INTRA-BEAM SCATTERING

A. Piwinski*)
CERN, Geneva, Switzerland

ABSTRACT

Intra-beam scattering is analysed and the rise times or damping times of the beam dimensions are derived. The theoretical results are compared with experimental values obtained on the CERN AA and SPS machines.

1. INTRODUCTION

Intra-beam scattering or multiple scattering is a Coulomb scattering between the particles within a bunch or within an unbunched beam. It can be compared with the scattering of gas molecules in a closed box. The focusing forces and the rf accelerating voltage of the storage ring play the same role as the walls of the box since they keep the particles together.

It is well known that the scattering of the molecules in a closed box leads to a Gaussian distribution of the three components of the velocities:

$$\phi(v_s,v_x,v_z) = \left(\frac{m}{2\pi kT}\right)^{3/2} \exp\left\{-\frac{m}{kT}\left(v_s^2 + v_x^2 + v_z^2\right)\right\}$$

where m is the mass of the molecules, T is the absolute temperature and k is Boltzmann's constant. This can be proved by assuming that the distributions of the three velocity components are independent.

The situation is different in the case of intra-beam scattering. Due to the dispersion a change in energy always causes a change in the betatron amplitude, and a coupling arises between the synchrotron oscillation and the betatron oscillation. Furthermore, above transition energy the particles behave as if they had a negative mass, i.e. an increase of energy reduces the revolution frequency. We will see that due to this behaviour an equilibrium distribution of the particles cannot exist above transition energy, and the intra-beam scattering will increase all three dimensions of the bunch in so far as they do not hit other limitations.

But even when an equilibrium distribution exists (below transition energy) the initial distribution will, in general, be different from the equilibrium distribution and the change of the distribution due to the intra-beam scattering can reduce the lifetime or the luminosity of the storage ring. The main purpose of the following investigation is to determine the rise times or damping times of the bunch dimensions. This is done as follows:

---

* On leave from DESY, Hamburg, Fed. Rep. Germany

a) The transformation of the momenta of two colliding particles into their centre-of-mass system.

b) The changes of the momenta due to the collision are calculated.

c) The changed momenta are then transformed back into the laboratory system.

d) The changes of the momenta give the changes of the emittances of the betatron oscillations and the change of the amplitude of the synchrotron oscillation.

e) At first the average is taken over all scattering angles using the Rutherford cross section.

f) Then it is averaged over all momenta and positions of the colliding particles assuming Gaussian distributions.

g) Now one can see that the result of the intra-beam scattering is different below and above transition energy.

h) The average values also give the rise times or damping times of the bunch dimensions.

i) Finally experimental results of accelerators at CERN are discussed.

2. CALCULATION OF RISE TIMES AND DAMPING TIMES

a) Lorentz transformation

The three momenta of the two particles before the collision are given by

$$\vec{p}_{1,2} = p_{1,2\,s}\ \{1,\ x'_{1,2},\ z'_{1,2}\}_{s,x,z}$$

in the coordinate system {s,x,z} (see Fig. 1). For the Lorentz-transformation a new coordinate system {u,v,w} is defined by

$$\vec{e}_u = (\vec{p}_1 + \vec{p}_2)/|\vec{p}_1 + \vec{p}_2|$$

$$\vec{e}_v = \vec{p}_1 \times \vec{p}_2/|\vec{p}_1 \times \vec{p}_2|$$

$$\vec{e}_w = \vec{e}_u \times \vec{e}_w$$

where $\vec{e}_u$, $\vec{e}_v$ and $\vec{e}_w$ are the unit vectors parallel to the coordinate axes. In this coordinate system the two momenta can be written as

$$\vec{p}_{1,2} = p_{1,2}\ \{\cos\alpha_{1,2},\ 0,\ \pm\sin\alpha_{1,2}\}_{u,v,w}$$

where α_1 and α_2 are shown in Fig. 1.

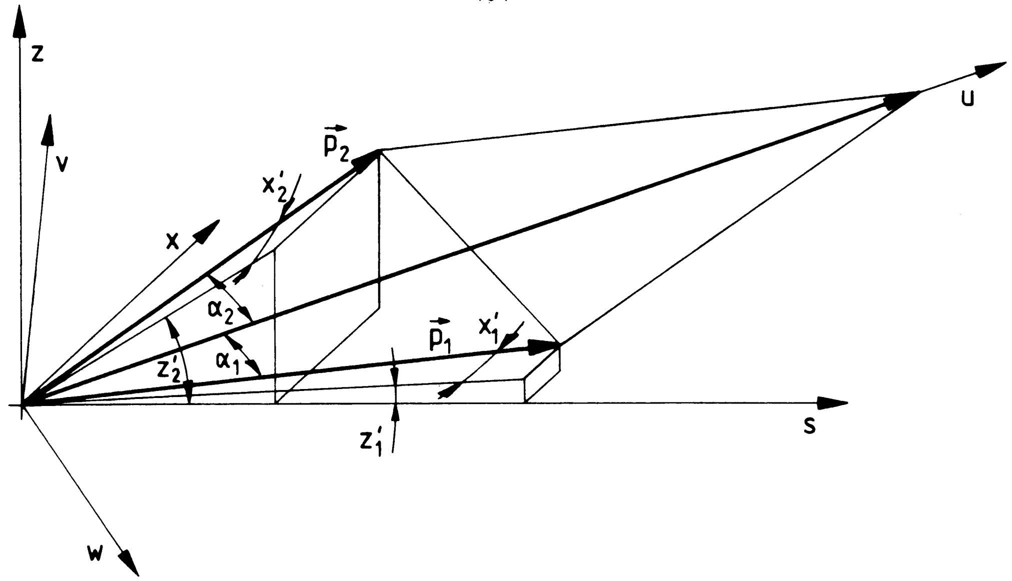


Fig. 1 Relationship of the (u,v,ω) coordinate system aligned on the centre-of-mass motion and the local coordinate system (s,x,z)

A Lorentz-transformation parallel to the u-axis can now be made and gives for the momenta in the centre of mass system

$$\vec{\bar{p}}_{1,2} = \pm \{\bar{p}_{\bar{u}}, 0, \bar{p}_{\bar{w}}\}_{\bar{u},\bar{v},\bar{w}}$$

with

$$\bar{p}_{\bar{u}} = \pm p_{1,2}\, \gamma_{\ell t} \left(\cos \alpha_{1,2} - \frac{\vec{\beta}_{\ell t}}{\beta_{1,2}}\right)$$

$$\bar{p}_{\bar{w}} = p_{1,2} \sin \alpha_{1,2}$$

where the bars denote all quantities in the centre of mass system. $\beta_{\ell t}$ is given by the condition

$$\vec{\bar{p}}_1 + \vec{\bar{p}}_2 = 0$$

or

$$p_1 \left(\cos \alpha_1 - \frac{\beta_{\ell t}}{\beta_1}\right) + p_2 \left(\cos \alpha_2 - \frac{\beta_{\ell t}}{\beta_2}\right) = 0$$

and

$$\beta_{\ell t} = \frac{\gamma_1 \beta_1 \cos \alpha_1 + \gamma_2 \beta_2 \cos \alpha_2}{\gamma_1 + \gamma_2} .$$

b) Change of the momenta

The changes of the momenta of the two colliding particles have, for symmetry reasons, in the centre of mass system the same absolute values but opposite signs. They can be described by the polar angle ψ and the azimuthal angle ϕ (see Fig. 2).

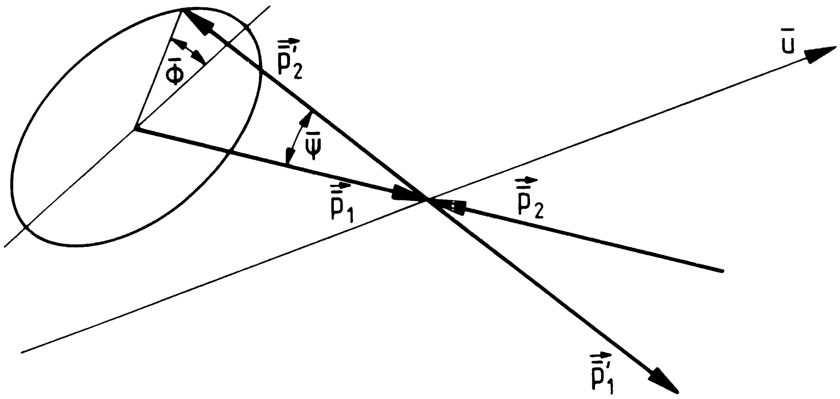


Fig. 2 Change of momenta in a two-particle collision in the centre-of-mass coordinate system

After the collision the momenta can be written in the form

$$\vec{\bar{p}}'_{1,2} = \pm \{\bar{p}_{\bar{w}} \sin \bar{\psi} \cos \bar{\phi} + \bar{p}_{\bar{u}} \cos \bar{\psi} \;,\; \bar{p} \sin \bar{\psi} \sin \bar{\phi} \;,\; \bar{p}_{\bar{w}} \cos \bar{\psi}$$

$$- \bar{p}_{\bar{u}} \sin \bar{\psi} \cos \bar{\phi}\}_{\bar{u},\bar{v},\bar{w}} \;.$$

We now assume that the particle velocities are non-relativistic in the centre of mass system:

$$\bar{\beta}^2 = \frac{\beta^2}{4} \left[\frac{(p_1-p_2)^2}{p^2} + \gamma^2 (\alpha_1 + \alpha_2)^2\right] \ll 1$$

where β is the mean velocity of the particles divided by the velocity of light and p is the mean momentum. This condition is satisfied very well in the SPS with $\alpha_{1,2} \simeq 3 \cdot 10^{-5} - 3 \cdot 10^{-4}$ at $\gamma = 300$. In this case one gets

$$\gamma_{\ell t} = \gamma$$

and

$$\cos \alpha_{1,2} - \frac{\beta_{\ell t}}{\beta_{1,2}} = \pm \frac{p_1 - p_2}{2\gamma^2 p} \;.$$

With the abbreviations

$$\frac{p_1 - p_2}{\gamma p} = \xi \,, \quad x'_1 - x'_2 = \Theta \,, \quad z'_1 - z'_2 = \zeta$$

$$\alpha_1 + \alpha_2 = 2\alpha = \sqrt{(x'_1 - x'_2)^2 + (z'_1 - z'_2)^2} = \sqrt{\Theta^2 + \zeta^2}$$

one gets for the changes of the momenta in the storage ring system the expression

$$\delta\vec{p}_{1,2} = \vec{p}\,'_{1,2} - \vec{p}_{1,2}$$

$$= \pm \frac{p}{2} \{2\alpha\gamma \sin \bar{\psi} \cos \bar{\phi} + \gamma\xi(\cos \bar{\psi} - 1) \,,$$

$$[\zeta\sqrt{1 + \frac{\xi^2}{4\alpha^2}} \sin \bar{\phi} - \frac{\xi\Theta}{2\alpha} \cos \bar{\phi}] \sin \bar{\psi} + \Theta (\cos \bar{\psi} - 1) \,,$$

$$[\Theta\sqrt{1 + \frac{\xi^2}{4\alpha^2}} \sin \bar{\phi} - \frac{\xi\zeta}{2\alpha} \cos \bar{\phi}] \sin \bar{\psi} + \zeta (\cos \bar{\psi} - 1)\}_{s,x,z} \,.$$

d) Change of the oscillation amplitudes

The betatron amplitudes are given by the emittances. Neglecting β'_x one gets for the change of the horizontal emittance:

$$\beta_x \delta\varepsilon_x = 2\, \delta x_\beta + \delta^2 x_\beta + \beta_x^2 \,(2x'\delta x' + \delta^2 x')$$

δx_β is given, as in the case of quantum fluctuation, by the change of the absolute value of the momentum, and $\delta x'$ is given by the change of the direction of the momentum:

$$\beta_x \delta\varepsilon_x = -2x_\beta\, D \frac{\delta p}{p} + D^2 \frac{\delta^2 p}{p^2} + \beta_x^2 \,(2x' \frac{\delta p_x}{p} + \frac{\delta^2 p_x}{p^2}) \,.$$

Here we have neglected also the derivative of the dispersion D'. For the change of the vertical emittance one obtains

$$\delta\varepsilon_z = \beta_z \,(2z' \frac{\delta p_z}{p} + \frac{\delta^2 p_z}{p^2})$$

since the vertical dispersion is usually zero.

For the invariants of the linearized longitudinal motion one gets

$$H = \begin{cases} \left(\frac{\Delta p}{p}\right)^2 + \frac{1}{\Omega_s^2}\left(\frac{d}{dt}\frac{\Delta p}{p}\right)^2 & \text{for bunched beams} \\ \left(\frac{\Delta p}{p}\right)^2 & \text{for unbunched beams} \end{cases}$$

where $\Omega_s/2\pi$ is the synchrotron frequency. The change of H due to a scattering event is in both cases

$$\delta H = 2\,\frac{\Delta p}{p}\,\frac{\delta p}{p} + \frac{\delta^2 p}{p^2}\ .$$

e) Averaging over all scattering angles

The distribution of the scattering angles ψ and ϕ is given by the Rutherford cross-section:

$$d\bar{\sigma} = \left(\frac{r_p}{4\beta^2 \sin^2\frac{\bar{\psi}}{2}}\right)^2 \sin\bar{\psi}\, d\bar{\psi}\, d\bar{\phi}$$

where r_p is the classical proton radius. Integration over all scattering angles gives

$$\int_{\bar{\psi}_m}^{\pi}\int_0^{2\pi} \delta\vec{p}_1\, d\bar{\sigma} = -\frac{\pi}{2}\,\frac{p r_p^2}{\bar{\beta}^4}\,\ln\frac{2\bar{\beta}^2\bar{d}}{r_p}\,\{\gamma\xi,\Theta,\zeta\}_{s,x,z}$$

$$\int_{\bar{\psi}_m}^{\pi}\int_0^{2\pi} \delta^2 p\, d\bar{\sigma} = \frac{\pi}{4}\,\frac{p^2 r_p^2}{\bar{\beta}^4}\,\ln\frac{2\bar{\beta}^2\bar{d}}{r_p}\,[\gamma^2(\Theta^2+\zeta^2) + (\xi^2+\zeta^2) + (\xi^2+\Theta^2)]\ .$$

The minimum scattering angle ψ_m is determined by the impact parameter $\bar{d}$ by

$$\tan\frac{\bar{\psi}_m}{2} = \frac{r_p}{2\bar{\beta}^2\bar{d}}$$

with

$$\bar{d} \approx \frac{1}{2}\ \text{beam height}\ .$$

Furthermore,

$$2\bar{\beta}^2\bar{d} \gg r_p$$

is assumed for the calculation of the integrals.

The change of the three invariants is then given by

$$\int_{\bar{\varphi}_m}^{\pi} \int_{0}^{2\pi} \begin{bmatrix} \delta H_1/\gamma^2 \\ \delta\varepsilon_{x_1}/\beta_x \\ \delta\varepsilon_{z_1}/\beta_z \end{bmatrix} d\bar{\sigma} =$$

$$\frac{\pi r_p^2}{4\bar{\beta}^4} \ln \frac{2\bar{d}\bar{\beta}^2}{r_p} \begin{bmatrix} -4\dfrac{\Delta p_1}{\gamma p}\xi + \theta^2 + \zeta^2 \\ -4x'_1\theta + \xi^2 + \zeta^2 - 4\dfrac{x_{\beta_1}}{\beta_x^2}\gamma\zeta + \dfrac{D^2\gamma^2}{\beta_x^2}(\theta^2 + \zeta^2) \\ -4z'_1\zeta + \xi^2 + \theta^2 \end{bmatrix}$$

f) Averaging over all particles

The relative velocity between two colliding particles in the centre of mass system is $2c\bar{\beta}$. The probability for a scattering into $\bar{\psi}$ and $\bar{\phi}$ per unit time in the centre of mass system is $2c\bar{\beta}\bar{\rho}\, d\bar{\sigma}$ where $\bar{\rho} = \rho/\gamma$ is the particle density. The probability for scattering into ψ and ϕ per unit time in the storage ring system is

$$P_{scat} = \frac{2c\bar{\beta}\rho\, d\bar{\sigma}}{\gamma^2}$$

since $dt = \gamma d\bar{t}$. In order to calculate the mean change of the invariants of all particles we have to average with respect to the 12 variables of the two colliding particles:

$$s_1, s_2, x_{\beta_1}, x_{\beta_2}, z_{\beta_1}, z_{\beta_2}, \Delta p_1/p, \Delta p_2/p, x'_1, x'_2, z'_1, z'_2$$

where three of the twelve variables are dependent:

$$s_1 = s_2 = s$$

$$x_{\beta_1} + D\frac{\Delta p_1}{p} = x_{\beta_2} + D\frac{\Delta p_2}{p}$$

$$z_{\beta_1} = z_{\beta_2} = z_\beta$$

which follows from the assumption that the two colliding particles have the same position. It is assumed that all variables have Gaussian density distribution except for the longitudinal distribution w_s which can have a Gaussian distribution or a continuous distribution (unbunched beams). The total density function is then

$$P = w_s^2(s) w_{x_\beta}(x_{\beta_1}) w_{x_\beta}(x_{\beta_2}) w_{z_\beta}^2(z_\beta) w_p\left(\frac{\Delta p_1}{p}\right) w_p\left(\frac{\Delta p_2}{p}\right) w_{x'}(x'_1) w_{x'}(x'_2) w_{z'}(z'_1) w_{z'}(z'_2) \ .$$

Before integrating over the three variables we make the following substitutions

$$x_{\beta 1,2} = x_\beta \mp D\gamma\xi/2 \qquad \frac{\Delta p_{1,2}}{p} = \eta \pm \gamma\xi/2$$

$$x'_{1,2} = x' \pm \Theta/2 \qquad z'_{1,2} = z' \pm \zeta/2$$

and the differential becomes

$$dV = \gamma ds dx_\beta dz_\beta d\eta d\xi dx' d\Theta dz' d\xi \ .$$

Now six of the nine integrals can be solved immediately and one obtains

$$\frac{d}{dt}\begin{bmatrix} \langle \delta H\rangle/\gamma^2 \\ \langle d\varepsilon_x\rangle/\beta_x \\ \langle \delta\varepsilon_2\rangle/\beta_z \end{bmatrix} = \int_V \frac{2c\bar{\beta}}{\gamma^2} \int_{\psi_m}^{\pi} \int_0^{2\pi} \begin{bmatrix} \delta H_1/\gamma^2 \\ \delta\varepsilon_{x_1}/\beta_x \\ \delta\varepsilon_{z_1}/\beta_z \end{bmatrix} d\bar{\sigma}\, dV =$$

$$2A \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left\{-\frac{\gamma^2\xi^2}{4}\left(\frac{1}{\sigma_\eta^2} + \frac{D^2}{\beta_1^2}\right) - \frac{\Theta^2}{4\sigma_{x'}^2} - \frac{\zeta^2}{4\sigma_x^2}\right\}$$

$$\begin{bmatrix} \Theta^2 + \zeta^2 - 2\xi^2 \\ \xi^2 + \zeta^2 - 2\Theta^2 + \frac{\gamma^2 D^2}{\beta_x^2}(\Theta^2 + \zeta^2 - 2\xi^2) \\ \xi^2 + \Theta^2 - 2\zeta^2 \end{bmatrix} \ln\left(2\frac{\bar{d}}{r_p}\bar{\beta}^2\right) \frac{d\xi d\Theta d\zeta}{(\xi^2 + \Theta^2 + \zeta^2)^{3/2}}$$

with

$$A = \frac{r_p^2 c N_b}{64\pi^2 \sigma_s \sigma_p \sigma_{x_\beta} \sigma_{z_\beta} \sigma_{x'} \sigma_{z'} \beta^3 \gamma^4} \qquad \text{(bunched)}$$

where N_b is the number of particles in the bunch. For an unbunched beam N_b/σ_s must be replaced by $2\sqrt{\pi}N/C$ where N is the number of particles in the beam and C is the circumference.

g) Invariants

Multiplying the equation for $\langle\delta H\rangle$ by $1-\gamma^2 D^2/\beta_x^2$ and adding it to the equations for $\langle\delta\varepsilon_x\rangle$ and $\langle\delta\varepsilon_z\rangle$ gives

$$\frac{d}{dt}\left[\langle\delta H\rangle\left(\frac{1}{\gamma^2}-\frac{D^2}{\beta_z^2}\right)+\frac{\langle\delta\varepsilon_x\rangle}{\beta_x}+\frac{\langle\delta\varepsilon_z\rangle}{\beta_z}\right]=0\ .$$

In a weak focusing machine the momentum compaction factor α_M is given by

$$\alpha_M=\frac{D^2}{\beta_x^2}\ .$$

After integration one obtains

$$\langle H\rangle\left(\frac{1}{\gamma^2}-\alpha_M\right)+\frac{\langle\varepsilon_x\rangle}{\beta_x}+\frac{\langle\varepsilon_z\rangle}{\beta_z}=\text{const}\ .$$

This equation shows that the behaviour of the particles is different below transition energy ($\gamma^2 < 1/\alpha_M$) and above transition energy ($\gamma^2 > 1/\alpha_M$). Below transition energy the sum of the three positive invariants is limited, i.e. the three oscillation amplitudes are limited. The particles behave like the molecules of a gas in a closed box (focusing corresponds to the walls of the box). They can only exchange their oscillation energy, but the total oscillation energy or the temperature is limited. In this case an equilibrium distribution must exist where the intra beam scattering does not change the beam dimensions.

Above transition energy the coefficient of $\langle H\rangle$ is negative and the total oscillation energy can increase as long as it does not exceed other limitations. In this case an equilibrium distribution does not exist.

h) Rise times

The rise times for the mean oscillation amplitudes, which determine the bunch dimensions, are given by[1,2)]

$$\frac{1}{\tau_p}=\frac{1}{2\sigma_p^2}\frac{d\sigma_p^2}{dt}=A\,\frac{\sigma_h^2}{\sigma_p^2}\,f(a,b,c)$$

$$\frac{1}{\tau_x}=\frac{1}{2\sigma_x^2}\frac{d\sigma_x^2}{dt}=A\left[f\left(\frac{1}{a},\frac{b}{a},\frac{c}{a}\right)+\frac{D^2\sigma_p^2}{\sigma_{x\beta}^2}\,f(a,b,c)\right]$$

$$\frac{1}{\tau}=\frac{1}{2\sigma_z^2}\frac{d\sigma_z^2}{dt}=A\,f\left(\frac{1}{b},\frac{a}{b},\frac{c}{b}\right)$$

with

$$a = \frac{\sigma_h}{\gamma\sigma_{x'}} , \quad b = \frac{\sigma_h}{\gamma\sigma_{z'}} , \quad c = \beta\sigma_h \sqrt{2\frac{d}{r_p}}$$

$$\frac{1}{\sigma_h^2} = \frac{1}{\sigma_p^2} + \frac{D^2}{\sigma_{x\beta}^2}$$

and

$$f(a,b,c) = 8\pi \int_0^1 [\ln (\frac{c^2}{2} (\frac{1}{\sqrt{p}} + \frac{1}{\sqrt{q}})) - 0.577 \ldots](1 - 3x^2) \frac{dx}{\sqrt{pq}}$$

$$p = a^2 + x^2 (1-a^2) , \quad q = b^2 + x^2 (1-b^2) .$$

The function f(a,b,c) is plotted in Fig. 3. For

$$a = b = 1$$

one obtains

$$f(1,1,c) = 0$$

i.e. for

$$\sigma_p \sqrt{\frac{1}{\gamma^2} - \frac{D^2}{\beta^2}} = \sigma_{x'} = \sigma_{z'}$$

an equilibrium is reached, and the Gaussian function remains stable. This is possible only below transition energy, i.e. $\gamma < \beta/D$.

3. EXPERIMENTAL RESULTS

The last section shows some experimental results obtained in the two CERN storage rings, the Super Proton Synchrotron (SPS) and the Antiproton Accumulator (AA). Investigations in the SPS have shown that the decay of the luminosity consists of three parts[3]. The first part is the increase of the dimensions of the proton bunches due to the intra-beam scattering, the second is the loss of protons due to the intra-beam scattering and the third is the loss of antiprotons due to the beam-beam interaction. The total decay time of the luminosity is then given by:

$$\frac{1}{\tau_{lum}} = \frac{1}{\tau^+_{x'}} + \frac{1}{\tau^+_{life}} + \frac{1}{\tau^-_{life}} .$$

The times $\tau^+_{x'}$, τ^+_{life} and τ^-_{life} are, very roughly, in the order of 60 h so that the decay time of the luminosity is in the order of 20 h and mainly determined by the intra-beam scattering.

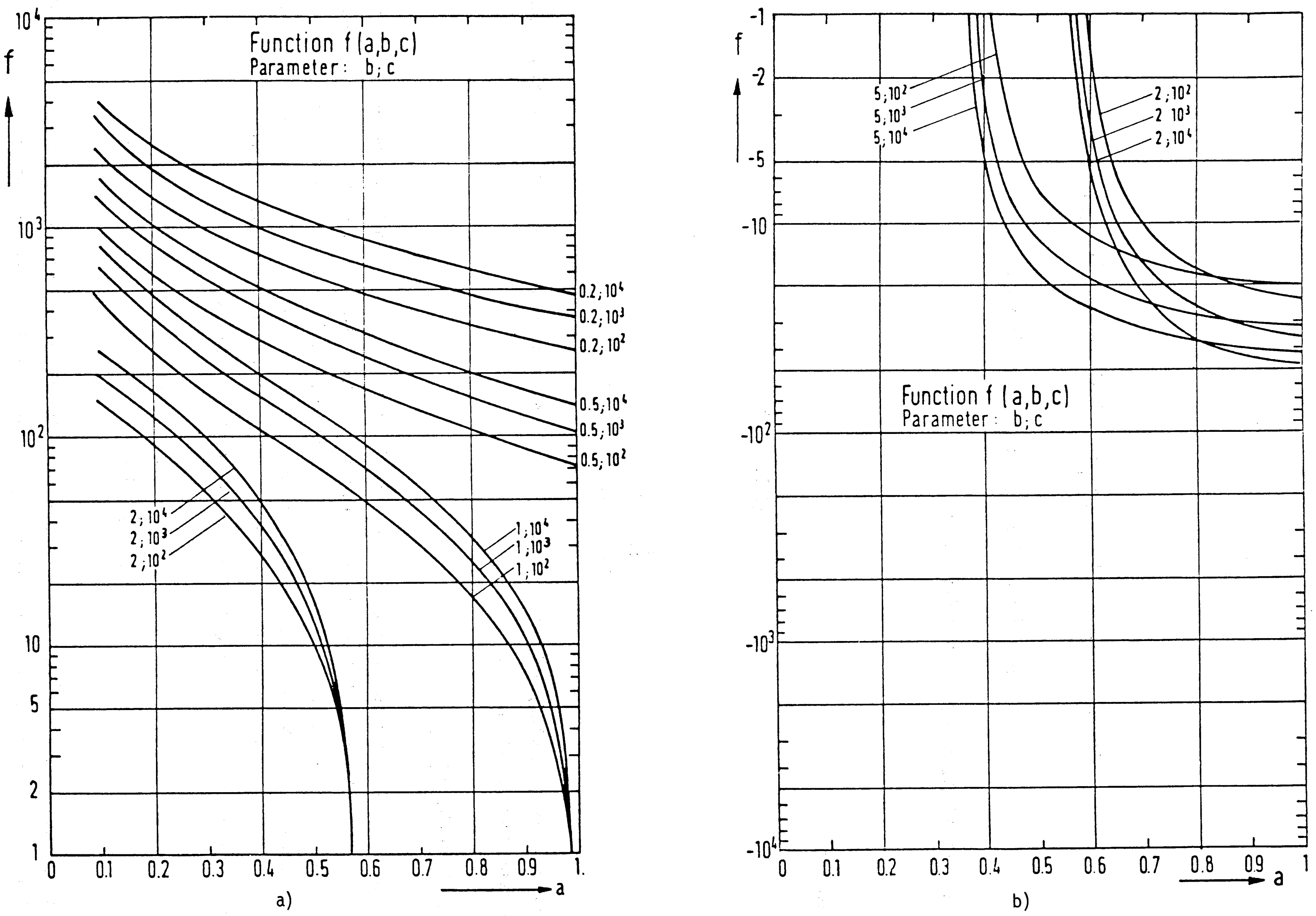
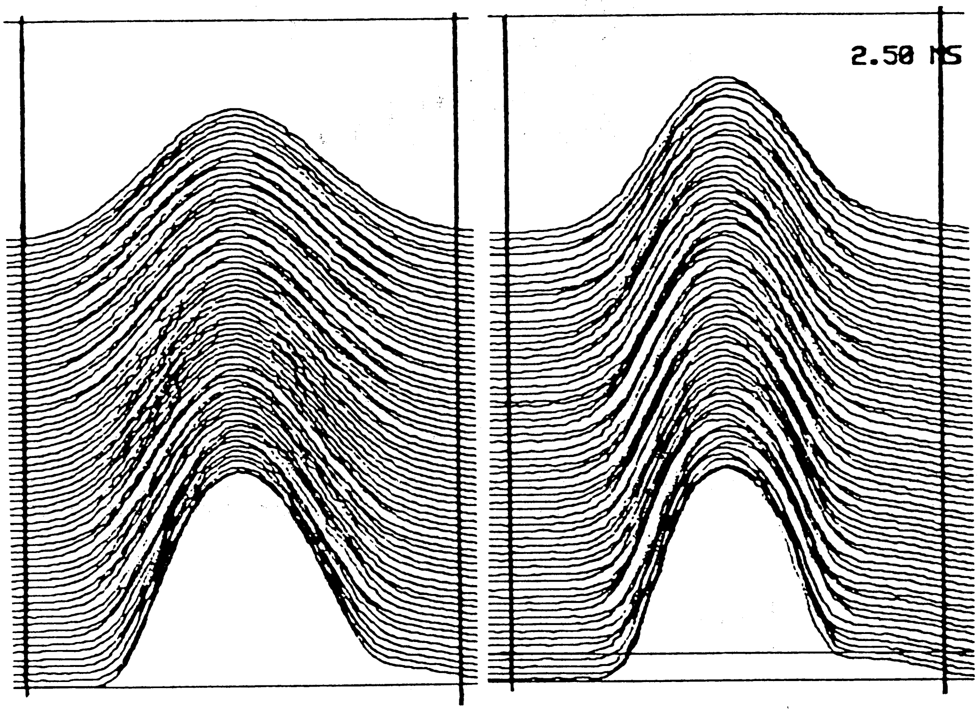


Fig. 3 Graphical representation of the function f(a,b,c)

Figure 4 shows the longitudinal distribution of proton and antiproton bunches measured at time intervals of 15 minutes[4]. It can be seen that in the case of antiprotons (b) the distribution remains almost constant whereas in the case of protons, which have a much larger density ($N^+ = 1.5 \times 10^{11}$, $N^- = 1.2 \times 10^{10}$), the bunch becomes longer and the density decreases. The rise times agree with the calculation[4].



a) b)

Fig. 4 Measurements made in the CERN SPS of the longitudinal distribution of (a) proton and (b) antiproton bunches

Figure 5 shows a direct comparison of the measured and calculated rise times for the horizontal emittance as a function of time[5]. The measured growth rate, which is the inverse of the rise time, is equal to the calculated growth rate plus a small constant. This constant is caused by the scattering of the protons by the residual gas[5].

In the AA, where the antiprotons are cooled, intra-beam scattering limits the maximum antiproton density which can be achieved by stochastic cooling[6,7]. In an experiment the cooling system was switched off after the antiprotons were cooled down and the growth times due to the intra-beam scattering were measured. Figure 6 shows the growth times for the momentum spread and the horizontal emittance[6].

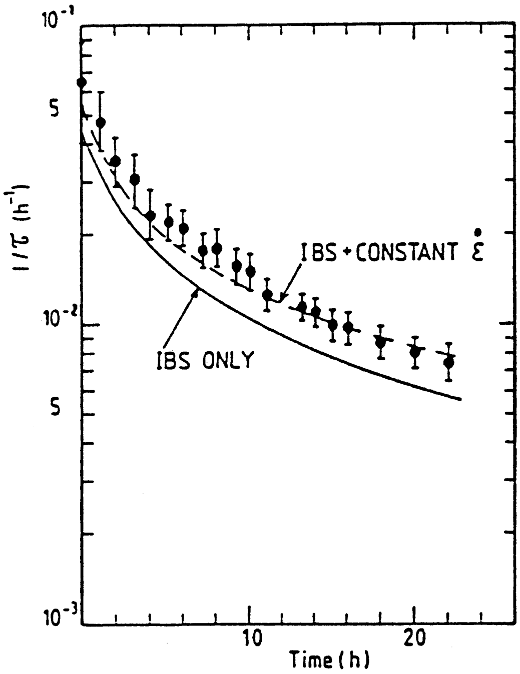


Fig. 5 Measured radial emittance growth rate in CERN SPS $\tau^{-1} = \dot{\varepsilon}/\varepsilon$ compared with the theoretical intra-beam scattering rate. The dotted line includes a correction for gas scattering.

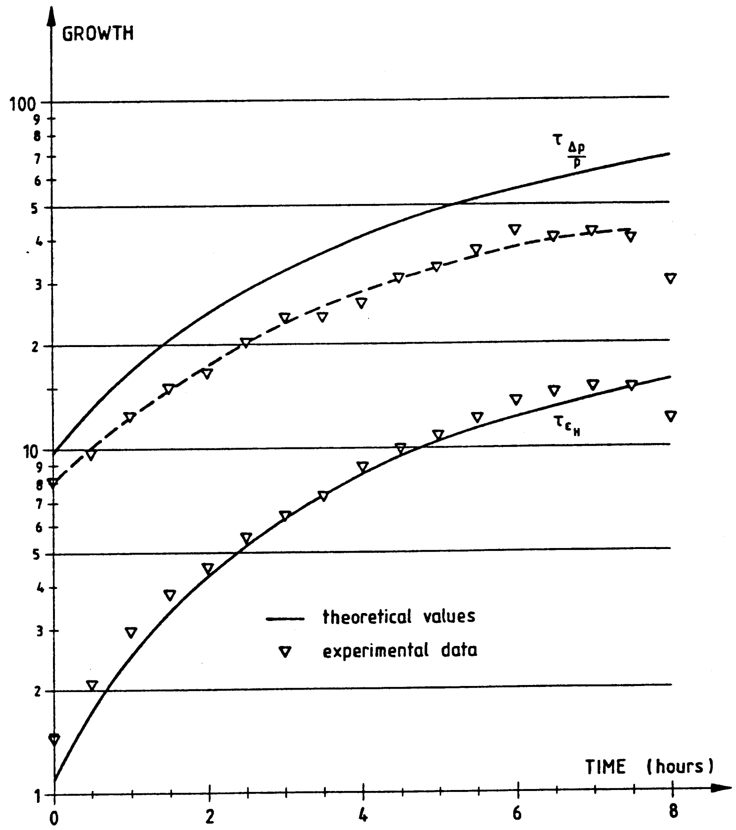


Fig. 6 Comparison of the measured and theoretical time constants for the growth of the momentum spread and horizontal emittance in the CERN AA with the cooling switched off

REFERENCES

1) A. Piwinski, Proc. 9th Int. Conf. on High Energy Accelerators, Stanford, 1974 (SLAC, Stanford, 1974) p. 405.

2) J.D. Bjorken and S.K. Mtingwa, Particle Accelerators, Vol. 13, p. 115 (1983).

3) L.R. Evans, CERN/SPS/83-30 (1983).

4) L.R. Evans, Private communication.

5) L.R. Evans and J. Gareyte, 1985 Particle Accelerator Conference, IEEE Trans. Nucl. Sci. NS-32, (1985).

6) M. Martini and S. van der Meer, PS/AA/ME/Note 75 (1984).

7) M. Martini, CERN PS/84-9 (AA) (1984).