathbf{L}$ (2.13) in a *solvable* basis, which is covariant with respect to $H_5 = \mathrm{USp}(8)$, the local symmetry of the $D = 5$ uplifted theory. Furthermore, $E(\lambda)$ is the coset representative of the rank-6 symmetric $D = 5$ scalar manifold

$$\frac{G_5}{H_5} = \frac{E_{6(6)}}{\mathrm{USp}(8)}, \ \dim_{\mathbb{R}} = 42. \tag{2.17}$$

JHEP02(2013)059

The 42 real $D=5$ scalars λ^x form the rank-4 self-dual antisymmetric skew-traceless irrep. **42** of USp(8). Note that (2.6) is consistent with the well known fact that the $N=8$, $D=5$ kinetic vector matrix $(a^{-1})_I{}^J$ is the square of the $D=5$ coset representative [16]. The scalar decomposition (1.4) in this case becomes

$$\begin{aligned} \mathrm{SU}(8) &\supset \mathrm{USp}(8); \\ \mathbf{70} &= \underset{\lambda^x}{\mathbf{42}} + \underset{a^I}{\mathbf{27}} + \underset{\phi}{\mathbf{1}}, \end{aligned} \tag{2.18}$$

where the axions a^I form a representation of $J_3^{\mathbb{O}_s}$, because

$$\begin{aligned} E_{6(6)} &\supset \mathrm{USp}(8); \\ \mathbf{27} &= \mathbf{27}. \end{aligned} \tag{2.19}$$

3 Relation between $\mathcal{M}$ and L

We now consider a further consequence of the symplectic structure of *generalized special geometry* [15], holding for every $D=4$ Maxwell-Einstein supergravity even beyond d-geometries. It can be useful in the present context and in view of applications to black holes. The black hole effective potential for dyonic charges $Q=(p^\Lambda,q_\Lambda)$ is given by [33]

$$V_{BH} = -\frac{1}{2}Q^t\mathcal{M}(\mathcal{N})Q =< Q, V_A >< Q, \overline{V}^A >= Z_A\overline{Z}^A \tag{3.1}$$

where the central charges $Z_A =< Q, V_A >$ are defined by the symplectic product

$$Z_A =< Q, V_A >= Q^T\Omega V_A = f^\Lambda{}_A q_\Lambda - h_{\Lambda\ A}p^\Lambda\,, \tag{3.2}$$

in terms of the symplectic invariant metric

$$\Omega = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}. \tag{3.3}$$

The matrix $\mathcal{M}$ is given by

$$\mathcal{M} = \begin{pmatrix} \mathbb{1} & -\mathrm{Re}\mathcal{N} \\ 0 & \mathbb{1} \end{pmatrix} \begin{pmatrix} \mathrm{Im}\mathcal{N} & 0 \\ 0 & (\mathrm{Im}\mathcal{N})^{-1} \end{pmatrix} \begin{pmatrix} \mathbb{1} & 0 \\ -\mathrm{Re}\mathcal{N} & \mathbb{1} \end{pmatrix} \equiv \mathcal{R}^T\mathcal{M}_D\mathcal{R}\,; \tag{3.4}$$

$$\mathcal{R} \equiv \begin{pmatrix} \mathbb{1} & 0 \\ -\mathrm{Re}\mathcal{N} & \mathbb{1} \end{pmatrix}; \tag{3.5}$$

$$\mathcal{M}_D \equiv \begin{pmatrix} \mathrm{Im}\mathcal{N} & 0 \\ 0 & (\mathrm{Im}\mathcal{N})^{-1} \end{pmatrix}, \tag{3.6}$$

where $\mathcal{N} = \mathbf{hf}^{-1}$ is the $D=4$ kinetic vector matrix.

In *generalized special geometry* [15] one introduces the $\mathrm{Sp}(2n_V+2)$ Hermitian matrix

$$\mathcal{C} \equiv \frac{1}{2}(\mathcal{M}+i\Omega)\,; \quad \mathcal{C}^\dagger = \mathcal{C}, \tag{3.7}$$

whose symmetric and antisymmetric parts are given by (3.4) and Ω respectively. $\mathcal{C}$ is related to the symplectic sections $(\mathbf{f}, \mathbf{h})$ by:

$$\mathcal{C} = \begin{pmatrix} -\mathbf{h}\mathbf{h}^\dagger & \mathbf{h}\mathbf{f}^\dagger \\ \mathbf{f}\mathbf{h}^\dagger & -\mathbf{f}\mathbf{f}^\dagger \end{pmatrix}, \tag{3.8}$$

and therefore its action on the vector V_A is given by

$$\frac{1}{2}(\mathcal{M} + i\Omega)V_A = i\Omega V_A \Leftrightarrow \mathcal{M}V_A = i\Omega V_A, \tag{3.9}$$

expressing a *twisted self-duality* [34], recently used in [35].

Using the above relations, since both $\mathcal{M}$ and $\mathbf{L}$ are given in terms of the sections $(\mathbf{f}, \mathbf{h})$, one can see that they can be related by [36, 37]

$$\mathcal{M} = -(\mathbf{L}^T)^{-1}\mathbf{L}^{-1} = -(\mathbf{L}\,\mathbf{L}^T)^{-1}; \tag{3.10}$$

$$\Updownarrow$$

$$\mathcal{M}\mathbf{L} = -(\mathbf{L}^T)^{-1} = \Omega\mathbf{L}\Omega, \tag{3.11}$$

where the last step in (3.11) follows from the symplecticity of $\mathbf{L}$ itself. Notice that, since also $\mathcal{M}$ is symplectic, (3.10) implies that $\mathcal{M} = -\tilde{\mathbf{L}}\tilde{\mathbf{L}}^T$, with $\tilde{\mathbf{L}} \equiv \Omega\mathbf{L}$.

To prove (3.10)–(3.11), one just notices that $\mathbf{L}$ (1.11) can be rewritten as (with $*$ here denoting complex conjugation)

$$\mathbf{L} = \frac{1}{\sqrt{2}}(\mathcal{B} + \mathcal{B}^*); \tag{3.12}$$

$$\mathcal{B} \equiv \begin{pmatrix} \mathbf{f} & i\mathbf{f} \\ \mathbf{h} & i\mathbf{h} \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ \mathbf{h} \end{pmatrix} (\mathbb{1}, i\mathbb{1}), \tag{3.13}$$

which, by (3.9) implies

$$\mathcal{M}\mathbf{L} = \mathcal{M}\frac{1}{\sqrt{2}}(\mathcal{B} + \mathcal{B}^*) = \frac{1}{\sqrt{2}} \begin{pmatrix} -i(\mathbf{h} - \mathbf{h}^*) & \mathbf{h} + \mathbf{h}^* \\ i(\mathbf{f} - \mathbf{f}^*) & -(\mathbf{f} + \mathbf{f}^*) \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \mathbf{L} \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} = \Omega\mathbf{L}\Omega \quad \blacksquare \tag{3.14}$$

By sandwiching (3.10) with the dyonic charge vector Q, one also obtains

$$V_{BH} = -\frac{1}{2}Q^t\mathcal{M}(\mathcal{N})Q = \frac{1}{2}\,(\mathbf{L}^{-1}Q)^T(\mathbf{L}^{-1}Q) = \frac{1}{2}\,Z^T \cdot Z \tag{3.15}$$

where the real central charge vector Z satisfies

$$Z = \mathbf{L}^{-1}Q\,, \tag{3.16}$$

JHEP02(2013)059

with the electric and magnetic real components of $Z = (Z^0_{(m)}, Z^a_{(m)}, Z^{(e)}_0, Z^{(e)}_a)^T$ given by universal formulae in terms of 5D axion and dilation fields

$$\begin{aligned}
Z^{(e)}_0 &= e^{-3\phi}(q_0 + q_I a^I + \frac{d}{2}p^0 - \frac{1}{2}p^I d_I)\,,\\
Z^{(e)}_I &= e^{-\phi}(q_I + \frac{1}{2}p^0 d_I - p^J d_{IJ})\,,\\
Z^0_{(m)} &= e^{3\phi}p^0\,,\\
Z^I_{(m)} &= e^{\phi}(p^I - p^0 a^I)\,,
\end{aligned} \tag{3.17}$$

which were derived in [9] for $N = 8$, but that we can here interpret as valid for all generalized d-geometries. The components with flat indices are obtained by

$$Z^{(e)}_a = Z^{(e)}_I (a^{-1/2})^I{}_a\,, \qquad Z^a_{(m)} = Z^I_{(m)}(a^{1/2})^a_I \tag{3.18}$$

so that the complex central charge vector with flat indices is

$$Z_A = \begin{pmatrix} Z_0 \\ Z_a \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} Z^{(e)}_0 + iZ^0_{(m)} \\ Z^{(e)}_a + iZ^a_{(m)} \end{pmatrix} \tag{3.19}$$

and the effective black hole potential is written as [9]

$$V_{BH} = |Z_0|^2 + Z_a \overline{Z}_a\,. \tag{3.20}$$

4 5D-covariant identities

In the 5D covariant formalism introduced in [9], it was found that the kinetic vector matrix $\mathcal{N}_{\Lambda\Sigma}$ in $N = 8$, $D = 4$ supergravity can be decomposed as:

$$\mathrm{Re}\mathcal{N} = \begin{pmatrix} \frac{d}{3} & -\frac{d_I}{2} \\ -\frac{d_J}{2} & d_{IJ} \end{pmatrix}\,, \quad \mathrm{Im}\mathcal{N} = \begin{pmatrix} -e^{6\phi} - e^{2\phi}a^I a^J a_{IJ} & a_{IJ}a^J \\ a_{IJ}a^I & -e^{2\phi}a_{IJ} \end{pmatrix}\,. \tag{4.1}$$

In virtue of the discussion of section 2, these formulae hold for any d-geometry. Note that $\mathrm{Im}\mathcal{N}$ depends on the axions a^I but not on d_{IJK}, whereas $\mathrm{Re}\mathcal{N}$ only depends on axions, and only through d_{IJK}. It is immediate to realize that this is a consequence of the solvable decomposition (2.2) of $\mathbf{L}$, as well as of the relation (3.10) between $\mathcal{M}$ and $\mathbf{L}$. Indeed, using (3.5), the matrix $\mathcal{A}$ (2.10) can be rewritten as

$$\mathcal{A} = \begin{pmatrix} \mathbb{1} & 0 \\ \mathrm{Re}\mathcal{N} & \mathbb{1} \end{pmatrix}\left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ a^I & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & -a^J \\ 0 & 0 & 0 & 1 \end{array}\right) \equiv (\mathcal{R})^{-1}\mathcal{A}_D(a^I)\,, \tag{4.2}$$

thus yielding

$$\mathbf{L} = (\mathcal{R})^{-1}\mathcal{A}_D \mathcal{D}G\,. \tag{4.3}$$

Then, since $\mathcal{D}G$ is a diagonal matrix, (3.10) implies

$$\mathcal{M} = -(\mathbf{L}^T)^{-1}\mathbf{L}^{-1} = -(\mathcal{R})^T\left[(\mathcal{A}^T_D)^{-1}(\mathcal{D}G)^{-1}(\mathcal{D}G)^{-1}\mathcal{A}^{-1}_D\right]\mathcal{R}\,. \tag{4.4}$$

Using (2.11), (2.12) and (4.2), one can check that

$$-(\mathcal{A}_D^T)^{-1}(\mathcal{D}G)^{-1}(\mathcal{D}G)^{-1}\mathcal{A}_D^{-1} = \begin{pmatrix} \mathrm{Im}\mathcal{N} & 0 \\ 0 & \mathrm{Im}\mathcal{N}^{-1} \end{pmatrix} . \tag{4.5}$$

As mentioned, this explains the dependence of Im$\mathcal{N}$ on axions alone and not on the d-tensor, and that of Re$\mathcal{N}$ on axions only through d_{IJK}.

5 A related case: $N = 4$, $D = 4$ pure supergravity

Although pure 4D $N = 4$ supergravity cannot be obtained from five dimensions by Kaluza-Klein reduction, which would always give rise to the coupling to matter multiplets, we mention it here because of the recent related work of [38] and as a simple instance of the splitting of scalar fields associated with (2.2). The vector kinetic matrix $\mathcal{N}_{\Lambda\Sigma}$ in this case reads [39] ($\Lambda, \Sigma = 1, \ldots, 6$)

$$\mathcal{N}_{\Lambda\Sigma} = -S\delta_{\Lambda\Sigma}, \tag{5.1}$$

where the axio-dilatonic complex scalar field S of the gravity multiplet, spanning the rank-1 symmetric coset $G/H = \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$, is defined as

$$S \equiv ie^{\phi} + a\,, \tag{5.2}$$

yielding

$$\mathrm{Re}\mathcal{N}_{\Lambda\Sigma} = -a\delta_{\Lambda\Sigma}\ , \qquad \mathrm{Im}\mathcal{N} = -e^{\phi}\delta_{\Lambda\Sigma}\,. \tag{5.3}$$

A solvable basis can be defined also for this theory as in (5.1), and it is given by the *axio-dilatonic symplectic frame* , where the relevant matrices read

$$\mathcal{M} = \begin{pmatrix} -e^{\phi} - a^2e^{-\phi} & -ae^{-\phi} \\ -a\,e^{-\phi} & -e^{-\phi} \end{pmatrix} ; \tag{5.4}$$

$$\mathbf{L} = \begin{pmatrix} 1 & 0 \\ -a & 0 \end{pmatrix}\begin{pmatrix} e^{-\phi/2} & 0 \\ 0 & e^{\phi/2} \end{pmatrix} = \begin{pmatrix} e^{-\phi/2} & 0 \\ -a\,e^{-\phi/2} & e^{\phi/2} \end{pmatrix}, \tag{5.5}$$

such that the coset representative $\mathbf{L}$ of $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ satisfies

$$\mathbf{L}^{-1}(a,\phi) = \mathbf{L}(-a,-\phi)\,. \tag{5.6}$$

In this case the axionic generator

$$\mathbf{A} \equiv \frac{\partial}{\partial a}\begin{pmatrix} 1 & 0 \\ -a & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \tag{5.7}$$

is nilpotent of order two rather than of order four, as for generic d-geometries:

$$\mathbf{A}^2 = 0. \tag{5.8}$$

The different degree of nilpotency is due to the fact that this theory does not admit a 5D uplift and thus it is not a d-geometry in absence of matter coupling.

6 *Vielbein* and *H*-connection in the axion basis

When the d-geometry is not only an homogeneous but a *symmetric* cosets G/H , the *Vielbein* P_μ and H-connection ω_μ in a solvable decomposition can be simply computed from the $(\mathfrak{g} \ominus \mathfrak{h})$-valued Maurer-Cartan 1-form $\mathbf{L}^{-1}d\mathbf{L}$ by standard methods

$$(\mathbf{L}^{-1}d\mathbf{L})_s = \frac{1}{2}\left(\mathbf{L}^{-1}d\mathbf{L} + (\mathbf{L}^{-1}d\mathbf{L})^T\right) = P_\mu\,; \tag{6.1}$$

$$(\mathbf{L}^{-1}d\mathbf{L})_a = \frac{1}{2}\left(\mathbf{L}^{-1}d\mathbf{L} - (\mathbf{L}^{-1}d\mathbf{L})^T\right) = \omega_\mu\,, \tag{6.2}$$

where subscripts "s" and "a" denote the symmetric and antisymmetric part, respectively.

The simplest example is provided by the axio-dilatonic coset $G/H = \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ treated above, whose coset representative is given by (5.5), with Maurer-Cartan 1-form

$$\mathbf{L}^{-1}d\mathbf{L} = \begin{pmatrix} -\frac{1}{2}d\phi & 0 \\ -e^{-\phi}da & \frac{1}{2}d\phi \end{pmatrix}, \tag{6.3}$$

leading to the *Vielbein* P_μ and U(1)-connection ω_μ respectively given by

$$P_\mu = \begin{pmatrix} -\frac{1}{2}d\phi & -\frac{1}{2}e^{-\phi}da \\ -\frac{1}{2}e^{-\phi}da & \frac{1}{2}d\phi \end{pmatrix}, \qquad \omega_\mu = \begin{pmatrix} 0 & \frac{1}{2}e^{-\phi}da \\ -\frac{1}{2}e^{-\phi}da & 0 \end{pmatrix}. \tag{6.4}$$

In particular, one sees that the U(1) connection ω_μ contains only the da differential. The kinetic term for the nonlinear σ-model $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ therefore reads [39]

$$\mathrm{Tr}\left(P^T P\right) = \frac{1}{2}\left(d\phi^2 + e^{-2\phi}da^2\right). \tag{6.5}$$

We now consider in particular $N = 8$ supergravity, where the Cartan decomposition for the $D = 4$ scalar manifold (2.16) reads

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{k}; \tag{6.6}$$

$$\mathfrak{g} = \mathfrak{e}_{7(7)};\ \mathfrak{h} = \mathfrak{su}(8);\ \mathfrak{k} = \mathbf{70} \text{ of } \mathfrak{su}(8). \tag{6.7}$$

According to (2.18)–(2.18), the following $\mathfrak{usp}(8)$-covariant branchings take place:

$$\mathfrak{k} : \mathbf{70} = \mathbf{1}_{\mathfrak{k}} + \mathbf{42}_{\mathfrak{k}} + \mathbf{27}_{\mathfrak{k}}; \tag{6.8}$$

$$\mathfrak{h} : \mathbf{63}_{\mathfrak{h}} = \mathbf{36}_{\mathfrak{h}} + \mathbf{27}_{\mathfrak{h}} \tag{6.9}$$

The coset *Vielbein* P_μ is given by the non-compact generators

$$\begin{aligned} \mathbf{1}_{\mathfrak{k}} &: \mathcal{D}^{-1}\partial\mathcal{D}; \\ \mathbf{42}_{\mathfrak{k}} &: [\mathcal{G}^{-1}\partial\mathcal{G}]_s; \\ \mathbf{27}_{\mathfrak{k}} &: \left[(\mathcal{D}\mathcal{G})^{-1}\partial T(a)(\mathcal{D}\mathcal{G})\right]_s, \end{aligned} \tag{6.10}$$

while the compact ones give the SU(8)-connection ω_μ

$$\begin{aligned} \mathbf{36}_{\mathfrak{h}} &\to [\mathcal{G}^{-1}\partial\mathcal{G}]_a; \\ \mathbf{27}_{\mathfrak{h}} &\to \left[(\mathcal{D}\mathcal{G})^{-1}\partial T(a)(\mathcal{D}\mathcal{G})\right]_a. \end{aligned} \tag{6.11}$$

The Maurer-Cartan 1-form gets generally decomposed as

$$\mathbf{L}^{-1}\partial\mathbf{L} = (\mathcal{D}\mathcal{G})^{-1}\partial T(a)(\mathcal{D}\mathcal{G}) + \mathcal{D}^{-1}\partial\mathcal{D} + \mathcal{G}^{-1}\partial\mathcal{G}\,. \tag{6.12}$$

From the definitions (2.10), (2.11) and (2.12), one can compute

$$\mathcal{D}^{-1}\partial\mathcal{D} = \left(\begin{array}{cc|cc} -3 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ \hline 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) d\phi \;= \left(\mathcal{D}^{-1}\partial\mathcal{D}\right)_s; \tag{6.13}$$

$$\mathcal{G}^{-1}\partial\mathcal{G} = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & E^{-1}dE & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -E^{-1}dE \end{array}\right); \tag{6.14}$$

$$(\mathcal{D}\mathcal{G})^{-1}\partial T(a)(\mathcal{D}\mathcal{G}) = e^{-2\phi}\left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ (a^{1/2})^a_I da^I & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -(a^{1/2})^b_I da^I \\ 0 & d_{IJK}(a^{-1/2})^J_a(a^{-1/2})^K_b da^I & 0 & 0 \end{array}\right). \tag{6.15}$$

This implies that the Maurer-Cartan 1-form $\mathbf{L}^{-1}\partial\mathbf{L}$ does not depend on the axions a^I explicitly, but only on their differential da^I.

According to (6.1) and (6.2), the *Vielbein* P_μ and SU(8)-connection ω_μ for the coset (2.16) are the symmetric and anti-symmetric part of (6.12), respectively. In particular, the component $\mathbf{27}_{\mathfrak{k}}$ of P_μ and the component $\mathbf{27}_{\mathfrak{h}}$ of ω_μ respectively read:

$$\mathbf{27}_{\mathfrak{k}} : \left[(\mathcal{D}\mathcal{G})^{-1}\partial T(a)(\mathcal{D}\mathcal{G})\right]_s =$$
$$= \frac{1}{2}e^{-2\phi}\left(\begin{array}{cc|cc} 0 & (a^{1/2})^b_I da^I & 0 & 0 \\ (a^{1/2})^a_I da^I & 0 & 0 & d_{IJK}(a^{-1/2})^J_a(a^{-1/2})^K_b da^I \\ \hline 0 & 0 & 0 & -(a^{1/2})^b_I da^I \\ 0 & d_{IJK}(a^{-1/2})^J_a(a^{-1/2})^K_b da^I & -(a^{1/2})^a_I da^I & 0 \end{array}\right); \tag{6.16}$$

$$\mathbf{27}_{\mathfrak{h}} : \left[(\mathcal{D}\mathcal{G})^{-1}\partial T(a)(\mathcal{D}\mathcal{G})\right]_a =$$
$$= \frac{1}{2}e^{-2\phi}\left(\begin{array}{cc|cc} 0 & -(a^{1/2})^b_I da^I & 0 & 0 \\ (a^{1/2})^a_I da^I & 0 & 0 & -d_{IJK}(a^{-1/2})^J_a(a^{-1/2})^K_b da^I \\ \hline 0 & 0 & 0 & -(a^{1/2})^b_I da^I \\ 0 & d_{IJK}(a^{-1/2})^J_a(a^{-1/2})^K_b da^I & (a^{1/2})^a_I da^I & 0 \end{array}\right). \tag{6.17}$$

7 Flat connections and axion basis

As shown in [17] and further investigated in [21], the defining identities of $N = 2$ special Kähler geometry can be viewed as the flatness condition of a non-holomorphic connection

$\mathcal{A}_I$ and can be encoded into a first-order matrix equation [21]

$$(\partial_i - \mathcal{A}_i)\,\mathbf{U} = 0\,, \tag{7.1}$$

where $\mathbf{U}$ is a non-holomorphic matrix $(V, D_iV, \overline{D}_{\bar{\imath}}\overline{V}, \overline{V})$ with $V = (X^\Lambda, F_\Lambda)$. One can further choose a gauge where $\mathcal{A}_i$ becomes holomorphic

$$\overline{\mathcal{A}}_i = 0 \Rightarrow \mathcal{A}_i = \mathtt{A}_i\ ,\ \bar{\partial}\mathtt{A}_i = 0, \tag{7.2}$$

such that (7.1) can be recast as follows:

$$(\partial_i - \mathtt{A}_i)\,\mathbf{V} = 0, \tag{7.3}$$

with now an holomorphic solution matrix $\mathbf{V}$ containing V in the first row. In turn, the holomorphic flat connection $\mathtt{A}_i$ can be decomposed as

$$\mathtt{A}_i = \mathbf{\Gamma}_i + \mathbf{C}_i, \tag{7.4}$$

where $\mathbf{\Gamma}_i$ is the diagonal part (which vanishes in *special coordinates*), and $\mathbf{C}_i$ generates an Abelian subalgebra of $\mathfrak{sp}(2n+2,\mathbb{R})$ that is nilpotent of order four:

$$\mathbf{C}_i\mathbf{C}_j\mathbf{C}_k\mathbf{C}_l = 0. \tag{7.5}$$

The case of special Kähler d-geometry in the axion basis basis is analysed in appendix C of [21]. In particular, by recalling (2.8), one can compute the axionic generators of the solvable parametrization of the $D = 4$ scalar manifold treated above as

$$\partial T(a)/\partial a^k = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ \delta^j_k & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -\delta^i_k \\ 0 & d_{ijk} & 0 & 0 \end{array}\right). \tag{7.6}$$

Up to relabelling of rows and columns, (7.6) matches the expression of $\mathbf{C}_i$ (for $n = 27$) given by (3.6) of [21].

For $N = 2$ special Kähler d-geometries (namely, for those special geometries admitting an uplift to $D = 5$) in the axion basis, this highlights the relation between the solvable parametrization of the $D = 4$ scalar manifold discussed in section 2 and the nilpotent connection of the reformulation *à la Strominger* in the holomorphic gauge (7.2).

8 $N = 2$ special Kähler d-geometry, symplectic sections and the unitary matrix M

In this section we are going to make contact with $N = 2$ special Kähler d-geometries [3] in the symplectic frame defined by the cubic prepotential (1.1). We recall for convenience some results of [7] and we build on them. It has already been remarked that $N = 2$ special Kähler d-geometry differs from the higher N-extended theories in that the n_V 5D axions a^i exactly combine with the 5D scalars $\lambda^i = \lambda^i(\lambda^x, \phi)$ in order to give complex 4D scalar fields

$\frac{X^i}{X^0} = z^i \equiv a^i - i\lambda^i$, where $X^\Lambda = X^0, X^i$. Moreover, in $N=2$ the central charge can be readily computed from the cubic prepotential $F(X)$ of eq. (1.1) by the usual formula (3.2)

$$Z = e^{\frac{K(z,\bar{z})}{2}}(X^\Lambda q_\Lambda - F_\Lambda p^\Lambda) \tag{8.1}$$

For $N=2$ cubic geometry one finds [7]

$$Z = \frac{1}{\sqrt{8\mathcal{V}}}[q_0 + q_i z^i + p^0 f(z) - p^i f_i(z)]\,; \tag{8.2}$$

$$D_i Z = \left(\partial_i + \frac{1}{2}\partial_i K\right) Z = \frac{1}{\sqrt{8\mathcal{V}}}\left[q_0\partial_i K + q_j(\delta_i^j + \partial_i K\, z^j) + \right. \\ \left. + p^0\left(f_i(z) + \partial_i K\, f(z)\right) - p^j\left(f_{ij}(z) + \partial_i K\, f_j(z)\right)\right]\,, \tag{8.3}$$

where

$$f(z) = \frac{1}{3!}d_{ijk}z^i z^j z^k\,, \quad f_i(z) = \frac{1}{2}d_{ijk}z^j z^k\,, \quad f_{ij}(z) = d_{ijk}z^k\,, \quad \mathcal{V} = \frac{1}{3!}d_{ijk}\lambda^i\lambda^j\lambda^k = e^{6\phi}\,, \tag{8.4}$$

with the (real) Kähler potential and its (purely imaginary) derivatives given by

$$K = -\ln(8\mathcal{V})\,; \qquad \partial_i K = -\frac{i}{4\mathcal{V}}d_{ijk}\lambda^j\lambda^k = -\partial_{\bar{\imath}}K\,. \tag{8.5}$$

Notice that i is a *curved* index of the 5D U-duality group G_5, and $\Lambda = (0,i)$. The connection with the universal basis is given by introducing n_V 5D scalars as $\hat{\lambda}^i = e^{-2\phi}\lambda^i$ so that they satisfy $d_{ijk}\hat{\lambda}^i\hat{\lambda}^j\hat{\lambda}^k = 1$. The n_V complex 4D scalar components are then $(a^i, \phi, \hat{\lambda}^i)$. The special Kähler metric is given by

$$g_{ij} = \frac{1}{4}\Big(\frac{1}{4}\kappa_i\kappa_j - \kappa_{ij}\Big)\mathcal{V}^{-2/3} = \frac{1}{4}\mathcal{V}^{-2/3}a_{ij} = \frac{1}{4}e^{-4\phi}a_{ij}\,, \tag{8.6}$$

$$\kappa_i = \mathcal{V}^{-2/3}d_{ijk}\lambda^j\lambda^k\,, \qquad \kappa_{ij} = \mathcal{V}^{-1/3}d_{ijk}\lambda^k\,. \tag{8.7}$$

One can assemble Z and $\overline{D_{\bar{\imath}}Z}$ into a symplectic central charge vector Z_α with a curved lower index

$$\mathcal{Z}_\alpha = \begin{pmatrix} Z \\ \overline{D_{\bar{\imath}}Z} \end{pmatrix} \equiv \langle Q, V_\alpha\rangle = Q^T\Omega V_\alpha = f^\Lambda{}_\alpha q_\Lambda - h_{\Lambda\,\alpha}p^\Lambda, \tag{8.8}$$

$$V_\alpha = \begin{pmatrix} f^\Lambda{}_\alpha \\ h_{\Lambda\alpha} \end{pmatrix}. \tag{8.9}$$

Then, from $\mathcal{Z}_\alpha$ in (8.2) and (8.3) one can read off the components of V_α, which are

$$\mathbf{f} \equiv f^\Lambda{}_\alpha = (f^\Lambda{}_0\,,\ f^\Lambda{}_{\bar{\jmath}}) = \frac{1}{\sqrt{8\mathcal{V}}}\begin{pmatrix} 1 & \partial_{\bar{\jmath}}K \\ z^i & \delta^{\bar{\imath}}_{\bar{\jmath}} + \partial_{\bar{\jmath}}K\,\bar{z}^{\bar{\imath}} \end{pmatrix}\,; \tag{8.10}$$

$$\mathbf{h} \equiv h_{\Lambda\alpha} = (h_{\Lambda 0}\,,\ h_{\Lambda\bar{\jmath}}) = \frac{1}{\sqrt{8\mathcal{V}}}\begin{pmatrix} -f(z) & -\overline{f}_{\bar{\jmath}}(\bar{z}) - \partial_{\bar{\jmath}}K\,\overline{f}(\bar{z}) \\ f_i(z) & \overline{f}_{\bar{\imath}\bar{\jmath}}(\bar{z}) + \partial_{\bar{\jmath}}K\,\overline{f}_{\bar{\imath}}(\bar{z}) \end{pmatrix}. \tag{8.11}$$

While it can be checked that

$$i(\mathbf{f}^\dagger\mathbf{h}-\mathbf{h}^\dagger\mathbf{f})_{\alpha\beta}=\mathcal{G}_{\alpha\beta}=\begin{pmatrix}1&0\\0&g_{ij}\end{pmatrix}, \tag{8.12}$$

we should better consider the normalized symplectic sections with flat tangent indices $A=(0,a)$, such that

$$i(\mathbf{f}^\dagger\mathbf{h}-\mathbf{h}^\dagger\mathbf{f})_{AB}=\delta_{AB}. \tag{8.13}$$

They are the components of $\mathcal{Z}_A=(Z,\overline{D_{\bar{a}}Z})$, and they can be obtained by flattening the curved indices i by the G_5- *Vielbein* e_i^a,[3] so that the orthonormalized symplectic sections ${f^\Lambda}_A$ and $h_{\Lambda A}$ are given by

$${f^\Lambda}_A={f^\Lambda}_\alpha{(G^{-1/2})^\alpha}_A\,,\qquad h_{\Lambda A}=h_{\Lambda\,\alpha}{(G^{-1/2})^\alpha}_A\,. \tag{8.14}$$

It was emphasized in [9] that the symplectic sections $\mathbf{f}$ and $\mathbf{h}$ of (generalized) special geometry are defined only up to the action

$$\mathbf{f}\to\mathbf{f}'\equiv\mathbf{f}M\,,\qquad\mathbf{h}\to\mathbf{h}'\equiv\mathbf{h}M\quad\Leftrightarrow\quad M=\mathbf{f}^{-1}\mathbf{f}'=\mathbf{h}^{-1}\mathbf{h}', \tag{8.15}$$

of a *unitary* matrix M, which preserves the form of the kinetic vector matrix $\mathcal{N}=\mathbf{h}\mathbf{f}^{-1}$ and the conditions (1.12) derived from symplectic invariance of $\mathbf{L}$. Actually, the matrix M found in [9] to connect $N=2$ with $N=8$ is exactly the necessary one to rotate the usual basis of special geometry into the axion basis of any d-geometry. It can be written as

$$M=\frac{1}{2}\begin{pmatrix}1&{(g^{-1/2})^{\bar{j}}}_{\tilde{a}}\partial_{\bar{j}}K\\-i\mathcal{V}^{-1/3}\lambda^i{(a^{1/2})_i}^a&\left(\mathcal{V}^{-1/3}\delta^i_{\bar{j}}+i\mathcal{V}^{-1/3}\lambda^i\partial_{\bar{j}}K\right){(a^{1/2})_i}^a{(g^{-1/2})^j}_{\tilde{a}}\end{pmatrix}; \tag{8.16}$$

$$MM^\dagger=\mathbb{1}, \tag{8.17}$$

where

$$\partial_{\bar{j}}K=2i\lambda^i g_{i\bar{j}}\,; \tag{8.18}$$

$$g_{ij}=\frac{1}{4}\mathcal{V}^{-2/3}a_{ij}\,; \tag{8.19}$$

$$(g^{-1/2})^{\bar{j}}_a\partial_{\bar{j}}K=2i\lambda^i(g^{1/2})^b_i\delta_{ab}\,; \tag{8.20}$$

$$(g^{-1/2})^i_a=2\mathcal{V}^{1/3}(a^{-1/2})^i_a\,. \tag{8.21}$$

By further rescaling the $D=4$ dilatons as

$$\lambda^i\equiv\mathcal{V}^{1/3}\hat{\lambda}^i,\ \ \frac{1}{6!}d_{ijk}\hat{\lambda}^i\hat{\lambda}^j\hat{\lambda}^k=1. \tag{8.22}$$

the matrix M (8.16) can be recast as follows:

$$M=\frac{1}{2}\begin{pmatrix}1&i\hat{\lambda}^i(a^{1/2})^b_i\delta_{ab}\\-i\hat{\lambda}^i{(a^{1/2})_i}^a&2\delta^a_{\tilde{a}}-\hat{\lambda}^i\hat{\lambda}^j{(a^{1/2})_i}^a{(a^{1/2})_j}^b\delta_{\tilde{a}b}\end{pmatrix}. \tag{8.23}$$

[3]Further below, in the explicit case of *stu* model, the *Vielbein* will be taken to be purely imaginary (cfr. appendix C).

Using (1.11), one can see that the action (8.15) of M induces the following transformation of the coset representative $\mathbf{L}$:

$$\mathbf{L} \to \mathbf{L}' = \mathbf{L} \begin{pmatrix} \mathrm{Re}M & -\mathrm{Im}M \\ \mathrm{Im}M & \mathrm{Re}M \end{pmatrix} \equiv \mathbf{L}\mathcal{Y}\left(\mathrm{Re}M, \mathrm{Im}M\right), \tag{8.24}$$

where the real symmetric and unitary matrix

$$\mathcal{Y} = \frac{1}{2}\left(\begin{array}{cc|cc} 1 & 0 & 0 & -\hat{\lambda}^i (a^{1/2})_i^b \delta_{ab} \\ 0 & 2\delta_b^a - \hat{\lambda}^i\hat{\lambda}^j (a^{1/2})_i^a (a^{1/2})_j^c \delta_{bc} & \hat{\lambda}^i (a^{1/2})_i^a & 0 \\ \hline 0 & \hat{\lambda}^i (a^{1/2})_i^b \delta_{ab} & 1 & 0 \\ -\hat{\lambda}^i (a^{1/2})_i^a & 0 & 0 & 2\delta_b^a - \hat{\lambda}^i\hat{\lambda}^j (a^{1/2})_i^a (a^{1/2})_j^c \delta_{bc} \end{array}\right); \tag{8.25}$$

$$\mathcal{Y}^* = \mathcal{Y}^\dagger = \mathcal{Y}^T = \mathcal{Y}^{-1} \Leftrightarrow \mathcal{Y}\mathcal{Y}^\dagger = \mathcal{Y}\mathcal{Y}^T = \mathcal{Y}^2 = \mathbb{1}, \tag{8.26}$$

does not depend on the volume modulus $\mathcal{V}$.

The symplecticity of $\mathbf{L}$ (and thus of $\mathbf{L}'$) yields

$$\mathbf{L}'^T \Omega \mathbf{L}' = \Omega \to \mathcal{Y}^T \Omega \mathcal{Y} = \Omega\,, \tag{8.27}$$

thus also $\mathcal{Y}$ is a symplectic matrix, as expected. Indeed, from its very definition (8.24), the symplectic condition (8.27) becomes

$$\mathrm{Im}M\,\mathrm{Re}M + \mathrm{Re}M\,\mathrm{Im}M = 0\,, \qquad \mathrm{Re}M^2 - \mathrm{Im}M^2 = \mathbb{1}\,, \tag{8.28}$$

which is identically satisfied since M is a unitary matrix, with $\mathrm{Re}M^T = \mathrm{Re}M$, and $\mathrm{Im}M^T = -\mathrm{Im}M$ (cfr. (8.16)–(8.17)).

The matrix $\mathcal{Y}\left(\mathrm{Re}M, \mathrm{Im}M\right)$ (8.24) provides a realization of the maximal symmetric embedding [10]

$$\mathrm{U}(28) \subset \mathrm{Sp}(56, \mathbb{R}). \tag{8.29}$$

Indeed, since $\mathbf{L}$ is symplectic, one has checked that also $\mathcal{Y}$ is symplectic, but given (8.26), this leads to

$$[\mathcal{Y}, \Omega] = 0\,. \tag{8.30}$$

An explicit computation of the matrices M (8.23) and $\mathcal{Y}$ (8.25) for the t^3 limit of the *stu* model is presented in appendix B.

9 Unitarity relations for M and induced relations on $\hat{M}$

The residual freedom in the definition of the symplectic section was found in [9] to imply that the symplectic vector $\mathcal{Z}_A = \left(Z, \overline{D_{\bar{a}} Z}\right)^T$ of $N = 2$ special geometry, with a flat index $A = (0, \bar{a})$, differs by a unitary transformation from the corresponding central charge vector $Z_A = (Z_0, Z_a)^T$ of the $N = 8$, $D = 4$ theory (3.19) in the $E_{6(6)}$-covariant symplectic frame (with $a = 1, \ldots, 27$),

$$Z_A = \mathcal{Z}_B M^B{}_A\,. \tag{9.1}$$

This is obvious from the fact that the $N = 2$ sections in (8.10) are not lower triangular, as required in the axion basis in (2.3) where the the symplectic section $\mathbf{f}$ is real. Notice that the $E_{6(6)}$ basis is related to the usual de Wit and Nicolai symplectic frame by a symplectic transformation [8] . However, under a change of symplectic basis, that is a duality transformation, the kinetic matrix transforms as $\mathcal{N}_{\Lambda\Sigma} \to (C + D\mathcal{N})(A + B\mathcal{N})^{-1}$, while the unitary transformation M leaves $\mathcal{N}_{\Lambda\Sigma}$ invariant.

M acts on the normalized sections, with a flat tangent index, as given by (8.15) (where now prime refers to $N = 8$ sections and unprimed sections are the $N = 2$ ones in the axion basis, discussed in section 8). On the other hand, one can define a matrix $\widehat{M}$ acting on (un-normalized) sections with a curved lower index as

$$\widehat{\mathbf{f}}' = \widehat{\mathbf{f}}\widehat{M}\,, \qquad \widehat{\mathbf{h}}' = \widehat{\mathbf{h}}\widehat{M} \qquad \Leftrightarrow \qquad \widehat{M} = \widehat{\mathbf{f}}^{-1}\widehat{\mathbf{f}}' = \widehat{\mathbf{h}}^{-1}\widehat{\mathbf{h}}'\,, \tag{9.2}$$

They can be obtained from (2.3) and (2.4) by multiplication with the appropriate *Vielbein*, that is

$$\widehat{f}^{\Lambda}{}_{\alpha} = f^{\Lambda}{}_{A}(A^{1/2})^{A}_{\alpha}\,, \qquad \widehat{h}_{\Lambda\alpha} = h_{\Lambda\,A}(A^{1/2})^{A}_{\alpha}\,, \tag{9.3}$$

with

$$A \equiv \left(\begin{array}{c|c} 1 & 0\dots 0 \\ \hline 0 & \\ \dots & a_{IJ} \\ 0 & \end{array}\right) \tag{9.4}$$

where a_{IJ} is the kinetic vector matrix of $N = 8$, $D = 5$ supergravity. In the $E_{6(6)}$-frame of 4D $N = 8$ supergravity, the symplectic section with curved indices $\widehat{\mathbf{f}}$ read [9]

$$\hat{f}^{\Lambda}{}_{\alpha} = \frac{1}{\sqrt{2}}\left(\begin{array}{c|c} e^{-3\phi} & 0 \\ \hline e^{-3\phi}a^{J} & e^{-\phi}\delta^{J}_{I} \end{array}\right)\,, \quad (\hat{f}^{-1})_{\Lambda}{}^{\alpha} = \sqrt{2}\left(\begin{array}{c|c} e^{3\phi} & 0 \\ \hline -e^{\phi}a^{I} & e^{\phi}\delta^{I}_{J} \end{array}\right)\,, \tag{9.5}$$

where, in the *symmetric* gauge [8], $\Lambda = 0, I$ and $\alpha = 0, I$, where here I is a *curved* index spanning the **27** of $E_{6(6)}$.

From (8.10), (9.5) and (9.3), one can compute the matrix [9]

$$\hat{M}^{\beta}_{\alpha} = (\hat{f}^{-1})_{\Lambda}{}^{\beta}\hat{f}'^{\Lambda}{}_{\alpha} = \frac{1}{2}\begin{pmatrix} 1 & \partial_{\bar{\jmath}}K \\ -i\lambda^{i}\mathcal{V}^{-1/3} & \mathcal{V}^{-1/3}\delta^{i}_{\bar{\jmath}} + i\mathcal{V}^{-1/3}\lambda^{i}\partial_{\bar{\jmath}}K \end{pmatrix}, \tag{9.6}$$

which does not depend on the axion fields. Moreover, using (8.14), (9.3) and (8.15), the relation between $\hat{M}$ and M is given by

$$\widehat{M} = \widehat{\mathbf{f}}^{-1}\widehat{\mathbf{f}}' = A^{-1/2}\mathbf{f}^{-1}\mathbf{f}'\mathcal{G}^{1/2} = A^{-1/2}M\mathcal{G}^{1/2} \Leftrightarrow M = A^{1/2}\widehat{M}\mathcal{G}^{-1/2}. \tag{9.7}$$

The unitarity of M entails the following identities for $\hat{M}$, namely:

$$MM^{\dagger} = Id \Leftrightarrow A\,\hat{M}\,\mathcal{G}^{-1}\hat{M}^{\dagger} = Id\,; \tag{9.8}$$

$$M^{\dagger}M = Id \Leftrightarrow \mathcal{G}^{-1}\,\hat{M}^{\dagger}A\,\hat{M} = Id\,. \tag{9.9}$$

10 Axion basis and the fake superpotential

In this section we show an interesting application of the axion basis to non-BPS extremal black holes. The unitary transformation M that rotates the usual $N=2$ basis of special geometry $\mathcal{Z}_A$ into the $E_{6(6)}$ basis Z_A allows to make a precise connection with the $N=2$ *stu* model, where the three complex scalar fields $z^i=\{s\,,t\,,u\}$ span the rank-3 coset space $\left[\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)}\right]^3$, with

$$f=stu\,,\quad e^{-K}=8\lambda^1\lambda^2\lambda^3=8\mathcal{V}\,, \tag{10.1}$$

viewed as a sub sector of the full $N=8$ theory [15, 22, 23]. The aim is to illustrate the computation of the fake superpotential for non-BPS solutions and (p^0,q_0) charge configuration in the stu-truncation of $N=8$ supergravity. This example was discussed from two different viewpoints: in [22] the fake superpotential was computed for generic charges in terms of duality invariants of the underlying special geometry, while in [26] Bossard, Michel and Pioline (BMP) provided a procedure based on nilpotent orbits which lead to the fake superpotential as solution of a sixth order polynomial.

The virtue of the axion basis is that, while showing the equivalence of the derivation of [26] and [22], we can read out the fake superpotential from the $N=8$ central charge in the skew symmetric form. Here we start from the formula for the central charge derived in [9] using 4D/5D special geometry relations, and we look for a suitable SU(8) transformation that brings it to the form given by eq. (2.68) of [26]

$$Z_{AB}^{CFG}\overset{\mathrm{SU}(8)}{\longrightarrow}Z_{AB}^{BMP} \tag{10.2}$$

In particular, we study the effect of such a rotation with respect to the decomposition $\mathbf{28}\to\mathbf{1}_{\mathbb{C}}+\mathbf{27}_{\mathbb{C}}$, which is common to the central charge normal frame of both [9] and [26]. We identify this transformation in the t^3-truncation where it depends only on one angle χ, purely given in terms of duality invariant quantities. When this rotation is used to match the central charge in [9] and that of [26], we consistently retrieve the non-BPS fake superpotential for the $N=2$ t^3 model, within the (p^0,q_0) charge configuration in presence of non zero axions. This is a non-trivial consistency check for the 4D/5D formalism based on the matrices $\widehat{M}$ and M [9] detailed in previous sections.

The key point of this analysis is that the $\mathbf{28}$ components of the $N=8$ central charge matrix Z_{AB} can be traded for the symplectic vectors Z_A (with flat lower index) or Z_α (with a curved one) reflecting the splitting $\mathbf{28}=\mathbf{1}_{\mathbb{C}}+\mathbf{27}_{\mathbb{C}}$ of the axion basis. Since Z_{AB} can always be brought to the skew-diagonal form

$$Z_{AB}=\begin{pmatrix} z_1 & 0 & 0 & 0\\ 0 & z_2 & 0 & 0\\ 0 & 0 & z_3 & 0\\ 0 & 0 & 0 & z_4\end{pmatrix}\otimes\epsilon\,, \tag{10.3}$$

one has to relate the eigenvalues z_1, z_2, z_3, z_4 with the complex components of $Z_\alpha=(Z_0,Z_I)$ [9], with $I=1,2,3$,

$$\begin{aligned} Z_0&=\frac{1}{\sqrt{2}}(Z_0^{(e)}+iZ^0_{(m)})\,,\\ Z_I&=\frac{1}{\sqrt{2}}(Z_I^{(e)}+ia_{IJ}Z^J_{(m)})\,. \end{aligned} \tag{10.4}$$

In fact, in light of the previous discussion, eqs. (8.15) and (9.2) yield

$$(Z\,,\overline{D}_{\bar{\imath}}\bar{Z}) = (Z_0, Z_i)\hat{M}\,, \tag{10.5}$$

where Z and D_iZ in the l.h.s. are given by (8.2) and (8.3). Using (9.2), one finds

$$Z\; = \frac{1}{2}(Z_0 - i\lambda^i Z_i \mathcal{V}^{-1/3})\,; \tag{10.6}$$

$$\overline{D}_{\bar{\imath}}\overline{Z}\; = \frac{1}{2}(\partial_{\bar{\imath}}K Z_0 + \mathcal{V}^{-1/3}Z_{\bar{\imath}} + i\mathcal{V}^{-1/3}\lambda^j Z_j \partial_{\bar{\imath}}K)\,. \tag{10.7}$$

In order to find the skew eigenvalues z_1, z_2, z_3, z_4 in (10.3), one needs the inverse metric, which in this case is factorized as

$$g^{s\bar{s}} = -(s-\bar{s})^2\,, \quad g^{t\bar{t}} = -(t-\bar{t})^2\,, \quad g^{u\bar{u}} = -(u-\bar{u})^2\,, \tag{10.8}$$

as well as the purely imaginary *Vielbein* (see appendix C)

$$(g^{-1/2})^{\bar{s}}{}_1 = (s-\bar{s})\,, \quad (g^{-1/2})^{\bar{t}}{}_2 = (t-\bar{t})\,, \quad (g^{-1/2})^{\bar{u}}{}_3 = (u-\bar{u})\,, \tag{10.9}$$

and the Kähler connection

$$\partial_{\bar{\imath}}K = \left(\frac{1}{s-\bar{s}}, \frac{1}{t-\bar{t}}, \frac{1}{u-\bar{u}}\right)^T\,. \tag{10.10}$$

Using (10.8)–(10.10) in (10.6)–(10.7), one obtains

$$Z\; = \frac{1}{2}(Z_0 - i\hat{\lambda}^i Z_i)\,; \tag{10.11}$$

$$\overline{D}_{\bar{s}}\bar{Z}\; = \frac{1}{2}\left(\frac{1}{s-\bar{s}}Z_0 + \mathcal{V}^{-1/3}Z_1 + i\mathcal{V}^{-1/3}\lambda^i Z_i \frac{1}{s-\bar{s}}\right)\,; \tag{10.12}$$

$$\overline{D}_{\bar{t}}\bar{Z}\; = \frac{1}{2}\left(\frac{1}{t-\bar{t}}Z_0 + \mathcal{V}^{-1/3}Z_2 + i\mathcal{V}^{-1/3}\lambda^i Z_i \frac{1}{t-\bar{t}}\right)\,; \tag{10.13}$$

$$\overline{D}_{\bar{u}}\bar{Z}\; = \frac{1}{2}\left(\frac{1}{u-\bar{u}}Z_0 + \mathcal{V}^{-1/3}Z_3 + i\mathcal{V}^{-1/3}\lambda^i Z_i \frac{1}{u-\bar{u}}\right)\,. \tag{10.14}$$

By recalling the definition $\lambda^i\mathcal{V}^{-1/3} = \lambda^i e^{-2\phi} \equiv \hat{\lambda}^i$ (cfr. section 8), and defining

$$e_1 \equiv \hat{\lambda}^1 Z_1\,, \quad e_2 \equiv \hat{\lambda}^2 Z_2\,, \quad e_3 \equiv \hat{\lambda}^3 Z_3\,, \tag{10.15}$$

one computes

$$g^{s\bar{s}}\overline{D}_{\bar{s}}\bar{Z}D_sZ = \frac{1}{4}\left|Z_0 - i\hat{\lambda}^1 Z_1 + i\hat{\lambda}^2 Z_2 + i\hat{\lambda}^3 Z_3\right|^2 = \frac{1}{4}\left|Z_0 + i\,(-e_1+e_2+e_3)\right|^2; \tag{10.16}$$

$$g^{t\bar{t}}\overline{D}_{\bar{t}}\bar{Z}D_tZ = \frac{1}{4}\left|Z_0 + i\hat{\lambda}^1 Z_1 - i\hat{\lambda}^2 Z_2 + i\hat{\lambda}^3 Z_3\right|^2 = \frac{1}{4}\left|Z_0 + i\,(e_1-e_2+e_3)\right|^2; \tag{10.17}$$

$$g^{u\bar{u}}\overline{D}_{\bar{u}}\bar{Z}D_uZ = \frac{1}{4}\left|Z_0 + i\hat{\lambda}^1 Z_1 + i\hat{\lambda}^2 Z_2 - i\hat{\lambda}^3 Z_3\right|^2 = \frac{1}{4}\left|Z_0 + i\,(e_1+e_2-e_3)\right|^2, \tag{10.18}$$

JHEP02(2013)059

from which the entries of the Z_{AB} matrix can be read off (in the conventions of e.g. (5.32) of [23])

$$z_1 = Z = \frac{i}{2}\left[-(e_1+e_2+e_3) - iZ_0\right] , \tag{10.19}$$

$$\begin{aligned} z_2 &= \overline{D}_{\bar{s}}\bar{Z}(g^{-1/2})^{\bar{s}}{}_1 = \frac{i}{2}\left(-iZ_0 - \hat{\lambda}^1 Z_1 + \hat{\lambda}^2 Z_2 + \lambda^3 Z_3\right) = \\ &= \frac{i}{2}\left[(e_2+e_3-e_1) - iZ_0\right] , \end{aligned} \tag{10.20}$$

$$\begin{aligned} z_3 &= \overline{D}_{\bar{t}}\bar{Z}(g^{-1/2})^{\bar{t}}{}_2 = \frac{i}{2}\left(-iZ_0 + \hat{\lambda}^1 Z_1 - \hat{\lambda}^2 Z_2 + \lambda^3 Z_3\right) = \\ &= \frac{i}{2}\left[(e_1+e_3-e_2) - iZ_0\right] , \end{aligned} \tag{10.21}$$

$$\begin{aligned} z_4 &= \overline{D}_{\bar{u}}\bar{Z}(g^{-1/2})^{\bar{u}}{}_3 = \frac{i}{2}\left(-iZ_0 + \hat{\lambda}^1 Z_1 + \hat{\lambda}^2 Z_2 - \lambda^3 Z_3\right) = \\ &= \frac{i}{2}\left[(e_1+e_2-e_3) - iZ_0\right] . \end{aligned} \tag{10.22}$$

The $4D/5D$ covariant splitting is thus manifest in the following form of the central charge matrix[4] [9]

$$Z_{AB} = \frac{i}{2}\epsilon \otimes \left[-iZ_0 \; id_4 + \begin{pmatrix} -e_1-e_2-e_3 & 0 & 0 & 0 \\ 0 & -e_1+e_2+e_3 & 0 & 0 \\ 0 & 0 & e_1-e_2+e_3 & 0 \\ 0 & 0 & 0 & e_1+e_2-e_3 \end{pmatrix} \right] . \tag{10.23}$$

This result, compared with formulæ (3.2) of [9], explains the definition

$$Z_{AB} = \frac{1}{2}\left(e_{AB} - iZ^0\Omega\right) ,$$

in which $\Omega = \epsilon \otimes id_4$, given in eq. (4.7) of the same reference; notice that the overall phase i is uninfluential.

10.1 Residual $\mathrm{U}(1)^3$ symmetry of the skew-diagonal Z_{AB}

The form of the central charge, as derived in the previous section, reflects the more general structure of the $\mathbf{28} \to \mathbf{1}_{\mathbb{C}} + \mathbf{27}_{\mathbb{C}}$ decomposition of $\mathrm{SU}(8) \supset \mathrm{USp}(8)$ representation.

The central charge matrix for the p^0, q_0 configuration in $N = 8$ Supergravity has been given in [26], in the same symplectic frame. The reason why this is a suitable frame to study the non-BPS orbit is related to the choice of orbit representative. The moduli space of the non-BPS p^0, q_0 solution is indeed the moduli space of the 5 dimensional theory, namely $E_{6(6)}/\mathrm{USp}(8)$. By solving a nonstandard diagonalization problem, the authors of [26] identify the fake-superpotential in the singlet of the axion-base decomposition of the central charge matrix. However, the form of Z_{AB} is unique up to SU(8) transformations,

[4] id_n denotes the $n \times n$ identity matrix throughout.

and the choice of symplectic frame is not covariant with respect to the action of SU(8), since the singlet is not left invariant by R-symmetry rotations.

Starting from the form of the central charge in (10.23), we look for the transformation that rotates Z_{AB} in such a way that the transformed matrix can be identified with the one of [26]. The goal is to determine the SU(8) rotation in terms of the scalar fields, and then read from the transformed singlet the explicit form of the fake superpotential.

Because of the residual USp(8) symmetry of the skew-diagonal central charge (10.3), we can restrict the analysis to the transformations of $U(1)^3 \subset SU(8)/USp(8)$.

10.1.1 The (p^0, q_0) configuration

In the non-BPS (p^0, q_0) charge configuration (corresponding to $D0-D6$ in Type II language), the dressed charges of the $N=8$ theory read (3.17)

$$Z_0 = \frac{1}{\sqrt{2}}\left(e^{-3\phi}q_0 + e^{-3\phi}p^0 a_1 a_2 a_3 + ie^{3\phi}p^0\right); \tag{10.24}$$

$$Z_i = \frac{1}{\sqrt{2}}p^0\left[e^{-\phi}\begin{pmatrix}\hat\lambda^1 a^2 a^3\\ \hat\lambda^2 a^1 a^3\\ \hat\lambda^3 a^1 a^2\end{pmatrix} - ie^{\phi}\begin{pmatrix}\frac{a^1}{\hat\lambda^1}\\ \frac{a^2}{\hat\lambda^2}\\ \frac{a^3}{\hat\lambda^3}\end{pmatrix}\right]. \tag{10.25}$$

Thus, the $N=8$ skew-diagonal Z_{AB} (10.3) in the (p^0, q_0) charge configuration can then be written as

$$Z_{AB}^{(p_0,q_0)} = \frac{1}{2\sqrt{2}}\epsilon\otimes\left[(e^{-3\phi}q_0 + \alpha_1\alpha_2\alpha_3\, p^0 e^{3\phi} + ip^0 e^{3\phi})\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix} + \right. \tag{10.26}$$

$$+p^0 e^{3\phi}\begin{pmatrix}-(\alpha_1+\alpha_2+\alpha_3)&0&0&0\\0&-\alpha_1+\alpha_2+\alpha_3&0&0\\0&0&\alpha_1-\alpha_2+\alpha_3&0\\0&0&0&\alpha_1+\alpha_2-\alpha_3\end{pmatrix}+$$

$$\left. +p^0 ie^{3\phi}(\alpha_1\alpha_2\alpha_3)\begin{pmatrix}-(\frac{1}{\alpha_1}+\frac{1}{\alpha_2}+\frac{1}{\alpha_3})&0&0&0\\0&(-\frac{1}{\alpha_1}+\frac{1}{\alpha_2}+\frac{1}{\alpha_3})&0&0\\0&0&(\frac{1}{\alpha_1}-\frac{1}{\alpha_2}+\frac{1}{\alpha_3})&0\\0&0&0&(\frac{1}{\alpha_1}+\frac{1}{\alpha_2}-\frac{1}{\alpha_3})\end{pmatrix}\right], \tag{10.27}$$

where $\alpha^i \equiv a^i/\lambda^i$ is the axion/dilaton ratio, with $\lambda^i = e^{2\phi}\hat\lambda^i$, and $\hat\lambda^1\hat\lambda^2\hat\lambda^3 = 1$. When $a^i = 0$, one recovers the KK solution studied in [9].

To proceed further, it is convenient to define the following quantities:

$$Y_0 = \frac{1}{\sqrt{2}}(q_0\, e^{-3\phi} + \alpha_1\alpha_2\alpha_3\, p^0 e^{3\phi}) + \frac{i}{\sqrt{2}}p^0 e^{3\phi}; \tag{10.28}$$

$$Y_i = -\frac{1}{\sqrt{2}}p^0 e^{3\phi}\left(\alpha_i + \frac{i}{2}\left|\epsilon_{ijk}\right|\alpha_j\alpha_k\right), \tag{10.29}$$

and

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{10.30}$$

We can write

$$id_2 \otimes id_2 = id_4 \,, \qquad id_2 \otimes \sigma_3 = \begin{pmatrix} 1 & 0 & & \\ 0 & 1 & & \\ & & -1 & 0 \\ & & 0 & -1 \end{pmatrix} , \qquad \sigma_3 \otimes id_2 = \begin{pmatrix} 1 & 0 & & \\ 0 & -1 & & \\ & & 1 & 0 \\ & & 0 & -1 \end{pmatrix} ;$$

$$\sigma_3 \otimes \sigma_3 = \begin{pmatrix} 1 & 0 & & \\ 0 & -1 & & \\ & & -1 & 0 \\ & & 0 & 1 \end{pmatrix} . \tag{10.31}$$

Thus, by recalling (10.26), Z_{AB} can be decomposed as

$$Z_{AB}^{(p^0,q_0)} \equiv Z_{AB}(Y_0, Y_i) = \frac{1}{2}\epsilon \otimes \left[Y_0 \ id_4 + Y_1 \ id_2 \otimes \sigma_3 + Y_2 \ \sigma_3 \otimes id_2 + Y_3 \ \sigma_3 \otimes \sigma_3 \right] . \tag{10.32}$$

This parametrization of the central charge matrix will allow us to perform the necessary rotation to identify the fake superpotential.

10.1.2 $U(1)^3$

The matrix Z_{AB} (10.32) has a residual $U(1)^3 \subset SU(8)/USp(8)$ symmetry. More precisely, $U(1)^3$ can be considered as the Cartan subalgebra of the symmetric, rank-3 compact manifold $SU(8)/USp(8)$ ($\dim_{\mathbb{R}} = 27$); indeed, $U(1)^3$-transformations do not generate off-diagonal elements, and they leave the skew-diagonal form of Z_{AB} invariant. We choose to parametrize such a $U(1)^3$ matrix as a 4×4 matrix acting on the diagonal part of Z_{AB}, namely ($\chi_i \in \mathbb{R}$)

$$\mathcal{U} \equiv \begin{pmatrix} e^{-i(\chi_1+\chi_2+\chi_3)} & & & \\ & e^{i(-\chi_1+\chi_2+\chi_3)} & & \\ & & e^{i(\chi_1-\chi_2+\chi_3)} & \\ & & & e^{i(\chi_1+\chi_2-\chi_3)} \end{pmatrix} \in U(1)^3 \subset SU(8)/USp(8) . \tag{10.33}$$

Note that, consistently, the sum of the four diagonal phases vanishes. Therefore, by the exponential mapping, one obtains

$$\mathcal{U} = \exp \begin{pmatrix} -i(\chi_1+\chi_2+\chi_3) & & & \\ & i(-\chi_1+\chi_2+\chi_3) & & \\ & & i(\chi_1-\chi_2+\chi_3) & \\ & & & i(\chi_1+\chi_2-\chi_3) \end{pmatrix} , \tag{10.34}$$

which, analogously to Z_{AB} (10.32), enjoys the following decomposition:

$$\begin{aligned} \mathcal{U} &= \exp\left[-i(\chi_1 \ \ id_2 \otimes \sigma_3 + \chi_2 \ \sigma_3 \otimes id_2 + \chi_3 \ \sigma_3 \otimes \sigma_3) \right] = \\ &= \exp\left[-i\chi_1 \ \ id_2 \otimes \sigma_3 \right] \cdot \exp\left[-i\chi_2 \ \sigma_3 \otimes id_2 \right] \cdot \exp\left[-i\chi_3 \ \sigma_3 \otimes \sigma_3 \right] = \\ &= \mathcal{U}_1 \cdot \mathcal{U}_2 \cdot \mathcal{U}_3 \end{aligned} \tag{10.35}$$

where all matrices are reciprocally commuting.

Under U(1)^3 (10.34), Z_{AB} (10.32) transforms as

$$Z_{AB} \to \mathcal{U} Z_{AB} \mathcal{U}^T \equiv \mathcal{U}^2 Z_{AB}. \tag{10.36}$$

Without loss of generality, one can therefore just redefine the χ_i's by a factor of 2, and consider the transformation

$$Z_{AB} \to \mathcal{U} Z_{AB}. \tag{10.37}$$

Each single $\mathcal{U}_i$ actually reads

$$\begin{aligned} \mathcal{U}_1 &= \exp\left[-i\chi_1 \quad id_2 \otimes \sigma_3\right] = \cos\chi_1 \; id_4 - i\sin\chi_1 \; id_2 \otimes \sigma_3; \\ \mathcal{U}_2 &= \exp\left[-i\chi_2 \; \sigma_3 \otimes id_2\right] = \cos\chi_2 \; id_4 - i\sin\chi_2 \; \sigma_3 \otimes id_2; \\ \mathcal{U}_3 &= \exp\left[-i\chi_3 \; \sigma_3 \otimes \sigma_3\right] = \cos\chi_3 \; id_4 - i\sin\chi_3 \sigma_3 \otimes \sigma_3, \end{aligned} \tag{10.38}$$

and induces the following transformation on Z_{AB} (10.32):

$$\begin{aligned} \mathcal{U}_1 \, Z_{AB} &\to \cos\chi_1 \; Z_{AB} - i\sin\chi_1 \; Z_{AB} \cdot id_2 \otimes \sigma_3; \\ \mathcal{U}_2 \, Z_{AB} &\to \cos\chi_2 \; Z_{AB} - i\sin\chi_2 \; Z_{AB} \cdot \sigma_3 \otimes id_2; \\ \mathcal{U}_3 \, Z_{AB} &\to \cos\chi_3 \; Z_{AB} - i\sin\chi_3 \; Z_{AB} \cdot \sigma_3 \otimes \sigma_3. \end{aligned} \tag{10.39}$$

Consequently, $\mathcal{U}$ (10.35) has a well defined action on the coefficients of the matrices (10.31); for example, by acting with only $\mathcal{U}_1$ gives rise to the following transformations of Y_0 and Y_i's:

$$\begin{aligned} Y_0 &\to \gamma_0 \equiv \cos\chi_1 \; Y_0 - i\sin\chi_1 \; Y_1; \\ Y_1 &\to \gamma_1 \equiv \cos\chi_1 \; Y_1 - i\sin\chi_1 \; Y_0; \\ Y_2 &\to \gamma_2 \equiv \cos\chi_1 \; Y_2 - i\sin\chi_1 \; Y_3; \\ Y_3 &\to \gamma_3 \equiv \cos\chi_1 \; Y_3 - i\sin\chi_1 \; Y_2, \end{aligned} \tag{10.40}$$

such that the $\mathcal{U}_1$-transformed central charge matrix (10.32) can be rewritten as

$$Z_{AB}(Y_0, Y_i) \to \mathcal{U}_1 Z_{AB}(Y_0, Y_i) = Z_{AB}(\gamma_0, \gamma_i)\,. \tag{10.41}$$

The complete action of $\mathcal{U}$ (10.35) on (10.32) reads

$$Z_{AB}(Y_0, Y_i) \to Z_{AB}(\zeta_0, \zeta_i) = \mathcal{U}_3 \mathcal{U}_2 \mathcal{U}_1 Z_{AB}(Y_0, Y_i)\,, \tag{10.42}$$

where the ζ_I's are defined as

$$\begin{aligned} \zeta_0 &\equiv A \; Y_0 + B \; Y_1 + C \; Y_2 + D \; Y_3; \\ \zeta_1 &\equiv B \; Y_0 + AY_1 + D \; Y_2 + C \; Y_3; \\ \zeta_2 &\equiv C \; Y_0 + D \; Y_1 + A \; Y_2 + B \; Y_3; \\ \zeta_3 &\equiv D \; Y_0 + C \; Y_1 + B \; Y_2 + A \; Y_3, \end{aligned} \tag{10.43}$$

with ($c_i \equiv \cos\chi_i$, $s_i \equiv \sin\chi_i$)

$$\begin{aligned} A &\equiv (c_1 c_2 c_3 - i s_1 s_2 s_3); \\ B &\equiv (-c_1 s_2 s_3 + i s_1 c_2 c_3); \\ C &\equiv (-s_1 c_2 s_3 + i c_1 s_2 c_3); \\ D &\equiv (-s_1 s_2 c_3 + i c_1 c_2 s_3). \end{aligned} \tag{10.44}$$

Within the same (p^0, q_0) axionful charge configuration, it is interesting to compare the $\mathrm{U}(1)^3$-transformed Z_{AB} (10.42)–(10.44) with the "non-standard" skew-diagonalized $Z_{AB}^{(BMP)}$ obtained by Bossard, Michel and Pioline (BMP) in [26]

$$Z_{AB}^{(BMP)} = \frac{1}{2}\epsilon \otimes \left[i(e^{i(\alpha-\pi/4)} + \sin 2\alpha e^{-i(\alpha-\pi/4)})\rho \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right. \\ \left. + e^{-i(\alpha-\pi/4)} \begin{pmatrix} \xi_1+\xi_2+\xi_3 & 0 & 0 & 0 \\ 0 & -\xi_1 & 0 & 0 \\ 0 & 0 & -\xi_2 & 0 \\ 0 & 0 & 0 & -\xi_3 \end{pmatrix} \right], \tag{10.45}$$

which can equivalently be recast in the following form:

$$\begin{aligned} Z_{AB}^{(BMP)} &= \frac{1}{2}\epsilon \otimes \left[i(e^{i\eta} + \cos 2\eta\, e^{-i\eta})\rho\, id_4 + \right. \\ &\left. + e^{-i\eta} \begin{pmatrix} -\mu_1-\mu_2-\mu_3 & 0 & 0 & 0 \\ 0 & -\mu_1+\mu_2+\mu_3 & 0 & 0 \\ 0 & 0 & \mu_1-\mu_2+\mu_3 & 0 \\ 0 & 0 & 0 & \mu_1+\mu_2-\mu_3 \end{pmatrix} \right] = \\ &= \frac{1}{2}\epsilon \otimes \left[\mu_0\ id_4 - e^{-i\eta}\mu_1\ id_2 \otimes \sigma_3 - e^{-i\eta}\mu_2\ \sigma_3 \otimes id_2 - e^{-i\eta}\mu_3\ \sigma_3 \otimes \sigma_3 \right] \end{aligned} \tag{10.46}$$

by introducing the quantities:

$$\begin{aligned} &\mu_0 \equiv i(e^{i\eta} + \cos 2\eta\, e^{-i\eta})\rho\,, \qquad \eta \equiv \alpha - \frac{\pi}{4}\,, \\ &\xi_1 \equiv \mu_1 - \mu_2 - \mu_3\,, \qquad \xi_2 \equiv -\mu_1 + \mu_2 - \mu_3\,, \qquad \xi_3 \equiv -\mu_1 - \mu_2 + \mu_3. \end{aligned} \tag{10.47}$$

By comparing (10.32) and (10.46), in order to match (10.45) with (10.42)–(10.44), a transformation $\mathcal{U} \in \mathrm{U}(1)^3$ should be found, such that

$$Y_0 \to \zeta_0 = \mu_0\,, \quad Y_i \to \zeta_i = -e^{-i\eta}\mu_i\,, \qquad i = 1,2,3\,. \tag{10.48}$$

This amounts to solving the system composed by (10.43)–(10.44) and (10.47)–(10.48). For simplicity's sake, we will here confine ourselves to solve such a system within the "t^3-degeneration" of the formalism under consideration, which amounts to choosing three equal phases χ_i's, corresponding to the diagonal $\mathrm{U}(1)_{\mathrm{diag}}$ inside $\mathrm{U}(1)^3$.

10.1.3 t^3 model

As mentioned, at the level of $\mathcal{U}$-transformation, the "degeneration" procedure from *stu* to t^3 model amounts to identifying

$$\chi_1 = \chi_2 = \chi_3 \equiv \chi. \tag{10.49}$$

JHEP02(2013)059

This corresponds to considering the action of $\mathrm{U}(1)_{\mathrm{diag}} \subset \mathrm{U}(1)^3 \subset \mathrm{SU}(8)/\mathrm{USp}(8)$, such that (recall (10.35))

$$\mathcal{U} = \mathcal{U}_1 \cdot \mathcal{U}_2 \cdot \mathcal{U}_3 \equiv \mathbb{U}_{\mathrm{diag}} = \exp \begin{pmatrix} -3i\chi & & & \\ & i\chi & & \\ & & i\chi & \\ & & & i\chi \end{pmatrix} . \tag{10.50}$$

The central charge matrix given by (10.26) and (10.32) thus acquires the following structure:[5]

$$\begin{aligned} Z_{AB}^{(p^0,q_0),t^3} &= \frac{1}{2\sqrt{2}} \epsilon \otimes \left[\left(e^{-3\phi} q_0 + p^0 e^{3\phi} (i + \alpha^3) \right) id_4 + p^0 \alpha e^{3\phi} (1 + i\alpha) \begin{pmatrix} -3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right] \\ &= \frac{1}{2} \epsilon \otimes \left[Y_0 \, id_4 + Y \ (id_2 \otimes \sigma_3 + \sigma_3 \otimes id_2 + \sigma_3 \otimes \sigma_3) \right] , \end{aligned} \tag{10.51}$$

where here ($\alpha_1 = \alpha_2 = \alpha_3 \equiv \alpha$)

$$\begin{aligned} Y_0 &\equiv \tfrac{1}{\sqrt{2}} \left(e^{-3\phi} q_0 + p^0 e^{3\phi} (i + \alpha^3) \right) ; \\ Y &\equiv -\tfrac{1}{\sqrt{2}} \left(p^0 e^{3\phi} \alpha (1 + i\alpha) \right) . \end{aligned} \tag{10.52}$$

On the other hand, the consistent "t^3-degeneration" of the central charge matrix (10.46)–(10.47) reads

$$\begin{aligned} Z_{AB}^{(BMP),t^3} &= \frac{1}{2} \epsilon \otimes \left[i(e^{i\eta} + \cos 2\eta \, e^{-i\eta}) \rho \; id_4 + e^{-i\eta} \mu \begin{pmatrix} -3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right] = \\ &= \frac{1}{2} \epsilon \otimes \left[\mu_0 \; id_4 - e^{-i\eta} \mu (\; id_2 \otimes \sigma_3 + \sigma_3 \otimes id_2 + \sigma_3 \otimes \sigma_3) \right] , \end{aligned} \tag{10.53}$$

where

$$\mu_0 \equiv i(e^{i\eta} + \cos 2\eta \, e^{-i\eta}) \rho \, , \quad \eta \equiv \alpha - \frac{\pi}{4} \quad \mu_1 = \mu_2 = \mu_3 \equiv \mu \equiv -\xi_1 = -\xi_2 = -\xi_3 \, . \tag{10.54}$$

We notice that, by denoting η_0 the phase of μ_0, it holds that

$$\tan \eta_0 = -\frac{1}{(\tan \eta)^3} \, . \tag{10.55}$$

Thus, in order to match (10.51)–(10.52) with (10.53)–(10.54), a phase χ should be determined such that it rotates the relevant quantities as follows ($\zeta_1 = \zeta_2 = \zeta_3 \equiv \zeta$, $Y_1 = Y_2 = Y_3 \equiv Y$)

$$Y_0 \to \zeta_0 = \mu_0 \, , \quad Y \to \zeta = -e^{-i\eta} \mu \, . \tag{10.56}$$

[5] In order to simplify the computation, we will henceforth choose $p^0 > 0$ and $q_0 > 0$. This does not imply any loss of generality, since all other sign choices are related to this by a duality rotation along the non-BPS ($Z_H \neq 0$) charge orbit of the *stu* model.

From the "t^3-degeneration" of (10.43), one gets

$$\zeta_0 = AY_0 + 3BY; \tag{10.57}$$
$$\zeta = (A+2B)Y + BY_0. \tag{10.58}$$

However, now A and B respectively simplifies down to

$$A \equiv B + e^{-i\chi}, \quad B \equiv \frac{i}{2}\sin(2\chi)e^{i\chi}, \tag{10.59}$$

thus allowing for the following re-writing of (10.57)–(10.58):

$$\begin{aligned} \zeta_0 &= e^{-i\chi}Y_0 + \frac{i}{2}\sin(2\chi)e^{i\chi}(3Y+Y_0); \\ \zeta &= e^{-i\chi}Y + \frac{i}{2}\sin(2\chi)e^{i\chi}(3Y+Y_0)\,. \end{aligned} \tag{10.60}$$

The action of $\mathrm{U}(1)_{\mathrm{diag}} \subset \mathrm{U}(1)^3$ implies that

$$e^{i\chi} = \frac{Y-Y_0}{\zeta-\zeta_0}. \tag{10.61}$$

As pointed out above, in order to match (10.51)–(10.52) with (10.53)–(10.54), we are interested in finding the phases of these parameters in terms of χ entering (10.50). Therefore, we can solve for $\tan\chi$, as we read from (10.55) and (10.56):

$$\eta_0 = \tan\psi(\zeta_0) = \frac{1}{[\tan\psi(\zeta)]^3}, \tag{10.62}$$

where $\psi(\zeta_0)$ and $\psi(\zeta)$ respectively denote the phases of ζ_0 and ζ.

From (10.60), one obtains

$$\tan\psi(\zeta_0) = \frac{1}{\tau^3}\frac{Y_{0I} - \tau^3 Y_{0R} - 3\tau^2 Y_I + 3\tau Y_R}{Y_{0I} + \frac{1}{\tau^3}Y_{0R} - \frac{3}{\tau^2}Y_I - \frac{3}{\tau}Y_R}; \tag{10.63}$$
$$\tan\psi(\zeta) = \frac{1}{\tau^3}\frac{Y_I - \tau^3 Y_R - \tau^2(2Y_I + Y_{0I}) + \tau(2Y_R + Y_{0R})}{Y_I + \frac{1}{\tau^3}Y_{0R} - \frac{1}{\tau^2}(2Y_I + Y_{0I}) - \frac{1}{\tau}(2Y_R + Y_{0R})}, \tag{10.64}$$

where

$$Y \equiv Y_R + iY_I, \quad Y_0 \equiv Y_{0R} + iY_{0I}, \quad \tau \equiv \tan\chi. \tag{10.65}$$

In order to find τ in terms of α, p^0, q_0, one needs to solve (10.62), which in virtue of (10.63)–(10.64) can be made explicit as

$$\frac{Y_{0I} - \tau^3 Y_{0R} - 3\tau^2 Y_I + 3\tau Y_R}{Y_{0I} + \frac{1}{\tau^3}Y_{0R} - \frac{3}{\tau^2}Y_I - \frac{3}{\tau}Y_R} = \tau^{12}\left[\frac{Y_I + \frac{1}{\tau^3}Y_{0R} - \frac{1}{\tau^2}(2Y_I + Y_{0I}) - \frac{1}{\tau}(2Y_R + Y_{0R})}{Y_I - \tau^3 Y_R - \tau^2(2Y_I + Y_{0I}) + \tau(2Y_R + Y_{0R})}\right]^3. \tag{10.66}$$

Further simplifications are possible. Indeed, by recalling (10.52), the dependence of (10.63)–(10.64) on $\alpha, e^{\phi}, p^0, q^0$ can be made manifest:

$$\begin{aligned}\eta_0 &= \tan\psi(\zeta_0) = \frac{-q_0 \tan\chi^3 + p^0 e^{6\phi}(1-\tan\chi\;\alpha)^3}{q_0 + p^0 e^{6\phi}(\tan\chi+\alpha)^3} \\ &= -\tan\chi^3 \frac{1-\frac{p^0}{q_0}e^{6\phi}(\frac{1}{\tan\chi}-\alpha)^3}{1+\frac{p^0}{q_0}e^{6\phi}(\tan\chi+\alpha)^3}; \end{aligned} \tag{10.67}$$

$$\begin{aligned}\tan\psi(\zeta) &= \frac{-q_0\tan\chi + p^0 e^{6\phi}(\tan\chi+\alpha)^2(1-\tan\chi\;\alpha)}{q_0\tan\chi^2 + p^0 e^{6\phi}(\tan\chi+\alpha)(1-\tan\chi\;\alpha)^2} \\ &= -\frac{1}{\tan\chi}\,\frac{1-\frac{p^0}{q_0}e^{6\phi}(\tan\chi+\alpha)^2(\frac{1}{\tan\chi}-\alpha)}{1+\frac{p^0}{q_0}e^{6\phi}(\tan\chi+\alpha)(\frac{1}{\tan\chi}-\alpha)^2}. \end{aligned} \tag{10.68}$$

As a consequence, (10.62) can be recast as

$$\frac{1-x^3}{1+y^3} = \frac{(1+x^2 y)^3}{(1-x\,y^2)^3}\,, \tag{10.69}$$

$$x \equiv \left(\frac{p^0}{q_0}\right)^{1/3} e^{2\phi}\left(\frac{1}{\tan\chi}-\alpha\right)\,, \qquad y \equiv \left(\frac{p^0}{q_0}\right)^{1/3} e^{2\phi}\,(\tan\chi+\alpha)\,, \tag{10.70}$$

and therefore solved for

$$x = y \qquad \text{or} \quad x \neq y\;,\quad xy = -1\,. \tag{10.71}$$

For real values of $\tan\chi$ the case $x = y$ is not allowed, so one is left with

$$xy = -1 \quad\Rightarrow\quad \left(\frac{p^0}{q_0}\right)^{2/3} e^{4\phi}\left[1-\alpha^2+\frac{2\alpha}{\tan 2\chi}\right] = -1\,. \tag{10.72}$$

Thus, the angle χ, which provides the $\mathrm{U}(1)_{\mathrm{diag}}$-rotation between the skew-eigenvalues of (10.51) and (10.53), is given by

$$\tan\chi = \frac{1}{2\nu^{2/3}\alpha}\left((1-\nu^{2/3}(\alpha^2+1)) \pm \sqrt{(1-\nu^{2/3}(\alpha^2+1))^2+4\nu^{2/3}}\right)\,, \tag{10.73}$$

$$\nu \equiv (p^0/q_0)e^{6\phi}. \tag{10.74}$$

For later convenience we explicite here the expression for χ

$$\chi = -\frac{1}{2}\arctan\left[\frac{2\alpha}{(\frac{q_0}{p^0})^{2/3}e^{4\phi}-1+\alpha^2}\right]\;; \tag{10.75}$$

we also recall the choice of $q_0 > 0$, $p^0 > 0$, in our computation.

10.1.4 Duality invariants

One can also relate the parameters entering the solution (10.73) to the duality invariants $\mathcal{I}_4$, i_1, i_2 and i_3 defined e.g. in [40]. Using the relations (3.6)-(3.10) of [22], one finds

$$\alpha = \frac{b}{3\sqrt{-\mathcal{I}_4}}\;; \tag{10.76}$$

$$(q^0)^2 e^{-6\phi} = \frac{1}{(-\mathcal{I}_4)}\left(4i_3\sqrt{-\mathcal{I}_4} \pm \sqrt{b^6 - \mathcal{I}_4(3b^4+16i_3^2)+3b^2(-\mathcal{I}_4)^2-\mathcal{I}_4^3}\right), \tag{10.77}$$

where $i_2 = b + 3i_1$, and the "±" choice has to be consistent with the positivity of $e^{6\phi}$. We notice that α is a duality invariant quantity by itself, as well as the combinations $q_0 e^{-3\phi}$ and $p^0 e^{3\phi}$ (recall $\sqrt{-\mathcal{I}_4} = p^0 q_0$). Thus, the expression (10.73) is explicitly duality invariant.

10.2 Recovering the non-BPS fake superpotential

In [26] it is shown that the non-BPS fake superpotential is given by

$$W = 2\rho, \tag{10.78}$$

where ρ enters the expression (10.46). From the same equation, one can also write μ_0 as

$$\mu_0 = 2\rho(-\sin\eta^3 + i\cos\eta^3)\,, \tag{10.79}$$

thus yielding

$$W = 2\rho = \frac{\mathrm{Im}\mu_0}{\cos\eta^3} \equiv \frac{\mathrm{Im}\zeta_0}{\cos\eta^3}\,. \tag{10.80}$$

Moreover, (10.52) and (10.60) imply

$$\begin{aligned} \mathrm{Im}\zeta_0 &= -\frac{1}{\sqrt{2}} e^{-3\phi}\cos\chi^3\left(q_0\tan\chi^3 - e^{6\phi}p^0(1-\tan\chi\,\alpha)^3\right) = \\ &= -\frac{1}{\sqrt{2}} e^{-3\phi} q_0 \sin\chi^3\left(1 - \nu\left(\frac{1}{\tan\chi} - \alpha\right)^3\right)\,. \end{aligned} \tag{10.81}$$

By using

$$\sin\chi^3 = \frac{\tan\chi^3}{(1+\tan\chi^2)^{3/2}}\,, \qquad \frac{1}{\cos\eta^3} = (1+\tan\phi(\zeta)^2)^{3/2}\,, \tag{10.82}$$

and (10.67)–(10.68), (10.62) and (10.71) yield

$$\nu^{2/3}(1/\tau - \alpha)(\tau + \alpha) = -1\,, \tag{10.83}$$

and one can rewrite

$$\begin{aligned} \tan\psi(\zeta) &= -\frac{1}{\tau}\frac{1+\nu^{1/3}(\tau+\alpha)}{1-\nu^{1/3}(1/\tau-\alpha)} \\ &\Downarrow \\ \frac{1}{\cos\phi(\zeta)^3} &= \frac{1}{\tau^3}\frac{\left((1-\nu^{1/3}(1/\tau-\alpha))^2 + (1+\nu^{1/3}(\tau+\alpha))^2\right)^{3/2}}{(1-\nu^{1/3}(1/\tau-\alpha))^3} = \\ &= \frac{(1+\tau^2)^{3/2}}{\tau^3}\frac{(1+2\alpha\nu^{1/3}+\nu^{2/3}(\alpha^2+1))^{3/2}}{(1-\nu^{1/3}(1/\tau-\alpha))^3}; \end{aligned} \tag{10.84}$$

$$\mathrm{Im}\zeta_0 = -q_0 e^{-3\phi}\frac{\tau^3}{(1+\tau^2)^{3/2}}\left(1 - \nu\,(1/\tau-\alpha)^3\right)\,. \tag{10.85}$$

Therefore, the non-BPS fake superpotential W (10.80) is given by

$$W = -\frac{1}{\sqrt{2}} q_0 e^{-3\phi}\frac{\left(1-\nu\,(1/\tau-\alpha)^3\right)}{\left(1-\nu^{1/3}\,(1/\tau-\alpha)\right)^3}(1+2\alpha\nu^{1/3}+\nu^{2/3}(\alpha^2+1))^{3/2}\,. \tag{10.86}$$

Substituting the expression of $\tau \equiv \tan\chi$ as in (10.73), one finds that

$$\frac{\left(1-\nu\left(1/\tau-\alpha\right)^3\right)}{\left(1-\nu^{1/3}\left(1/\tau-\alpha\right)\right)^3} = \frac{1-\alpha\nu^{1/3}+\nu^{2/3}(\alpha^2+1)}{1+2\alpha\nu^{1/3}+\nu^{2/3}(\alpha^2+1)}, \tag{10.87}$$

which yields the following explicit expression:

$$\begin{aligned} W &= -\frac{1}{\sqrt{2}} q_0 e^{-3\phi}\sqrt{1+2\alpha\nu^{1/3}+\nu^{2/3}(\alpha^2+1)}\left(1-\alpha\nu^{1/3}+\nu^{2/3}(\alpha^2+1)\right) = \\ &= -\frac{1}{\sqrt{2}} e^{-3\phi}\sqrt{(q_0^{1/3}+(p^0)^{1/3}\alpha e^{2\phi})^2+e^{4\phi}(p^0)^{2/3}} \ \cdot \\ &\qquad \cdot\left((q_0^{1/3}+(p^0)^{1/3}\alpha e^{2\phi})^2+e^{4\phi}(p^0)^{2/3}-3(q_0p^0)^{1/3}\alpha e^{2\phi}\right). \end{aligned} \tag{10.88}$$

Notice that the overall minus in (10.88) is totally irrelevant, since it can be eliminated with a $\mathrm{U}(1)_{\mathrm{diag}}$-rotation through the matrix $-\epsilon\otimes id_4$.

Equation (10.88), up to a factor of 1/2, coincides with the formula of the non-BPS fake superpotential for the (p^0, q_0) configuration in the t^3 model computed in [22]. The difference of a factor 1/2 is simply due to the different normalization used for the normal form central charge in our notation (which coincides, for example, with the one in eq. (3.13) of [15]) with respect to the one used in [26], as one can read from eq. (2.11) therein. This implies that the correct identification would be $\mathrm{Im}\mu_0 = \frac{1}{2}\mathrm{Im}\zeta_0$. Consequently, the correctly normalized fake superpotential becomes finally

$$\begin{aligned} W = \frac{1}{2\sqrt{2}} e^{-3\phi}\sqrt{(q_0^{1/3}+(p^0)^{1/3}\alpha e^{2\phi})^2+e^{4\phi}(p^0)^{2/3}} \ \cdot \\ \cdot\left((q_0^{1/3}+(p^0)^{1/3}\alpha e^{2\phi})^2+e^{4\phi}(p^0)^{2/3}-3(q_0p^0)^{1/3}\alpha e^{2\phi}\right). \end{aligned} \tag{10.89}$$

This computation is a non-trivial consistency check for the formalism based on the axion-independent matrices M and $\widehat{M}$ introduced in sections 8 and 9, as well as for the results on the phase χ obtained above.

Acknowledgments

We are glad to thank R. Kallosh for enlightening discussions and remarks. We also acknowledge P. Aschieri and G. Bossard for useful discussions. The work of A.C. and S.F. was supported by the European ERC Advanced Grant no. 226455 "Supersymmetry, Quantum Gravity and Gauge Fields" (SUPERFIELDS). The work of A.C. is also supported by the Italian MIUR-PRIN contract 2009KHZKRX-007 "Symmetries of the Universe and of the Fundamental Interactions". The work of A.G. has been supported by the Padova University Project CPDA105015/10.

JHEP02(2013)059

J_3	$\frac{G_4}{H_4}$	$\frac{G_5}{H_5}$	q	N
$J_3^{\mathbb{O}}$	$\frac{E_{7(7)}}{\mathrm{SU}(8)}$	$\frac{E_{6(6)}}{\mathrm{USp}(8)}$	8	8
$J_3^{\mathbb{O}_s}$	$\frac{E_{7(-25)}}{E_{6(-78)}\times\mathrm{U}(1)}$	$\frac{E_{6(-26)}}{F_{4(-52)}}$	8	2
$J_3^{\mathbb{H}}$	$\frac{\mathrm{SO}^*(12)}{\mathrm{SU}(6)\times\mathrm{U}(1)}$	$\frac{\mathrm{SU}^*(6)}{\mathrm{USp}(6)}$	4	2 or 6
$J_3^{\mathbb{H}_s}$	$\frac{\mathrm{SO}(6,6)}{\mathrm{SO}(6)\times\mathrm{SO}(6)}$	$\frac{\mathrm{SL}(6,\mathbb{R})}{\mathrm{SO}(6)}$	4	0
$J_3^{\mathbb{C}}$	$\frac{\mathrm{SU}(3,3)}{\mathrm{SU}(3)\times\mathrm{SU}(3)\times\mathrm{U}(1)}$	$\frac{\mathrm{SL}(3,\mathbb{C})}{\mathrm{SU}(3)}$	2	2
$J_3^{\mathbb{C}_s}$	$\frac{\mathrm{SL}(6,\mathbb{R})}{\mathrm{SO}(6)}$	$\left[\frac{\mathrm{SL}(3,\mathbb{R})}{\mathrm{SO}(3)}\right]^2$	2	0
$J_3^{\mathbb{R}}$	$\frac{\mathrm{Sp}(6,\mathbb{R})}{\mathrm{SU}(3)\times\mathrm{U}(1)}$	$\frac{\mathrm{SL}(3,\mathbb{R})}{\mathrm{SO}(3)}$	1	2
$\mathbb{R}$ (t^3 model)	$\frac{\mathrm{SL}(2,\mathbb{R})}{\mathrm{U}(1)}$	–	$-2/3$	2
$\mathbb{R}\oplus\mathbf{\Gamma}_{m-1,n-1}$	$\frac{\mathrm{SL}(2,\mathbb{R})}{\mathrm{U}(1)}\times\frac{\mathrm{SO}(m,n)}{\mathrm{SO}(m)\times\mathrm{SO}(n)}$	$\mathrm{SO}(1,1)\times\frac{\mathrm{SO}(m-1,n-1)}{\mathrm{SO}(m-1)\times\mathrm{SO}(n-1)}$	$(m+n-4)/3$	2 (m or $n=2$) 4 (m or $n=6$) 0 otherwise

Table 1. Rank-3 Euclidean Jordan algebras J_3, and corresponding symmetric scalar manifolds for vector multiplets in $D=4$ and $D=5$, with the parameter q and the number of supersymmetries N.

A Some results on exponential matrices

Let us recall the decomposition (4.2):

$$\mathcal{A}=\begin{pmatrix}\mathbb{1} & 0\\ \mathrm{Re}\mathcal{N} & \mathbb{1}\end{pmatrix}\left(\begin{array}{cc|cc}1 & 0 & 0 & 0\\ a^I & 1 & 0 & 0\\ \hline 0 & 0 & 1 & -a^J\\ 0 & 0 & 0 & 1\end{array}\right)=(\mathcal{R})^{-1}\mathcal{A}_D(a^I)\,, \tag{A.1}$$

where $\mathcal{A}(a)=\exp(T(a))$ (cfr. (2.7)).

Thus, by defining

$$\mathcal{A}_D\equiv\exp(T_D),\ \ \mathcal{R}\equiv\exp(T_{\mathcal{R}}), \tag{A.2}$$

and

$$T(a)\ =T_D(a)+T_d(a,d); \tag{A.3}$$

$$T_D(a)\equiv\left(\begin{array}{cc|cc}0 & 0 & 0 & 0\\ a^I & 0 & 0 & 0\\ \hline 0 & 0 & 0 & -a^J\\ 0 & 0 & 0 & 0\end{array}\right),\quad T_d(a,d)\equiv\left(\begin{array}{cc|cc}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ \hline 0 & 0 & 0 & 0\\ 0 & 0 & d_{IJ} & 0\end{array}\right), \tag{A.4}$$

one obtains that

$$\mathcal{A}(a)=\exp[T_d+T_D]=\exp[-T_{\mathcal{R}}]\cdot\exp[T_D]\ , \tag{A.5}$$

with

$$T_{\mathcal{R}}(d) \equiv \begin{pmatrix} 0 & 0 \\ -\mathrm{Re}\mathcal{N} & 0 \end{pmatrix} . \tag{A.6}$$

This allows us to describe how the matrix $\mathrm{Re}\mathcal{N}$ is constructed from the algebra perspective, as

$$\begin{pmatrix} 1\!\!1 & 0 \\ -\mathrm{Re}\mathcal{N} & 1\!\!1 \end{pmatrix} \equiv \mathcal{R} = \exp\left[a^I(\hat{T}_D)_I\right] \exp\left[-a^I((\hat{T}_D)_I + (\hat{T}_d)_I)\right] ,$$

where the generators

$$(\hat{T}_D)_I = \frac{\partial}{\partial a^I} T_D , \qquad (\hat{T}_d)_I = \frac{\partial}{\partial a^I} T_d$$

do not depend on the axions, since

$$(\hat{T}_D)_I \equiv \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ \delta_I^J & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -\delta_I^J \\ 0 & 0 & 0 & 0 \end{array}\right) , \qquad (\hat{T}_d)_I \equiv \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & d_{IJK} & 0 \end{array}\right) .$$

B M and $\mathcal{Y}$ in the t^3 model

We now explicitly compute the matrices M (8.23) and $\mathcal{Y}$ (8.25) for the special geometry defined by the holomorphic prepotential

$$F = \frac{(X^1)^3}{X^0}, \tag{B.1}$$

corresponding to the t^3 model of $N = 2$, $D = 4$ supergravity, where the unique complex scalar field is defined as

$$\frac{X^1}{X^0} \equiv t = a - i\lambda. \tag{B.2}$$

In this model, which uplifts to $N = 2$, $D = 5$ "pure" supergravity (thus with no scalars in $D = 5$), the matrices M (8.23) and $\mathcal{Y}$ (8.25) are simply numerical matrices.

From the analysis of [7], it follows that

$$\partial_i K = 6\lambda^2 , \qquad g_{t\bar{t}} = 12\lambda , \tag{B.3}$$

with $\lambda = e^{2\phi}$. Since

$$a_{t\bar{t}} = \frac{1}{4} g_{t\bar{t}} e^{-4\phi} , \tag{B.4}$$

it then follows that

$$(g^{1/2})_t^{\bar{t}} = 2\sqrt{3}e^{2\phi} , \qquad (a^{1/2})_t^{\bar{t}} = \sqrt{3}. \tag{B.5}$$

Thus, the matrices M (8.23) and $\mathcal{Y}$ (8.25) can be computed to be

$$M = \frac{1}{2} \begin{pmatrix} 1 & \sqrt{3}i \\ -\sqrt{3}i & -1 \end{pmatrix} = \sin(\theta_{t^3})\,\sigma_3 + \cos(\theta_{t^3})\,\sigma_2; \tag{B.6}$$

$$\mathcal{Y} = \frac{1}{2} \left(\begin{array}{cc|cc} 1 & 0 & 0 & -\sqrt{3} \\ 0 & -1 & \sqrt{3} & 0 \\ \hline 0 & \sqrt{3} & 1 & 0 \\ -\sqrt{3} & 0 & 0 & -1 \end{array}\right) , \tag{B.7}$$

where σ_2 and σ_3 are the Pauli σ-matrices (such that the constraints (8.28) are trivially satisfied), and

$$\theta_{t^3} = \frac{\pi}{6}, \tag{B.8}$$

such that (cfr. (B.5))

$$(a^{-1/2})_{\bar{t}}^{t} = \tan(\theta_{t^3}) \,. \tag{B.9}$$

C On the complex *Vielbein* for the *stu* parametrization of $N = 8$ supergravity

The "*stu* parametrization" of $N = 8$, $D = 4$ supergravity is based on the following correspondence between the skew-eigenvalues of the $N = 8$ central charge matrix Z_{AB} and the (flattened) scalar-dressed charges of the $N = 2$, $D = 4$ *stu* model, which is a common sector of all rank-3 symmetric special Kähler geometries [15, 22, 23]:

$$Z_{AB} = \begin{pmatrix} z_1\epsilon & 0 & 0 & 0 \\ 0 & z_2\epsilon & 0 & 0 \\ 0 & 0 & z_3\epsilon & 0 \\ 0 & 0 & 0 & z_4\epsilon \end{pmatrix} = \\ = \begin{pmatrix} Z\epsilon & 0 & 0 & 0 \\ 0 & -i(g^{s\bar{s}})^{1/2}\bar{D}_{\bar{s}}\bar{Z}\epsilon & 0 & 0 \\ 0 & 0 & -i(g^{t\bar{t}})^{1/2}\bar{D}_{\bar{t}}\bar{Z}\epsilon & 0 \\ 0 & 0 & 0 & -i(g^{u\bar{u}})^{1/2}\bar{D}_{\bar{u}}\bar{Z}\epsilon \end{pmatrix} . \tag{C.1}$$

The square root of $g_{i\bar{j}}$ can in principle be chosen with real entries as

$$(g_{s\bar{s}})^{1/2} = \pm\frac{i}{s-\bar{s}} \,, \tag{C.2}$$

and analogously for the $t\bar{t}$ and $u\bar{u}$ components of $g^{1/2}$. Thus, in this symplectic frame, the rank-3 C-tensor reads

$$C_{stu} = \frac{i}{(s-\bar{s})(t-\bar{t})(u-\bar{u})} \tag{C.3}$$

can be written as

$$C_{stu} = \mp\,(g_{s\bar{s}})^{1/2}(g_{t\bar{t}})^{1/2}(g_{u\bar{u}})^{1/2} \,, \tag{C.4}$$

consistent with the choice made in (C.2). This choice affects the attractor equations since

$$\begin{aligned} \bar{Z}D_tZ &= -iC_{stu}g^{s\bar{s}}g^{u\bar{u}}\bar{D}_{\bar{s}}\bar{Z}\bar{D}_{\bar{u}}\bar{Z} = \\ &= (\mp)(-i)(g_{t\bar{t}})^{1/2}(g^{s\bar{s}})^{1/2}(g^{u\bar{u}})^{1/2}\bar{D}_{\bar{s}}\bar{Z}\bar{D}_{\bar{u}}\bar{Z} \,, \\ &\Downarrow \\ Z(g^{t\bar{t}})^{1/2}\bar{D}_{\bar{t}}\bar{Z} &= \mp i(g^{s\bar{s}})^{1/2}D_sZ\,(g^{u\bar{u}})^{1/2}D_uZ \,, \end{aligned} \tag{C.5}$$

which, using the notations of (C.1), can be recast as

$$z_1z_3 = \pm\overline{z_2z_4} \,, \tag{C.6}$$

where only the choice "$-$" allows the attractor equation from special geometry to be embedded into the $N = 8$ theory. Thus, we are lead to choose the minus sign in (C.2), and correspondingly the *Vielbein* is fixed to be purely imaginary:

$$\mathbf{e} = -i\mathbf{g}^{1/2} = \begin{pmatrix} (s-\bar{s})^{-1} & 0 & 0 \\ 0 & (t-\bar{t})^{-1} & 0 \\ 0 & 0 & (u-\bar{u})^{-1} \end{pmatrix} = -\bar{\mathbf{e}}. \tag{C.7}$$

D U-duality invariants for the $D0 - D6$ $i_3 = 0$ configuration

Following the definitions in [23, 40], one can write the following U-duality invariant expressions in stu model within the (p^0, q_0) configuration with $i_3 = 0$ (recall (C.1) and (10.26)):

$$\begin{aligned} i_1 &= |Z|^2 = 2e^{-6\phi} q_0 \left[q_0 + p^0 \alpha_1\alpha_2\alpha_3 - e^{4\phi} p^0 (\alpha_1 + \alpha_2 + \alpha_3) \right] ; \\ i_2^s &= |D_s Z|^2 = 2e^{-6\phi} q_0 \left[q_0 + p^0 \alpha_1\alpha_2\alpha_3 + e^{4\phi} p^0 (-\alpha_1 + \alpha_2 + \alpha_3) \right] ; \\ i_2^t &= |D_t Z|^2 = 2e^{-6\phi} q_0 \left[q_0 + p^0 \alpha_1\alpha_2\alpha_3 + e^{4\phi} p^0 (\alpha_1 - \alpha_2 + \alpha_3) \right] ; \\ i_2^u &= |D_u Z|^2 = 2e^{-6\phi} q_0 \left[q_0 + p^0 \alpha_1\alpha_2\alpha_3 + e^{4\phi} p^0 (\alpha_1 + \alpha_2 - \alpha_3) \right] . \end{aligned} \tag{D.1}$$

It is worth remarking that that these four invariants collapse to a single one, in the axionless case ($\alpha_i \equiv a^i/\lambda^i = 0$).

The black hole potential for this system is given in terms of the invariants by

$$V_{BH} = i_1 + i_2^s + i_2^t + i_2^u \,, \tag{D.2}$$

and it admits the fake superpotential [22, 26, 41]

$$W = \frac{1}{2}\left(\sqrt{i_1} + \sqrt{i_2^s} + \sqrt{i_2^t} + \sqrt{i_2^u} \right) ; \tag{D.3}$$

this case is usually referred to as the non-BPS "doubly-extremal" phase. Actually, one can show that (D.3) satisfies

$$V_{BH} = W^2 + 4 g^{i\bar{j}} \partial_i W \bar{\partial}_{\bar{j}} W \tag{D.4}$$

only in the case $i_3 = 0$. Indeed, by their very definitions, using the special geometry relations (cfr. e.g. eqs. (2.24)–(2.26) of [23])

$$\begin{aligned} D_s i_1 &= D_s i_2^s = \bar{Z} D_s Z \,; \\ D_s i_2^t &= D_s i_2^u = i C_{stu} g^{t\bar{t}} g^{u\bar{u}} D_{\bar{t}} \bar{Z} D_{\bar{u}} \bar{Z} \,, \end{aligned} \tag{D.5}$$

as well as the analogous ones concerning derivatives with respect to the scalars t and u, and by recalling that (recall (C.4))

$$C_{stu}{}^2 = g_{s\bar{s}}\, g_{t\bar{t}}\, g_{u\bar{u}} \,,$$

one can compute that

$$4D_sW\overline{D}_{\bar{s}}Wg^{s\bar{s}} = \frac{1}{4}\Bigg[(\sqrt{i_1}+\sqrt{i_2^s})^2+(\sqrt{i_2^t}+\sqrt{i_2^u})^2+ \\ +i\frac{(\sqrt{i_1}+\sqrt{i_2^s})(\sqrt{i_2^t}+\sqrt{i_2^u})}{\sqrt{i_1 i_2^s i_2^t i_2^u}}(z_1z_2z_3z_4-\bar{z}_1\bar{z}_2\bar{z}_3\bar{z}_4)\Bigg] . \tag{D.6}$$

By definition (cfr. e.g. (1.12) of [22])

$$i(z_1z_2z_3z_4-\bar{z}_1\bar{z}_2\bar{z}_3\bar{z}_4) = i_4 \Rightarrow i_4 = -\sqrt{4i_1i_2^si_2^ti_2^u - i_3^2}, \tag{D.7}$$

thus

$$V_{BH} = W^2 + 4g^{i\bar{j}}\partial_iW\partial_{\bar{j}}W+ \tag{D.8}$$
$$-\left(\sqrt{i_1i_2^s}+\sqrt{i_1i_2^t}+\sqrt{i_1i_2^u}+\sqrt{i_2^si_2^t}+\sqrt{i_2^si_2^u}+\sqrt{i_2^ti_2^u}\right)\left(1-\sqrt{1-\frac{i_3^2}{4i_1i_2^si_2^ti_2^u}}\right),$$

which gives the required relation (D.4) in the case $i_3 = 0$. We also notice that the expression (D.8) is non-singular, since none of the four invariants i_1, i_2^s, i_2^t, i_2^u vanishes for this solution.

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JHEP02(2013)059