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INFRA-RED ASYMPTOTIC BEHAVIOUR OF ONE-FERMION GREEN'S FUNCTION IN A SCALAR MODEL WITH ISOSPIN

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ABSTRACT

In a theory where massive fermions interact with massless scalar field of isospin 1, the behaviour of the one-fermion Green's function is found to differ from the free Green's function by a factor

$$\left(1-\frac{2}{\pi^2}\, g^2\, \ln m\,|x-y|\right)^{-3/8}$$

in the limit of large separation $|x-y|$.

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It was found by Abrikosov, Landau and Khalatnikov [1] that the one-electron Green's function in quantum electrodynamics, in the Landau gauge, behaves like

$$(1+p^2/m^2)^{-1-\frac{3e^2}{8\pi^2}} \tag{1}$$

near the mass shell $p^2+m^2=0$.

The answer to the corresponding problem for Yang-Mills theory [2] is not known, because of the non-linear self-interaction and the isospin structure of the interaction with the fermion field. Thus it may be interesting to look for simple models which have a non-trivial infra-red asymptotic behaviour different from that of QED.

In this note, we consider the interaction of a massive fermion ψ of isospin $\frac{1}{2}$ with a massless scalar boson φ_a of isospin 1. More precisely, the action functional in Euclidean space is

$$S=-\int d^4x\left[\bar{\psi}(\gamma_\mu\partial_\mu+m+g\tau_a\varphi_a)\psi+\tfrac{1}{2}\partial_\mu\varphi_a\partial_\mu\varphi_a\right]. \tag{2}$$

For this theory, we shall find that in the Euclidean region

$$G(x-y)=\langle\psi(x)\bar{\psi}(y)\rangle\sim G_0(x-y)\left(1+\frac{2g^2}{\pi^2}\ln m|x-y|\right)^{-3/8} \tag{3}$$

for large $|x-y|$. In principle, the corresponding formula in momentum space can be obtained by Fourier transform but the appearance of the logarithmic factor makes the result less explicit.

We can come to (3) in different ways. It was first obtained by summing the perturbation series. However, it is more elegant to derive (3) by the functional integral approach, where

$$\langle\psi(x)\bar{\psi}(y)\rangle=\frac{\int\psi(x)\bar{\psi}(y)\,e^S\prod_x d\bar{\psi}(x)\,d\psi(x)\prod_a d\varphi_a(x)}{\int e^S\prod_x d\bar{\psi}(x)\,d\psi(x)\prod_a d\varphi_a(x)}. \tag{4}$$

After the integration over the fermion fields, we obtain

$$\langle\psi(x)\bar{\psi}(y)\rangle = \frac{\int G(x,y|\varphi)\, e^{S_0[\varphi]} \det[I + g(\gamma_\mu \partial_\mu + m)^{-1} \tau_a \varphi_a] \prod_{x,a} d\varphi_a(x)}{\int e^{S_0[\varphi]} \det[I + g(\gamma_\mu \partial_\mu + m)^{-1} \tau_a \varphi_a] \prod_{x,a} d\varphi_a(x)}, \tag{5}$$

where $S_0[\varphi]$ is the actional functional of the free φ field, and $G(x,y|\varphi)$ is the classical one-fermion Green's function in an external field φ.

Let the φ field be separated into the "slow" and "fast" parts

$$\varphi_a(x) = \varphi_a^{(0)}(x) + \varphi_a^{(1)}(x),$$

where the Fourier transform $\tilde{\varphi}_a^{(0)}(k)$ of $\varphi_a^{(0)}(x)$ vanishes for $|k| > k_0$, while that of $\varphi_a^{(1)}(x)$ vanishes for $|k| \leq k_0$, k_0 being a fixed momentum value. We apply the following approximations [3),4)], previously used in QED, to the right-hand side of (5)

$$\det[I + g(\gamma_\mu \partial_\mu + m)^{-1} \tau_a \varphi_a] \sim 1 \tag{6}$$

and

$$G(x,y|\varphi) \sim G(x,y|\varphi^{(1)})\, T \exp\left(g \int_x^y \tau_a \varphi_a^{(0)}(z)\, dz\right), \tag{7}$$

where the symbol T denotes ordering the exponential along a straight line between x and y. With these approximations, the desired Green's function is given by

$$G(x-y) \sim \left\langle T \exp g \int_x^y \tau_a \varphi_a^{(0)}(z)\, dz \right\rangle_0 \left\langle G(x,y|\varphi^{(1)}) \right\rangle_1, \tag{8}$$

where $\langle\ \rangle_0$ and $\langle\ \rangle_1$ are functional averages with the weight factors $e^{S_0[\varphi^{(0)}]}$ and $e^{S_0[\varphi^{(1)}]}$, respectively.

The non-trivial long range behaviour is due to the first "infra-red" factor in (8). The second one is a quantum Green's function of a fermion interacting only with the "fast" field $\varphi_a^{(1)}$. In p space, it is equal to

$$a_{k_0}(m_{k_0} + i\not{p})(m_{k_0}^2 + p^2)^{-1} \tag{9}$$

near the mass shell $p^2 = -m^2$. Here

$$m_{k_0} = m + \Delta m\,(k_0) \tag{10}$$

is a renormalized fermion mass, a_{k_0} is a renormalization constant of one-fermion Green's functions. In x space we will have

$$\langle G(x,y\,|\,\varphi^{(1)})\rangle_1 \sim G_0(x-y)\, a_{k_0}\, \exp\,(-\Delta m(k_0)\, r), \tag{11}$$

where $r = |x-y|$.

Using the notation

$$\langle\langle F(u)\rangle\rangle = \langle T\, F(u)\, \exp g \int_x^y \tau_a\, \varphi_a(\xi)\, d\xi \rangle\, a_{k_0}\, \exp(-\Delta m(k_0) r), \tag{12}$$

we can write

$$G(x-y) = G_0(x-y)\, \langle\langle 1 \rangle\rangle = G_0(x-y)\, f(r). \tag{13}$$

The function $f(r)$ must not depend on a parameter k_0 at all. We will obtain a differential equation for $f(r)$ and solve it in the "leading log region"

$$g^2 \ll 1 \qquad\qquad g^2 \log mr \sim 1. \tag{14}$$

First of all, we have

$$\begin{aligned} \frac{df}{dr} &= -\Delta m\, f + g\, \langle\langle \tau_a\, \varphi_a(r) \rangle\rangle \\ &= -\Delta m\, f + g^2 \int_0^r \mathcal{D}(\xi)\, \langle\langle \tau_a\, \tau_a(\xi) \rangle\rangle\, d\xi \end{aligned} \tag{15}$$

where

$$\mathcal{D}(z) = \frac{1}{(2\pi)^4} \int\limits_{|k|<k_0} \frac{e^{ikq}}{k^2} d^4k = \frac{1 - J_0(k_0 z)}{4\pi^2 z^2} \tag{16}$$

is a Green's function of a "slow" field. Let us suppose for a moment that τ_a matrices commute with each other. In this case we will have $\langle\langle \tau_a \tau_a(z)\rangle\rangle =$ $= \tau_a \tau_a \langle\langle 1 \rangle\rangle = 3f$ and come to the equation

$$\frac{df}{dr} = \left(-\Delta m + 3g^2 \int_0^r \mathcal{D}(z)\, dz\right) f,$$

$$\int_0^r \mathcal{D}(z)\, dz = \int_0^\infty \mathcal{D}(z)\, dz - \int_r^\infty \mathcal{D}(z)\, dz \sim (4\pi^2)^{-1}(k_0 - r^{-1}).$$

The k_0 term must cancel Δm, and we obtain the equation

$$\frac{df}{dr} = -\frac{3g^2}{4\pi^2 r} f.$$

Its solution

$$f = \exp\left(-\frac{3g^2}{4\pi^2} \log mr\right)$$

is similar to that in QED.

The non-commutativity of τ_a matrices changes the situation radically. We can take it into account, integrating by parts

$$\int_0^r \mathcal{D}(z) \langle\langle \tau_a \tau_a(z)\rangle\rangle\, dz = 3\left[u(0) - u(z)\right] f - \int_0^r dz\, u(z) \frac{d}{dz} \langle\langle \tau_a \tau_a(z)\rangle\rangle,$$

$$u(z) = \int_z^\infty \mathcal{D}(t)\, dt. \tag{17}$$

We have

$$\frac{d}{dz} \langle\langle \tau_a \tau_a(z)\rangle\rangle = g \langle\langle \tau_a [\tau_a, \tau_b \varphi_b(z)]\rangle\rangle = 2ig\, \varepsilon_{abc} \langle\langle \tau_a \varphi_b(z) \tau_c(z)\rangle\rangle$$

$$= 2ig^2 \varepsilon_{abc} \int_0^r dz_1\, \mathcal{D}(z - z_1) \langle\langle \tau_a \tau_b(z_1) \tau_c(z)\rangle\rangle. \tag{18}$$

So we arrive at the equation

$$\frac{df}{dr} = [-\Delta m + 3g^2(u(0) - u(r))]f$$

$$+ 2ig^4\varepsilon_{abc}\int_0^r\int_0^r dz\, dz_1\, u(z)\, D(z-z_1)\, \langle\langle \tau_a \tau_b(z_1)\tau_c(z)\rangle\rangle. \qquad (19)$$

The second term in the right-hand side may be written in the following form

$$ig^4\varepsilon_{abc}\int_0^r\int_0^r dz\, dz_1\, [u(z) - u(z_1)]\, D(z-z_1)\, \langle\langle \tau_a \tau_b(z_1)\tau_c(z)\rangle\rangle. \qquad (20)$$

We have

$$[u(z) - u(z_1)]\, D(z-z_1) \sim (4\pi^2)^{-2}\, [z z_1 (z_1 - z)]^{-1} \qquad (21)$$

if z, z_1, $|z-z_1|$ are much larger than k_0^{-1}.

In the "leading log region" (14) the main contributions in the double integral (20) are due to the domains

$$k_0^{-1} \ll |z - z_1| \ll r, \quad k_0^{-1} \ll z \ll r, \quad k_0^{-1} \ll z_1 \ll r. \qquad (22)$$

In the first of these domains we have

$$i\varepsilon_{abc}\langle\langle \tau_a \tau_b(z_1)\tau_c(z)\rangle\rangle \sim 2\,\varepsilon(z - z_1)\,\langle\langle \tau_a \tau_a(z)\rangle\rangle. \qquad (23)$$

The contribution to Eq. (20) is equal to

$$2g^4\int_{k_0}^r \frac{dz}{z^2}\langle\langle \tau_a \tau_a(z)\rangle\rangle \int \frac{dz_1}{|z - z_1|}. \qquad (24)$$

The inner integral in (24) is equal to

$$\int \frac{dz_1}{|z - z_1|} \sim 2 \log mr \qquad (25)$$

(with a logarithm accuracy).

Inner integrals for the two other domains (22) are equal to

$$\int \frac{d\zeta}{\zeta} = \int \frac{d\zeta_1}{\zeta_1} \sim \log\, mr \tag{26}$$

As a result, we can rewrite (20) as

$$\frac{8g^4 \log mr}{(4\pi^2)^2} \int_{k_0}^{r} \frac{d\zeta}{\zeta^2} \langle\langle \tau_a\, \tau_a(\zeta) \rangle\rangle . \tag{27}$$

According to Eq. (15) this expression is equal to

$$- \frac{2g^2}{\pi^2}(\log\, mr)\, \frac{df}{dr}$$

up to an addendum, which is proportional to $k_0 f$ and must cancel k_0 dependent terms in the equation. So we come to the equation

$$\frac{df}{dr} = - \frac{3g^2}{4\pi^2 r} f - \frac{2g^2}{\pi^2} (\log\, mr) \frac{df}{dr} . \tag{28}$$

Its solution

$$\left(1 + \frac{2g^2}{\pi^2} \log\, mr \right)^{-3/8} \tag{29}$$

implies (3) to be valid. For a theory with N φ fields, we have to replace $-3/8 \to -N/8$ in (29). It is interesting that here we have no leading-log exponentiation.

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