DESCRIPTION OF THE MATHEMATICS INVOLVED IN THE RECONSTRUCTION OF SPARK POSITIONS IN AN ACOUSTICAL SPARK CHAMBER

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(presented by P. Zanella)

1. INTRODUCTION

This paper describes mathematical methods to find a spark position in an acoustical spark chamber. Programmes based on such methods have been written in FORTRAN for the IBM 7090 computer and are used at CERN for the analysis of data from acoustical spark chamber experiments.

2. THE PROBLEM: DEFINITIONS AND ASSUMPTIONS

In general one measures the time-of-flight T of the pressure wave formed by the spark, from the spark position to a suitable detector. The detector will be called a probe. The distance R between the spark and the probe is, of course, given by

$$R = \int_0^T v(t)dt \tag{1}$$

where $v(t)$ is the rapidly varying velocity of a shock wave, asymptotically decreasing to the velocity of sound, v_o.

The computations described here are all based on the assumption that the probes are kept sufficiently far from the active spark chamber area, so that the velocity with which the pressure wave reaches the probes, can be safely approximated by v_o.

Figure 1 shows approximately the behaviour of the velocity of a shock wave as a function of time, i.e. distance from the spark. The two curves refer to different spark energies. The higher energy gives rise to a higher initial velocity, but both curves tend to v_o at large distances.

If a measured time T is outside the shock wave region, one may rewrite equation (1) as follows: (Fig. 2)

$$R = \int_0^T v_o dt + \int_0^T (v(t) - v_o)dt = v_o T + \Delta R = v_o(T + \Delta T) \qquad (2)$$

The sound velocity varies with temperature and depends on the gas mixture. Let us call δv_o the uncertainty in the value of v_o. Similarly, let us indicate with $\delta(\Delta T)$ the uncertainty in the estimate of ΔT, which varies with the spark energy.

Assuming that T, which is measured electronically, is known to a greater accuracy than v_o and ΔT, we can write the following expression for the relative error of R :

$$\frac{\partial R}{R} = \frac{\partial v_o}{v_o} + \frac{\partial(\Delta T)}{T+\Delta T} \qquad (3)$$

To make an example, if we assume that during a run the variations of v_o are within ± 0.2% and that ΔT may have uncertainties of the order of ± 2 μsec, then the error in R can be as large as ± 2 mm for a spark 50 cm from the probe.

Now, let us define a linear chamber simply as two probes positioned sufficiently far from the active area, both probes being used to calculate the X-coordinate of the spark along the line connecting the detectors. If 2a is the length of the segment between probe 1 and probe 2 and if X is measured from the centre of this segment, we have :

$$X = \frac{R_1^2 - R_2^2}{4a} = \frac{v_o^2}{4a}\left[(T_1 + \Delta T)^2 - (T_2 + \Delta T)^2\right] \qquad (4)$$

and we can write the following expression for the relative error on X :

$$\frac{\partial X}{X} = 2\,\frac{\partial v_o}{v_o} + 2\,\frac{\partial(\Delta T)}{T_1 + T_2 + 2\Delta T} \qquad (5)$$

which gives for a spark position 50 cm from probe 1, under the assumptions made in the example above, an error of ± 0.65 mm in the X-coordinate of the spark if the probes are placed 80 cm apart, i.e. spark 10 cm from the centre of the chamber.

If the two probes are, for example, 160 cm apart, the error becomes ± 1.9 mm under the same conditions. In this case the spark is 30 cm from the centre of the chamber.

Finally, let us define a two-dimensional chamber as a set of 4 probes positioned sufficiently far from the area of the chamber where we want to calculate spark's coordinates, as to allow the following assumptions to be valid :

i) The velocity with which the pressure wave reaches any probe is v_0.

ii) ΔT has the same value for all the probes.

From the assumptions above, the following four basic equations can be written for any two-dimensional chambers :

$$\begin{cases} R_1 = v_0 \; (T_1 + \Delta T) \\ R_2 = v_0 \; (T_2 + \Delta T) \\ R_3 = v_0 \; (T_3 + \Delta T) \\ R_4 = v_0 \; (T_4 + \Delta T) \end{cases} \quad \text{where} \quad R_i = \sqrt{(x - x_i)^2 + (y - y_i)^2} \quad (i = 1,2,\ldots 4) \qquad (6)$$

The calculations of spark position (x,y) from these equations are discussed below.

3. SOLUTION OF THE BASIC EQUATIONS

The methods of solution fall into any of the following four classes, depending on whether the variables v_0 and/or ΔT are considered known or unknown :

CLASS	KNOWN	UNKNOWN
I	v_0, ΔT	x, y
II	ΔT	x, y, v_0
III	v_0	x, y, ΔT
IV		x, y, v_0, ΔT

3.1 Class I

The spark position is overdetermined. To make good use of the four equations, one can split the two-dimensional chamber into two linear chambers, considering probes 1-3 and 2-4 as separate sets. Coordinate X can be calculated from the first probe set and y from the second, using equation (4).

The errors due to v_0 and ΔT indetermination have been already discussed for linear chambers. Since they increase with the distance of the spark from the centre of the chamber, this very simple method is generally unsuitable for large chambers.

3.2 Class II

Two velocity independent equations can be obtained from the basic set (6), as follows :

$$\begin{cases} \dfrac{R_1}{R_3} = \dfrac{T_1 + \Delta T}{T_2 + \Delta T} = C_1 & \text{(constant)} \\[2ex] \dfrac{R_2}{R_4} = \dfrac{T_2 + \Delta T}{T_4 + \Delta T} = C_2 & \text{(constant)} \end{cases} \qquad (7)$$

Hence, introducing x and y :

$$\begin{cases} x^2 + y^2 - 2\,\dfrac{x_1 - c_1^2 x_3}{1 - c_1^2}\,x - 2\,\dfrac{y_1 - c_1^2 y_3}{1 - c_1^2}\,y + \dfrac{x_1^2 + y_1^2 - c_1^2\,(x_3^2 + y_3^2)}{1 - c_1^2} = 0 \\[3ex] x^2 + y^2 - 2\,\dfrac{x_2 - c_2^2 x_4}{1 - c_2^2}\,x - 2\,\dfrac{y_2 - c_2^2 y_4}{1 - c_2^2}\,y + \dfrac{x_2^2 + y_2^2 - c_2^2\,(x_4^2 + y_4^2)}{1 - c_2^2} = 0 \end{cases}$$

$$\begin{cases} x^2 + y^2 - 2A_1 x - 2B_1 y + D_1 = 0 \\[2ex] x^2 + y^2 - 2A_2 x - 2B_2 y + D_2 = 0 \end{cases} \qquad (8)$$

These are the equations of two circles in the plane of the chambers. Each circle is defined by two probes and its centre lies on the line connecting these two probes. This circle is the geometrical locus of the positions that the spark determined by these two probes can assume by letting the velocity v_0 vary continuously.

The intersections of the two circles are solutions of the problem. One can easily choose from the two solutions the one of physical interest by imposing physically reasonable limits to the values of x, y and v_0.

The explicit solution is

$$\begin{cases} x = \dfrac{D_2 - D_1 - 2(B_2 - B_1)y}{A_2 - A_1} \\ y = \dfrac{L \pm \sqrt{L^2 - HM}}{H} \end{cases} \quad \text{where} \quad \begin{vmatrix} H = (A_2 - A_1)^2 + 4(B_2 - B_1)^2 \\ L = (A_2 - A_1)^2 \cdot B_2 + 2(B_2 - B_1)(D_2 - D_1) - 2A_2(A_2 - A_1)(B_2 - B_1) \\ M = (A_2 - A_1)^2 \cdot D_2 + (D_2 - D_1)^2 - 2(A_2 - A_1)(D_2 - D_1) \cdot A_2 \end{vmatrix} \quad (9)$$

The spark position computed by this method is free from errors due to velocity estimate and it is only affected by errors due to ΔT indetermination.

3.3 Class III

Two ΔT-independent equations can be extracted from the basic equations (6) :

$$\begin{cases} R_1 - R_3 = v_o \ (T_1 - T_2) = \text{const.} \\ R_2 - R_4 = v_o \ (T_2 - T_4) = \text{const.} \end{cases} \quad (10)$$

This case presents many analogies with the preceding one : The equations (10) represent two hyperbolae each being determined by a couple of probes.

The spark defined by two probes describes one such hyperbole when ΔT assumes all possible values. The solution of the problem has to be found among the intersections of the two hyperbolae.

The spark position, as calculated by this method, is free from errors due to an error in the value of ΔT common to all probes. The inaccuracy is caused by errors in the imposed value of v_o.

3.4 Class IV

It is possible to construct two equations independent of v_o and ΔT :

$$\begin{cases} \dfrac{R_1 - R_2}{R_3 - R_4} = \dfrac{T_1 - T_2}{T_3 - T_4} = K \ \text{(constant)} \\ \dfrac{R_1 - R_4}{R_2 - R_3} = \dfrac{T_1 - T_4}{T_2 - T_3} = I \ \text{(constant)} \end{cases}$$

Equations (11) are of the fourth degree in x and y. The behaviour of the curves of constant K and I within the active area of a chamber is shown in Fig. 3 and Fig. 4 respectively. It depends only on the probe positions. The times-of-flight T_1, T_2, T_3, T_4 define a value of K and a value of I, which select two curves in the families represented by equations (11). The spark position is at the intersection point of these two curves within the active area of the chamber. Since the direct solution of equations (11) is rather difficult, we have calculated the spark positions by the following method:

i) First a good approximation to x and y is obtained using the method described under Section 3.2

ii) Then the solution is iteratively improved up to the desired accuracy by linearizing the equations (11) in a small region around the approximate spark position.

iii) To avoid regions where $K = \infty$ or $I = \infty$, the inverse of K or I are used whenever their values are greater than 1.

The solution found by this method is free from errors due to v_0 and ΔT estimates. Its accuracy will be discussed in the next section.

3.5 Applicability and accuracy of the method of Class IV

The last method provides a solution of the spark position problem which is only dependent on the values of the times-of-flight.

Three major questions arise in this connection:

a) For a given probe configuration and with infinite accuracy on the T_i, does geometry alone dictate any limits on the active area ?

b) How accurately can the spark position be calculated if a certain inaccuracy is allowed in the measured times-of-flight ?

c) Which probe configuration gives most accurate coordinate determination for a given spark chamber size ?

Concerning question a), since the spark position is calculated by intersecting the two curves K and I, one can find a geometrical solution wherever the angle between these curves is different from zero. Figs. 5 and 6 show how this angle varies in the plane of a chamber. Obviously the region near zero degrees must be excluded from the active area. Or one can consider, in that region, the intersection of one of the curves above with the curve of constant J where

$$J = \frac{R_1 - R_3}{R_2 - R_4} = \frac{T_1 - T_3}{T_2 - T_4} = f\ (K,I).$$

Alternatively one can try to displace the probes such as to push the bad regions out of the active area.

Figures 7 and 8 show an example of a successful operation of this kind. Probes 1 and 3 were displaced by 12 cm in opposite directions.

Where the accuracy is concerned (question b), an IBM 7090 FORTRAN programme has been written, which produces a map of the values of

$$\frac{dS}{dR_i} \; (i = 1,4)$$

according to the following formulae:

$$\frac{dS}{dR_i} = \sqrt{\left(\frac{dx}{dR_i}\right)^2 + \left(\frac{dy}{dR_i}\right)^2} \qquad (12)$$

where

$$\frac{dx}{dR_i} = \frac{\frac{\partial I}{\partial y}\frac{\partial K}{\partial R_i} - \frac{\partial K}{\partial y}\frac{\partial I}{\partial R_i}}{\frac{\partial K}{\partial x}\frac{\partial I}{\partial y} - \frac{\partial I}{\partial x}\frac{\partial K}{\partial y}}$$

$$\frac{dy}{dR_i} = \frac{\frac{\partial I}{\partial x}\frac{\partial K}{\partial R_i} - \frac{\partial K}{\partial x}\frac{\partial I}{\partial R_i}}{\frac{\partial K}{\partial y}\frac{\partial I}{\partial x} - \frac{\partial I}{\partial y}\frac{\partial K}{\partial x}} \qquad (13)$$

S, as defined in (12) represents the distance between the actual and the measured spark positions. Its value is zero if there is no error in the times-of-flight.

Figure 9 shows the propagation, through the left half of a rectangular chamber (160 × 85 cm), of the error in the time T_1 measured by Probe 1.

Figure 10 shows the effect of errors in T_2 in the same chamber.

Figure 11 refers to another rectangular chamber (75 × 85) and shows the propagation of the error due to Probe 1 through the active area of the entire chamber.

Now, assuming that :

$$dR_1 = dR_2 = dR_3 = dR_4 = dR \quad (\text{max. error in } R_i) \tag{14}$$

from

$$dx = \frac{\partial x}{\partial R_1} dR_1 + \frac{\partial x}{\partial R_2} dR_2 + \frac{\partial x}{\partial R_3} dR_3 + \frac{\partial x}{\partial R_4} dR_4 \tag{15}$$

follows

$$\frac{dx}{dR} = \frac{\partial x}{\partial R_1} + \frac{\partial x}{\partial R_2} + \frac{\partial x}{\partial R_3} + \frac{\partial x}{\partial R_4} \tag{16}$$

and similarly

$$\frac{dy}{dR} = \frac{\partial y}{\partial R_1} + \frac{\partial y}{\partial R_2} + \frac{\partial y}{\partial R_3} + \frac{\partial y}{\partial R_4} \tag{17}$$

hence, one can define $\frac{dS}{dR}$ as follows :

$$\frac{dS}{dR} = \sqrt{\left(\frac{dx}{dR}\right)^2 + \left(\frac{dy}{dR}\right)^2} \tag{18}$$

The computer can provide a map of $\frac{dS}{dR}$ over the entire spark chamber.

The above equations (14) through (18), show that one can determine the probe configuration to be chosen for a given spark chamber area and for a given requirement on the accuracy of coordinate determination.

On the other hand, they can also be used to show the limitations on useful areas in already existing chambers.

4. DISCUSSION OF THE TWO SPARKS PROBLEM

All the considerations above refer to acoustical spark chambers built to handle a single spark. We want to add some remarks on the possibility of extending the detection capabilities of these devices to the handling of two simultaneous sparks. The chambers should in principle be able to detect the positions of two sparks provided that a sufficient number of probes is placed around the gap. The unknowns of the problem are seven :

x_1, y_1, x_2, y_2, ΔT_1, ΔT_2 and v_o

Therefore seven microphones should be enough. Where are these detectors to be placed ? An important condition must be obeyed: that the pressure wave from a spark reaches at least three probes.

One way of solving the problem could then be that of determining first which four probes have detected the same spark, and use the corresponding times-of-flight to calculate x_1, y_1, ΔT_1 and v_0. The remaining three times-of-flight would then be used to detect the second spark, making use of the known velocity v_0.

By pure geometrical considerations one can construct an active area which fulfils the condition above. Fig. 12 shows how to construct such an area from any given probe configuration. To increase the dimensions of the active area, the number of probes should be augmented. Fig. 13 shows an example of rectangular active area realized with eight probes. Wherever the two sparks occur in that region there will always be at least three probes receiving the wave from each spark.

We may mention that programmes are being written at CERN by Mr. Lefebvres, to deal with the two sparks, and that a chamber with eight probes has been constructed by Maglić for the tests.

In view of the limitations on the choice of experiments to be done with acoustical spark chambers, due to the restriction to only one spark per gap, we feel that this is a very important problem to solve.

ACKNOWLEDGEMENT

We would like to thank Dr. J.P. Scanlon, who initiated the study of the errors of spark location, for help with computations and for enlightening discussions.

Figure captions

Fig. 1 Approximate behaviour of the velocity of a shock wave as a function of time. The two curves refer to different spark energies.

Fig. 2 Graphical representation of $R = v_0 (T + \Delta T)$.

Fig. 3 Example of curves of constant K.

Fig. 4 Example of curves of constant I.

Fig. 5 Maps of the angle between curves of constant K and I.

Fig. 6 ditto.

Fig. 7 Effects of the displacement of 2 probes on the angles between curves of constant K and I.

Fig. 8 ditto.

Fig. 9 Propagation of errors in spark position determination, due to inaccurate measurement of time-of-flight by Probe 1. The curves are labelled with numbers giving the error in position as fraction of the error in the computed R_1. Chamber dimensions are 160 cm × 85 cm.

Fig. 10 Propagation of errors in spark position determination, due to inaccurate measurement of time-of-flight by Probe 2. The curves are labelled with numbers giving the error in position as fraction of the error in the computed R_2. Chamber dimensions are 160 cm × 85 cm.

Fig. 11 Propagation of errors in spark position determination as described for Fig. 9. Chamber dimensions are 75 cm × 85 cm.

Fig. 12 Construction of the useful area for two sparks for a given 7 probes configuration.

Fig. 13 Rectangular useful area for detection of two sparks, realized with an 8 probe configuration.

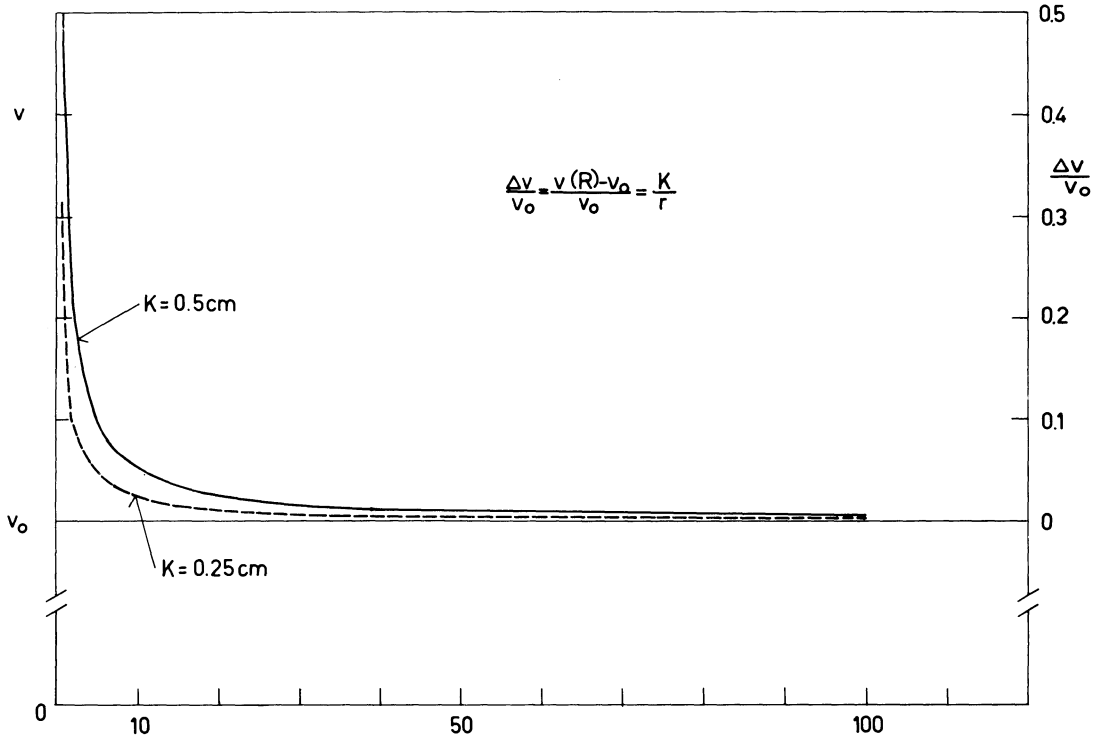

v
v_o
0
10
50
100
0.5
0.4
0.3
0.2
0.1
0
$\frac{\Delta v}{v_o}$
$\frac{\Delta v}{v_o} = \frac{v(R)-v_o}{v_o} = \frac{K}{r}$
K = 0.5cm
K = 0.25cm

Fig. 1

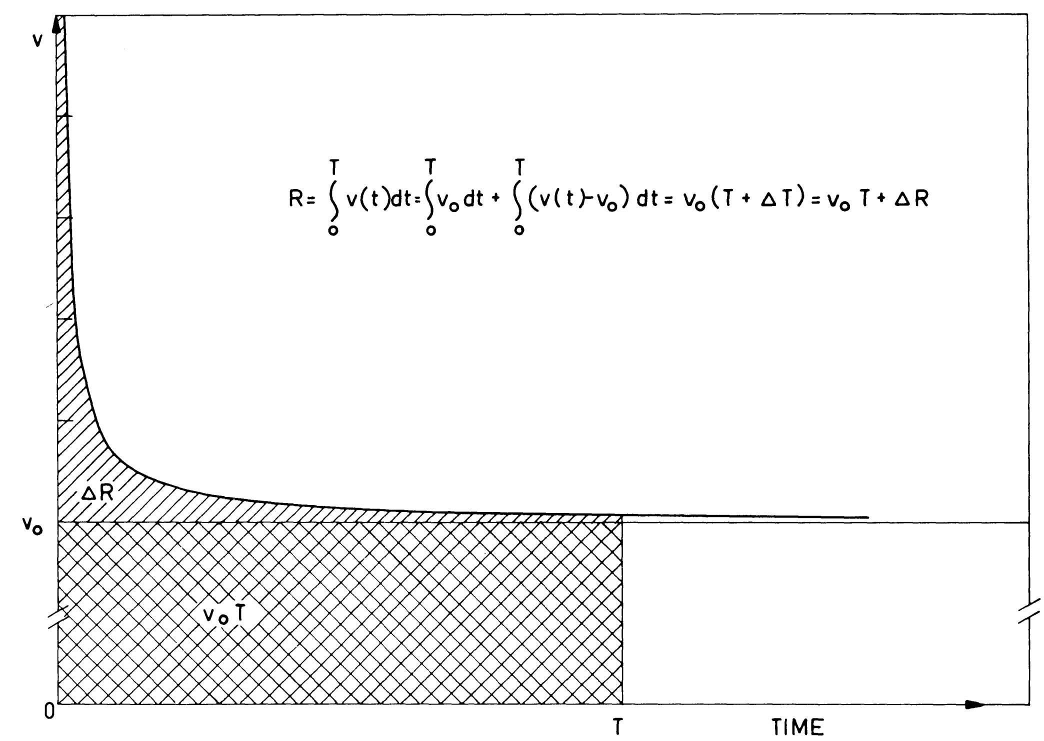

v
v_o
0
$R = \int_0^T v(t)dt = \int_0^T v_o dt + \int_0^T (v(t)-v_o)\,dt = v_o(T + \Delta T) = v_o T + \Delta R$
ΔR
$v_o T$
T
TIME

Fig. 2

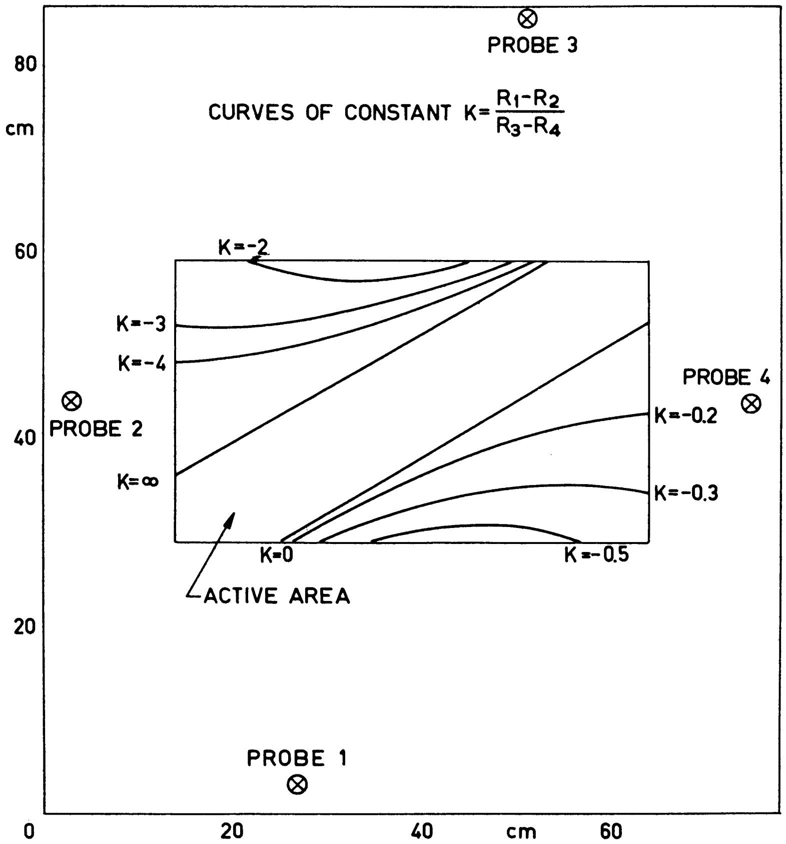

PROBE 3
CURVES OF CONSTANT $K=\frac{R_1-R_2}{R_3-R_4}$
K=-2
K=-3
K=-4
PROBE 4
PROBE 2
K=-0.2
K=∞
K=-0.3
K=0
K=-0.5
ACTIVE AREA
PROBE 1
80
cm
60
40
20
0
20
40
cm
60

Fig. 3

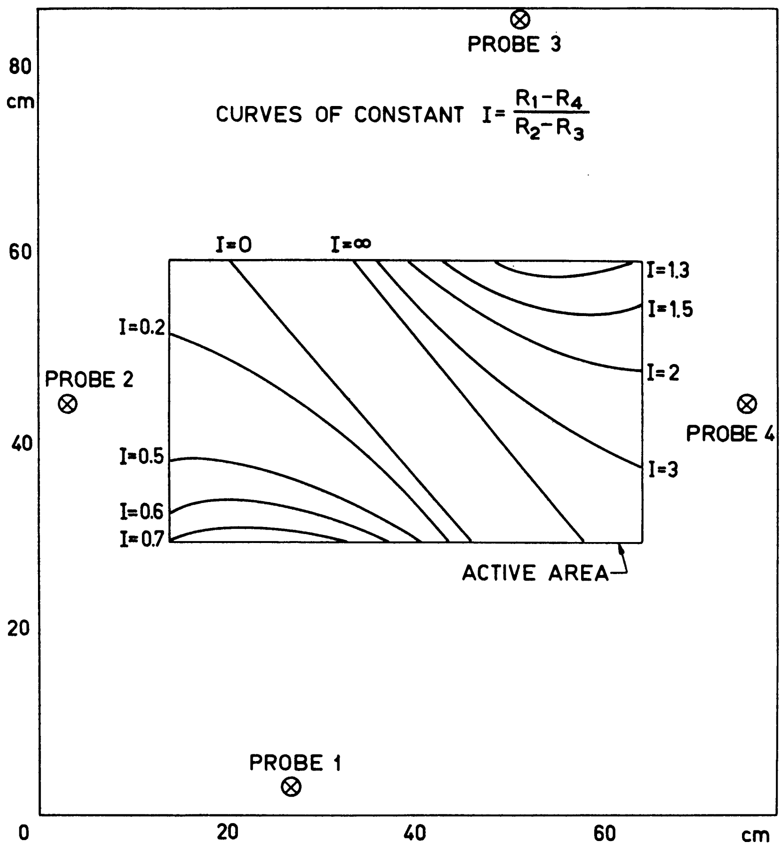

PROBE 3
CURVES OF CONSTANT $I=\frac{R_1-R_4}{R_2-R_3}$
I=0
I=∞
I=1.3
I=1.5
I=0.2
I=2
PROBE 2
PROBE 4
I=0.5
I=3
I=0.6
I=0.7
ACTIVE AREA
PROBE 1
80
cm
60
40
20
0
20
40
60
cm

Fig. 4

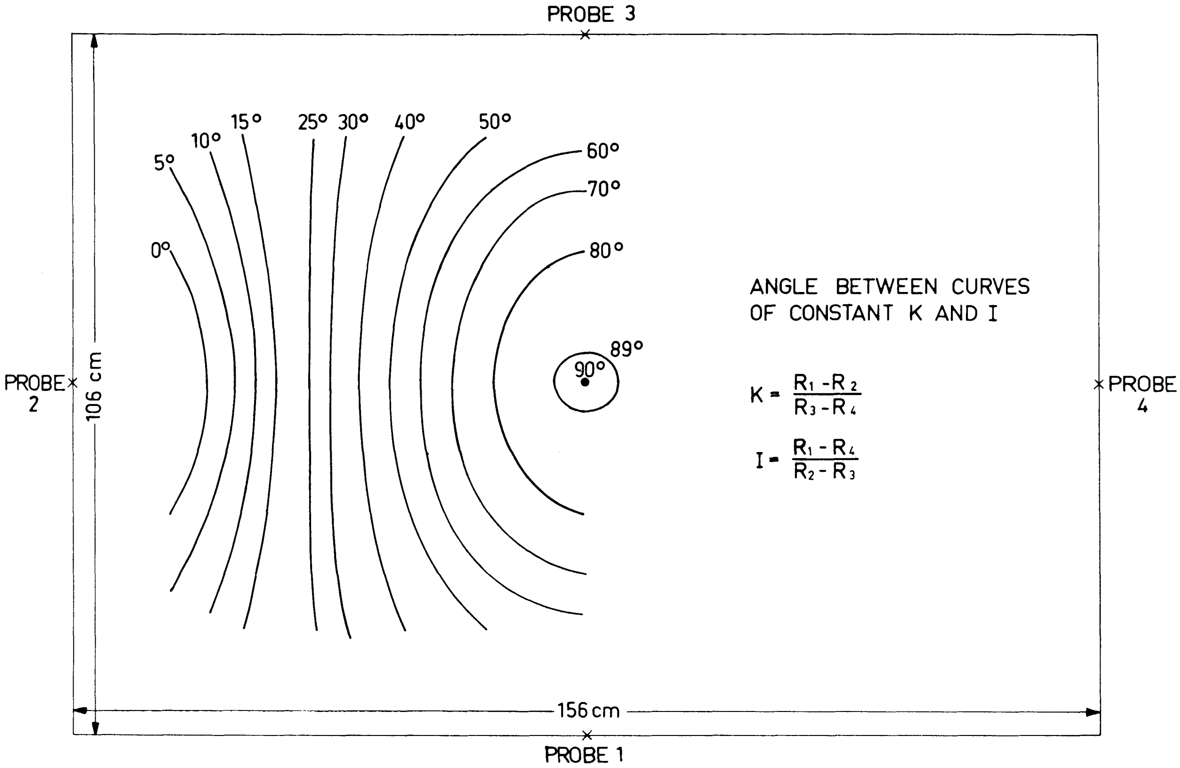

PROBE 3
PROBE 2
PROBE 4
PROBE 1
106 cm
156 cm
0°
5°
10°
15°
25°
30°
40°
50°
60°
70°
80°
89°
90°
ANGLE BETWEEN CURVES OF CONSTANT K AND I
$K = \frac{R_1 - R_2}{R_3 - R_4}$
$I = \frac{R_1 - R_4}{R_2 - R_3}$

Fig. 5

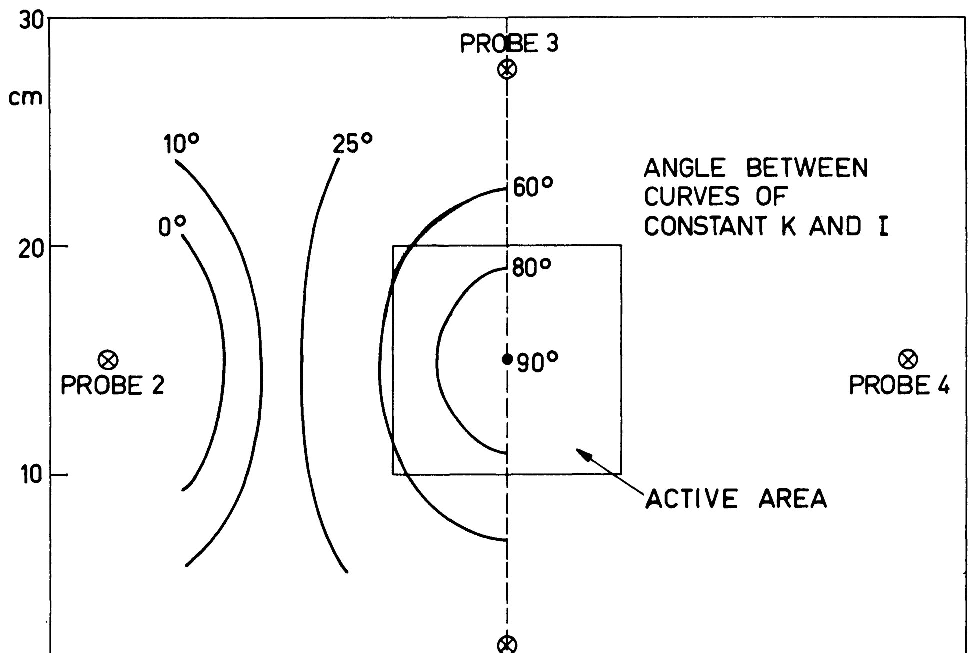

PROBE 3
PROBE 2
PROBE 4
PROBE 1
0°
10°
25°
60°
80°
90°
ANGLE BETWEEN CURVES OF CONSTANT K AND I
ACTIVE AREA
cm
0
10
20
30
40

Fig. 6

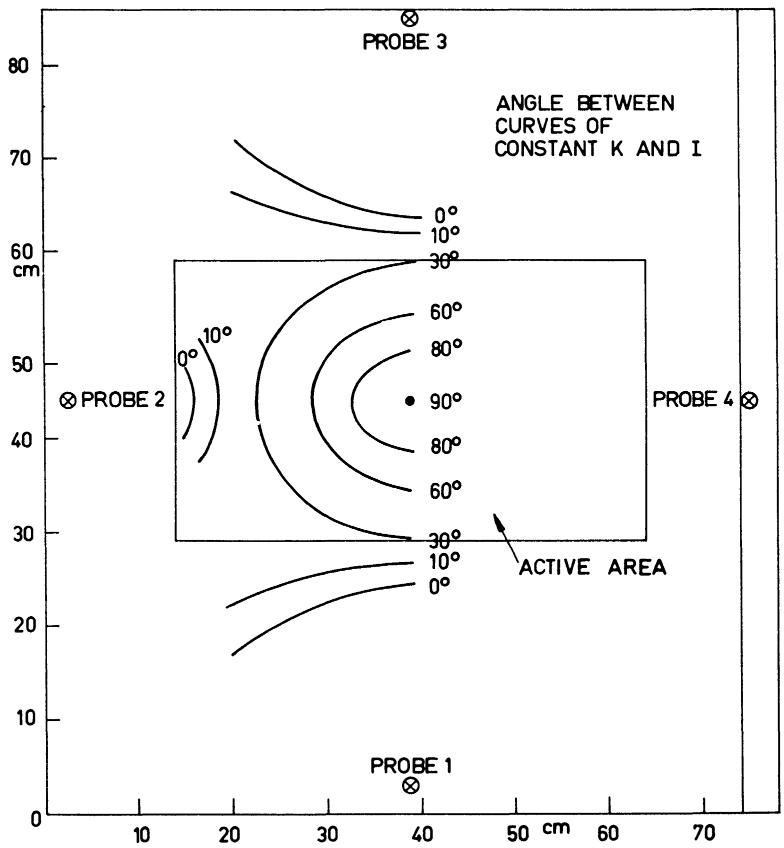

PROBE 3
ANGLE BETWEEN
CURVES OF
CONSTANT K AND I
0°
10°
30°
60°
80°
90°
80°
60°
30°
10°
0°
10°
0°
PROBE 2
PROBE 4
ACTIVE AREA
PROBE 1
80
70
60
cm
50
40
30
20
10
0
10
20
30
40
50 cm
60
70

Fig. 7

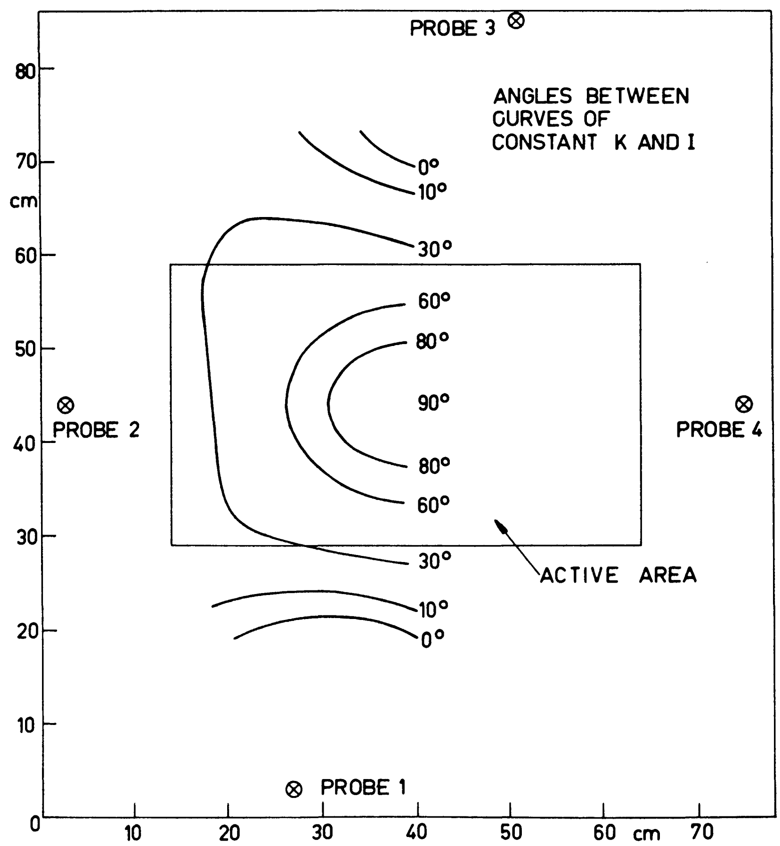

PROBE 3
ANGLES BETWEEN
CURVES OF
CONSTANT K AND I
0°
10°
30°
60°
80°
90°
80°
60°
30°
10°
0°
PROBE 2
PROBE 4
ACTIVE AREA
PROBE 1
80
70
cm
60
50
40
30
20
10
0
10
20
30
40
50
60 cm
70

Fig. 8

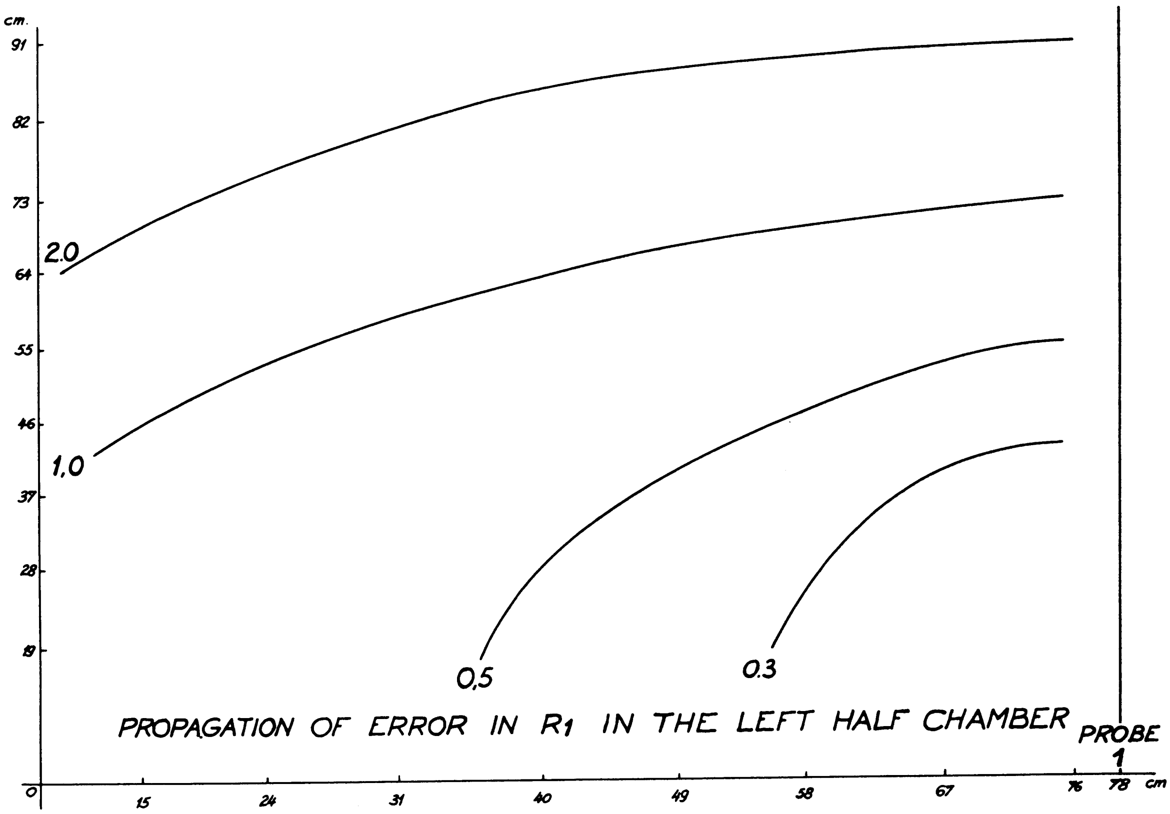

cm.
91
82
73
64
55
46
37
28
19
0
2.0
1,0
0,5
0,3
PROPAGATION OF ERROR IN R1 IN THE LEFT HALF CHAMBER
PROBE
1
15
24
31
40
49
58
67
76
78
cm

Fig. 9

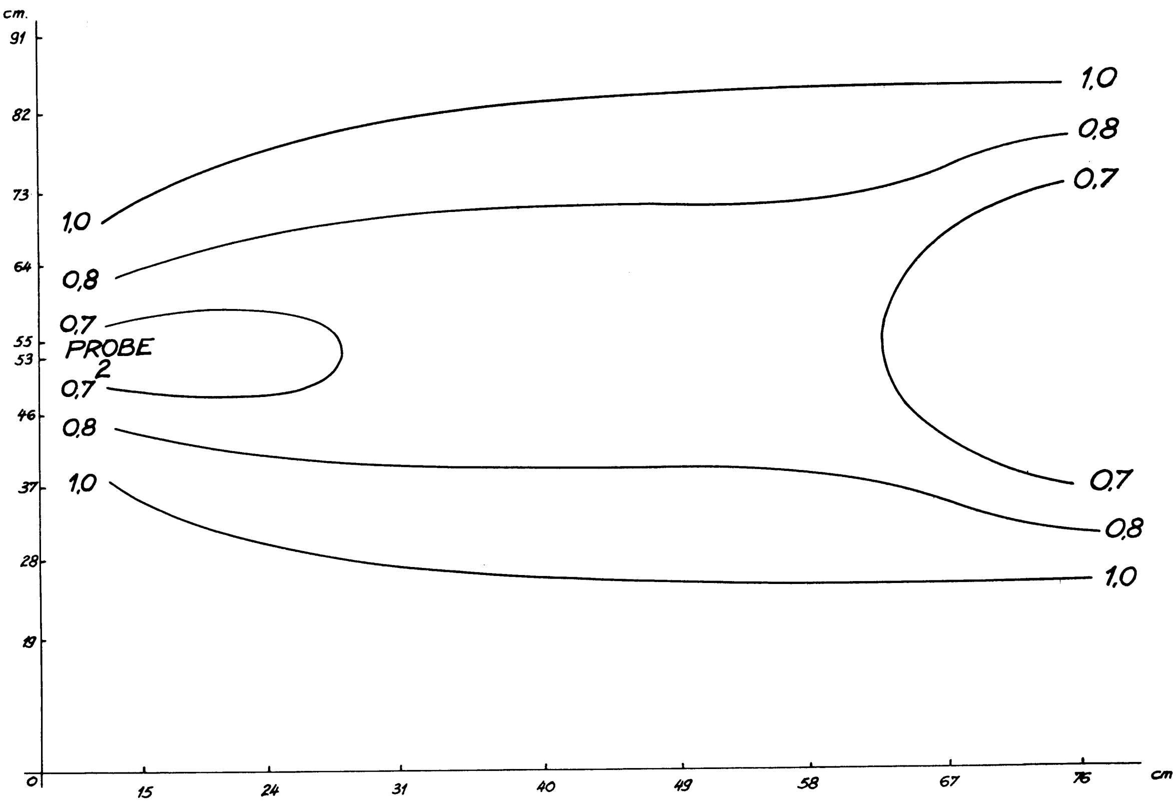

cm.
91
82
73
64
55
53
46
37
28
19
0
1,0
0,8
0,7
PROBE
2
0,7
0,8
1,0
1,0
0,8
0,7
0,7
0,8
1,0
15
24
31
40
49
58
67
76
cm

Fig. 10

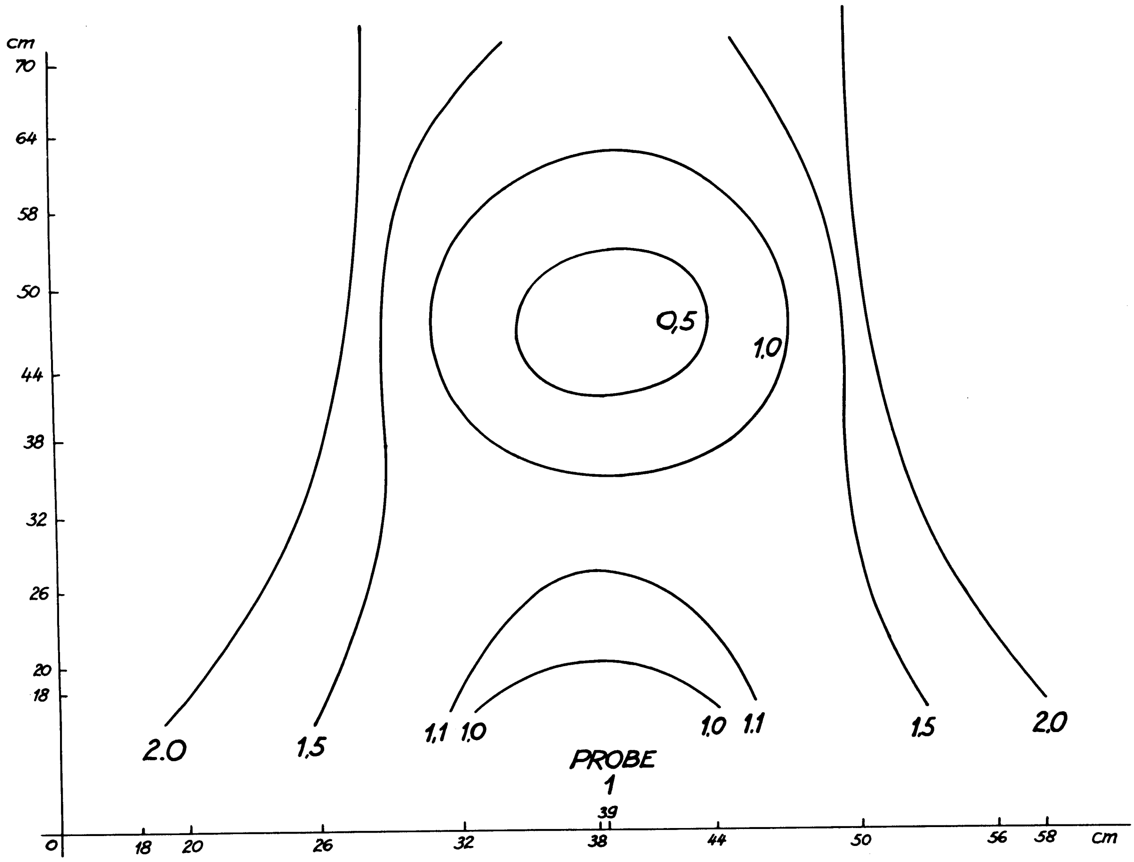
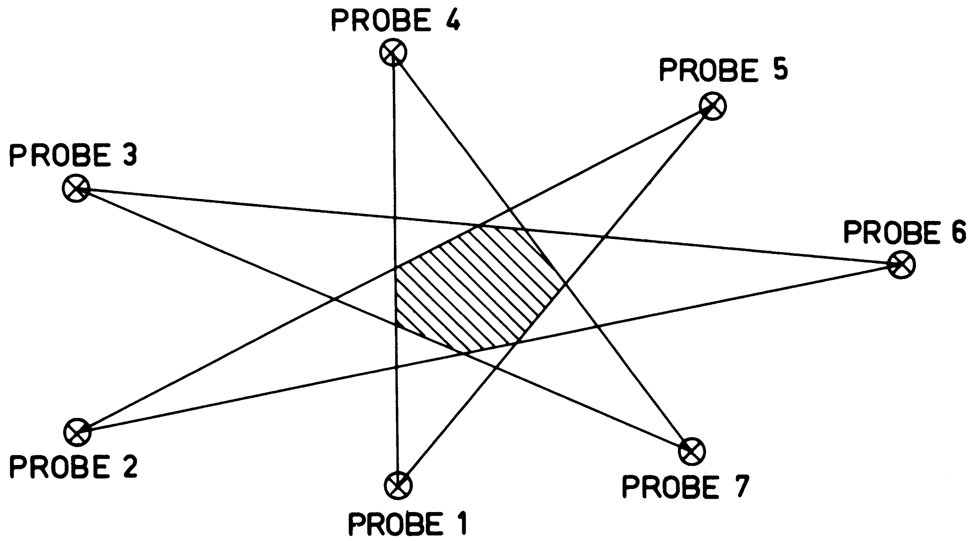


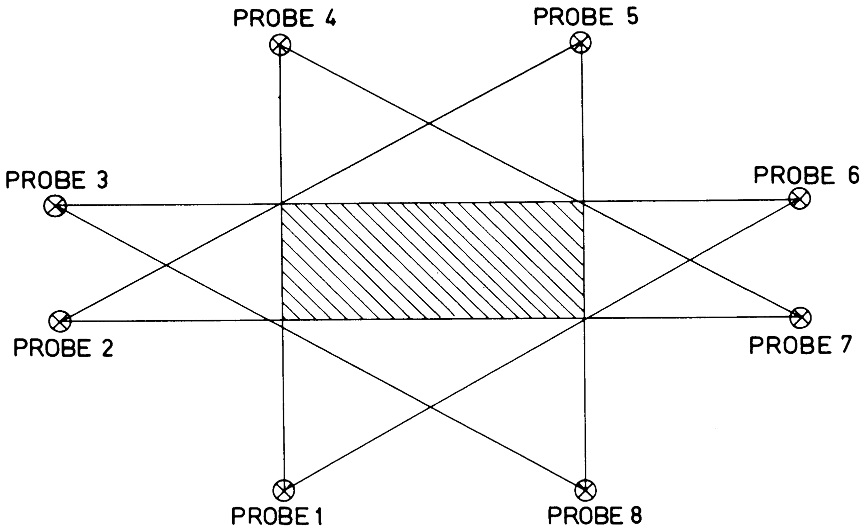
Fig. 11



POSSIBLE ACTIVE AREA FOR TWO SPARKS

(WITH 7 PROBES)

Fig. 12



EXAMPLE OF RECTANGULAR POSSIBLE ACTIVE AREA
FOR TWO SPARKS (WITH 8 PROBES)

Fig. 13

DISCUSSION

ELLISON: I am a bit worried to know if when you have done these computations you do in fact get physically reasonable values of ΔT and V_0.

ZANELLA: What we do is to keep on testing V_0 and ΔT and if they are too far from physically meaningful values we reject the solution.

LINDENBAUM: I was going to say this tomorrow, but in answer to your question, we actually started the other way. Being ignorant, we got a computer programme to find the solutions exactly. We found that you could trade large differences between V_0 and ΔT. For example, we could get a solution for X and Y which was physically sensible and not too far from the actual X and Y, but we got a ΔT which we knew wasn't true.

MAGLIC: As Taylor showed, ΔT is not common to all four probes. In fact $\Delta T \propto \log T_i$. That is why an exact solution is hard to obtain. The iterative procedure, however, averages ΔT_i's.

ROBERTS: For seeing more than one spark, microphones with good recovery might be preferable to special programmes and indefinite increase in the number of microphones. This must certainly be possible. I don't know any theoretical reason why one shouldn't be able to build such microphones.

ZANELLA: Yes, we have thought about this possibility. However, since we have existing programmes to handle things we just try to increase the number of microphones and limit the active region.

ROBERTS: I should like to see what happens with five sparks.

ZANELLA: At present we think that two sparks is already a difficult problem to solve and we don't think of more sparks.

ANDREWS: We have a microphone where you can clearly see two sparks, but we haven't yet got any electronics that will allow you to sort the two out, so it may in fact be easier to use single response microphones and the simpler electronics than to devise more complicated electronics for microphones with two pulses.

R.H. MILLER: It is interesting that this two spark solution completely sidesteps the question as to whether you get in trouble when one sound wave crosses another. I wonder if anybody has yet demonstrated that you can let one sound wave cross another without getting into some interesting kind of non-linearity or apparent velocity variation as it does.

BARDON: We have made an intense region of sparking with a source in one area of the spark chamber and detected sound waves passing through that area, and have been able to locate the spark.

MAGLIC: I didn't understand this last sentence.

BARDON: We are able to locate the spark but I am afraid that this is not really an answer to the question. I think that all we have proved is that the disturbance caused by the local heating of the gas does not interfere with the spark measurement.

MAGLIC: If the spark is far away from the region of interference, the shock wave is a plane wave and can pass the obstacle without being disturbed. The scattering of sound by sound has been studied in artillery and the effect is basically the same as putting an obstacle between the receiver and the emitter.