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TRANSVERSE MODES IN PERIODIC CYLINDRICAL CAVITIES

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ABSTRACT

Transverse forces acting on charged particles passing off-axis through resonant structures can lead to instabilities and beam break-up in high-energy particle accelerators or storage rings. These forces can be calculated from the resonant transverse modes, which are calculated here for a structure consisting of periodically repeating circular-cylindrical cavities connected by concentric side tubes. Results for the dipole mode are presented graphically for a wide variety of geometries, as well as Brillouin diagrams for m=1 and m=2.

1. INTRODUCTION

We investigate a perfectly conducting structure consisting of circular cylindrical cavities of radius b and length 2g, connected by concentric tubes of radius a, and repeated periodically with period $2\pi R$. The "longitudinal" i.e. axi-symmetric resonances have been obtained previously[1] by matching the expansion coefficients of the fields in cylindrical subregions. The "transverse" resonances, i.e. fields with azimuthal dependance, are complicated by the fact that they do no longer separate into the familiar TM and TE modes, but are in general "hybrid modes" for which all 6 field components are present. Further complications arise for the "synchronous case" when the phase-velocity of the travelling wave equals the light-velocity.

Nevertheless, the lowest deflecting (dipole) modes in such structures have been obtained in the past for the design of RF particle separators[2),3)]. For stability calculations, however, one needs a large number of transverse resonances, and the techniques had to be refined in order to obtain all resonances in a given frequency interval within reasonable time on a high-speed computer[4)].

The results of these computations are presented in graphical form as a function of cavity dimensions for the synchronous case, and in the form of Brillouin diagrams for a particular geometry. An application of this program to the Stanford Linear Accelerator structure is discussed in another paper at this Conference.

2. THE HERTZ VECTORS

Electro-magnetic resonances for a loss-less structure can be obtained by looking for the existence of fields in the absence of any excitation. In this case it is advantageous to derive the 6 electro-magnetic field components from the electric and magnetic Hertz vectors Π_E and Π_M, for which only the two axial components are non-zero in any uniform cylindrical region. The field components can then be found by pure differentiations

$$E = \text{curl curl } \Pi_E - \text{curl } \frac{1}{c}\frac{\partial \Pi_M}{\partial t},$$

$$Z_o H = \text{curl curl } \Pi_M + \text{curl } \frac{1}{c} \frac{\partial \Pi_E}{\partial t} . \tag{1}$$

We divide the geometry into subregions with uniform cross-section in the z-direction, one consisting of the infinite circular cylinder $r < a$ which can support travelling waves, and the others of the annular cylinders $a < r < b$, $-g + n\pi R < z < g + n\pi R$ in which only standing waves are possible. Using the Floquet condition for periodic structures, we need to calculate the fields only in one of these annular regions if we prescribe either the phase-shift per period or the phase velocity of the travelling wave.

The Hertz vectors are determined by the wave-equation, which reduces to the Helmholtz equation for a single frequency ω (or wave number $k = \frac{\omega}{c}$). We use cylindrical coordinates and expand the Hertz vectors into infinite series of product-solutions which fulfill a number of boundary conditions. For region I ($0<r<a$) we take for a given azimuthal mode-number m

$$\Pi^{I}_{\begin{Bmatrix}E\\M\end{Bmatrix}} = -\sum_{n=-\infty}^{\infty} \begin{Bmatrix}A_n\\B_n\end{Bmatrix} \frac{I_m(x_n r)}{x_n^2 I_m(x_n a)} e^{-j\beta_n z} \begin{Bmatrix}\cos\\ \sin\end{Bmatrix} m\theta \tag{2}$$

where I_m are modified Besselfunctions of order m. These expressions fulfill the Floquet-condition if

$$\beta_o = \beta_n + \frac{n}{R} \tag{3}$$

where β_o is the phase shift per cell divided by the period (or $\beta_o = \beta k$ for the synchronous case with $v = \beta c$). The Helmholtz equation requires

$$x_n^2 = \beta_n^2 - k^2 \tag{4}$$

The denominator $x_n^2 I_m(x_n a)$ is completely arbitrary and only taken for later convenience. In region II ($a < r < b$, $-g < z < g$) we take

$$\Pi^{II}_E = -\sum_{s=0}^{\infty} \frac{C'_s}{\Gamma_s^2} \frac{R_m(\Gamma_s r)}{R_m(\Gamma_s a)} \cos \alpha_s (z+g) \cos m\theta$$

$$\Pi^{II}_M = j\sum_{s=1}^{\infty} \frac{D'_s}{\Gamma_s^2} \frac{S_m(\Gamma_s r)}{S_m(\Gamma_s a)} \sin \alpha_s (z+g) \sin m\theta \tag{5}$$

where R_m and S_m are combinations of the modified Besselfunctions I_m and K_m which fulfil

$$R_m(\Gamma b) = S'_m(\Gamma b) = 0 \tag{6}$$

This satisfies the boundary conditions at $r = b$, and we have to take $\alpha_s = \frac{\pi s}{2g}$ (7)

in order to fulfil the boundary conditions at $z = \pm g$. The Helmholz equation is satisfied if

$$\Gamma_s^2 = \alpha_s^2 - k^2 \tag{8}$$

The four infinite sets of expansion coefficients A_n, B_n, C'_s, D'_s are still completely free and will be determined by field-matching at $r = a$. In order to obtain purely real equations, we take C, D equal C', D' for even s and equal jC', jD' for odd s.

3. FIELD MATCHING

At the common boundary of the two regions $r = a$, $|z| < g$ the 4 tangential field-components have to be equal. In addition, the tangential electric field components have to vanish at the (perfectly conducting) tube-wall $r = a$, $g < |z| < \pi R$. Using the orthogonality of $\exp(-i\beta_n z)$ in $(-\pi R, \pi R)$ we obtain

$$A_n = \alpha \sum_{s=0}^{\infty} N_{ns} C_s ,$$

$$P_n A_n + I_n B_n = \alpha \sum_{s=0}^{\infty} M_{ns} (Q_s C_s + S_s D_s) , \tag{9}$$

where

$$\alpha = \frac{g}{\pi R} \tag{10}$$

is the circumference factor,

$$\left.\begin{matrix} M_{ns} \\ N_{ns} \end{matrix}\right\} = \frac{\begin{Bmatrix} \alpha_s \\ \beta_n \end{Bmatrix}}{g(\beta_n^2 - \alpha_s^2)} \begin{cases} \sin\beta_n g ; & s \text{ even} \\ \cos\beta_n g ; & s \text{ odd} \end{cases} , \tag{11}$$

and

$$P_n = \frac{m\beta_n}{x_n^2 a} , \quad I_n = \frac{kI'_m}{x_n I_m}\Bigg|_{x_n a} ,$$

$$Q_s = \frac{m\alpha_s}{\Gamma_s^2 a} , \quad S_s = \frac{kS'_m}{\Gamma_s S_m}\Bigg|_{\Gamma_s a} . \tag{12}$$

Similarly, using the orthogonality of $\sin\alpha_s(z+g)$ or $\cos\alpha_s(z+g)$ we find

$$R_s C_s + Q_s D_s = 2 \sum_{n=-\infty}^{\infty} \tilde{N}_{ns} (I_n A_n + P_n B_n) ,$$

$$D_s = 2 \sum_{n=-\infty}^{\infty} \tilde{M}_{ns} B_n , \tag{13}$$

where

$$R_s = \frac{1+\delta_{so}}{\Gamma_s} \frac{kR'_m}{R_m}\Bigg|_{\Gamma_s a} . \tag{14}$$

We thus have 4 infinite sets of real equations for the unknown expansion coefficients. We can rewrite them in matrix notation if we remember that the index n varies between $-\infty$ and $+\infty$, while the index s runs from 0 to ∞ (or from 1 to ∞ since $M_{no} = 0$). We thus obtain

$$\begin{pmatrix} 1 & 0 \\ P & I \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \alpha \begin{pmatrix} N & 0 \\ 0 & M \end{pmatrix} \begin{pmatrix} 1 & 0 \\ Q & S \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix}$$

$$\begin{pmatrix} R & Q \\ 0 & 1 \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = 2 \begin{pmatrix} \tilde{N} & 0 \\ 0 & \tilde{M} \end{pmatrix} \begin{pmatrix} I & P \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} \tag{15}$$

where 1 stands for the unit matrix, and $\tilde{M}$, $\tilde{N}$ are the transpose matrices of M and N.

Substitution of the first into the second equation yields a single homogeneous equation for the vector $\begin{pmatrix} C \\ D \end{pmatrix}$. Since some of the matrix elements diverge for $x_n = 0$ or $\Gamma_s = 0$, we transform this equation into one for the vector $\begin{pmatrix} QC^{-C} + SD \end{pmatrix}$. Since it is still homogenous, it only has solutions when

$$\det\,(G - 2\alpha\tilde{K}HK) = 0 \tag{16}$$

$$\text{where } G = \begin{pmatrix} Q^2S^{-1}-R & QS^{-1} \\ QS^{-1} & S^{-1} \end{pmatrix}, \quad H = \begin{pmatrix} P^2I^{-1}-I & PI^{-1} \\ PI^{-1} & I^{-1} \end{pmatrix} \text{ and } K = \begin{pmatrix} N & 0 \\ 0 & M \end{pmatrix} \tag{17}$$

The equations are symmetric, and the coefficients no longer diverge for Γ or x vanishing (the latter happens when the phase-velocity equals the light velocity).

The computer program "TRANSVERSE" searches for zeros of the truncated determinant by stepping up in frequency after dividing out known poles. These zeros are the resonant frequencies, with which we can then obtain the expansion coefficients (except for an arbitrary common factor). One can then calculate the stored energy for each resonance, and hence the loss-factors and the geometrical factor R/Q

$$k = \frac{VV^*}{4U} = \frac{\omega}{4}\left(\frac{R}{Q}\right) \tag{18}$$

where V is the accelerating voltage per cell seen by a synchronous particle with v=c at r=a, θ=0. The results of these calculations are illustrated in the figures which show the ω-β diagram for two fixed geometries, and the resonant frequencies as well as R/Q as a function of a/b for a number of cavity and period lengths for the two lowest dipole modes.

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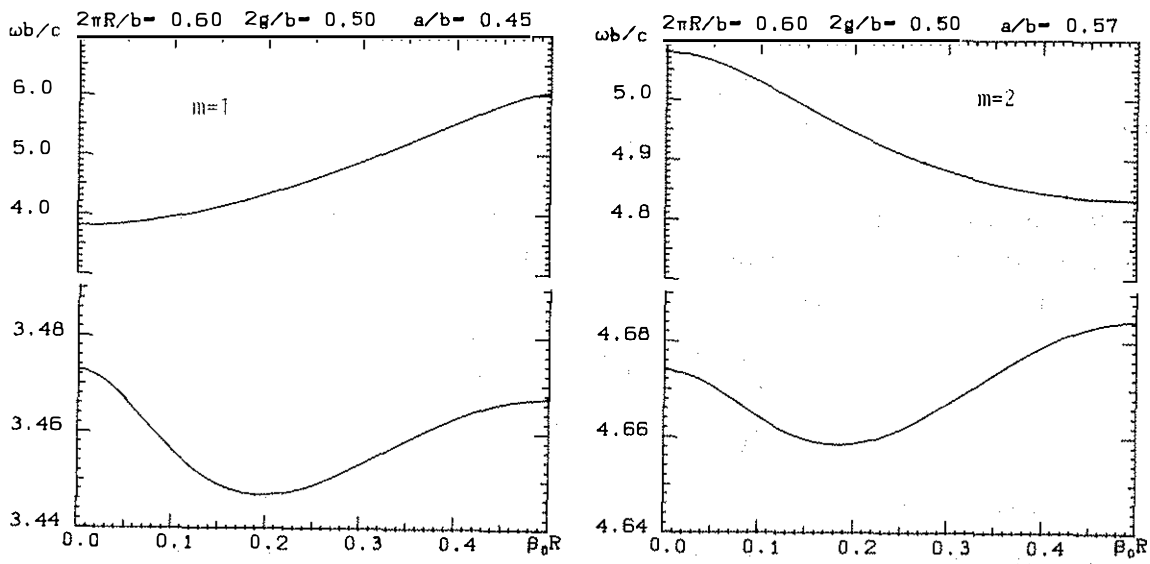

Fig. 1 Brillouin diagrams of the two lowest modes for m=1 and m=2 in a particular geometry.

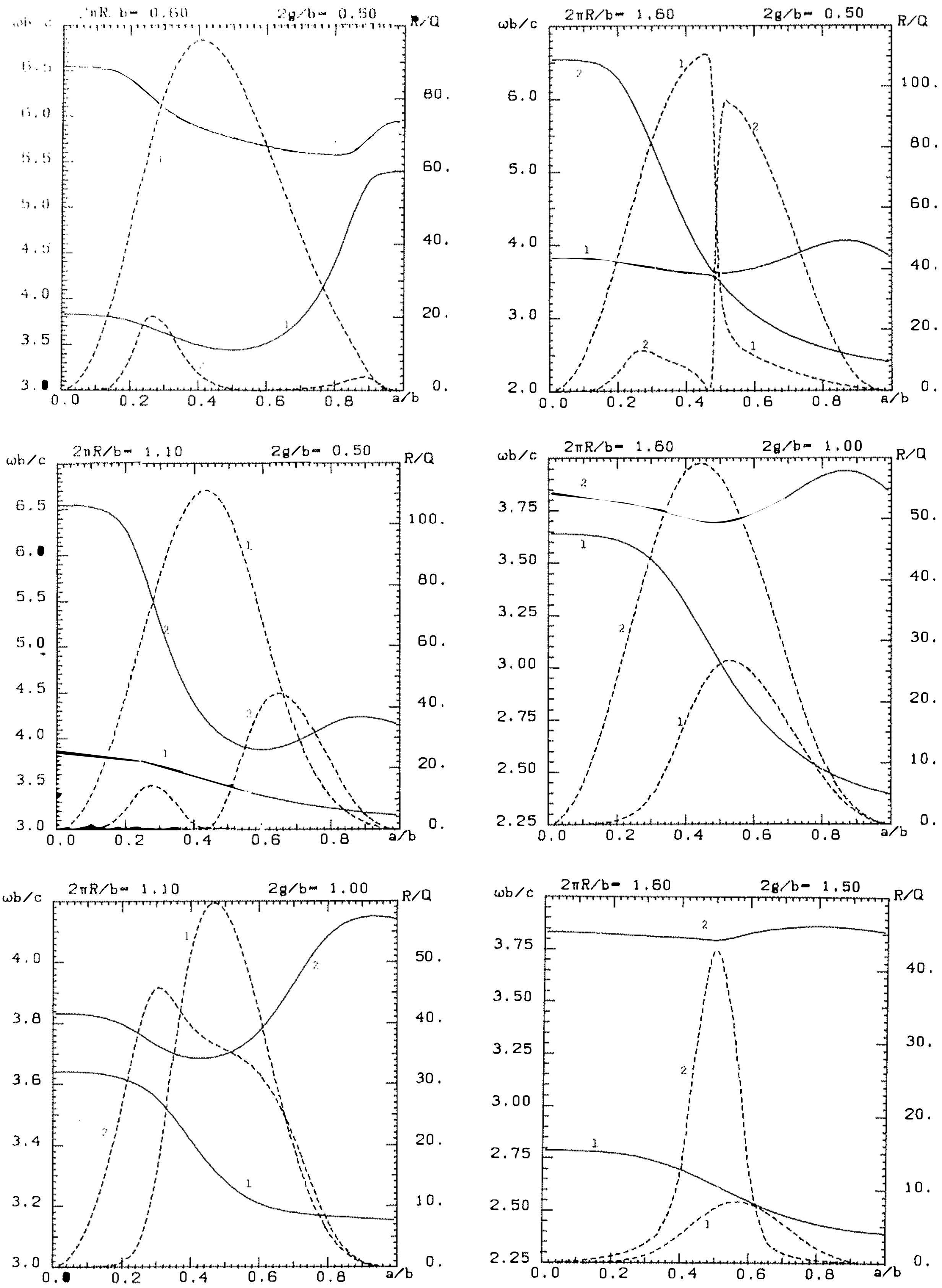

Fig. 2 Normalized frequency ωb/c (full line) and transverse R/Q (in Ω)(dashed line) versus the ratio of the tube cavity to tube diameter a/b for several values of cavity and period length for the first two dipole modes (label 1 and 2) for the synchronous case (v_p=c).