ACCELERATOR STUDIES AT A.E.R.E., HARWELL

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I. F.F.A.G. CYCLOTRONS

1. Introduction

The original proposal of FFAG fields by K. R. Symon introduced a new family of accelerators, including spiral ridge cyclotrons. Much work has been done by Kerst, Symon and others in the MURA group and at Oak Ridge, the Carnegie Institute of Technology and elsewhere. Work at Berkeley, California on the "clover leaf" cyclotron is also relevant.

We became interested in the spiral ridge cyclotron because it obviously offers the possibility of high beam currents at energies somewhat higher than those economically obtainable with synchrocyclotrons. There is even the possibility of continuous (constant frequency or CW) operation at quite high energies.

In the following sections a linear theory is developed for the particle dynamics of spiral ridge cyclotrons. This theory has been checked against orbits computed by the digital computers at Manchester University and the National Physics Laboratory, and the results are in good agreement, for oscillations of small amplitude.

Estimates are made of the beam " blow-up " in passing both half-integral and integral resonances (see Appendices II-IV). The latter appear to be disastrous and therefore place an upper limit ($E/E_0 \simeq 2$) on the energy obtainable with C.W. operation. The half-integral resonances are also serious but present estimates suggest that they could be passed with very accurate shimming.

The frequency and field tolerances in C.W. cyclotrons, in order to avoid serious phase-slip, are also estimated. In practice, phase-slip may further limit the energy obtainable in C.W. machines to perhaps a few hundred Mev.

For higher energies, certainly above $\sim$ 900 Mev protons, it is necessary to introduce some degree of frequency modulation (FM), and to establish a fixed 'working point' in the later stages of acceleration by using a power-law field. It may be expedient to introduce a small degree of F.M. even in the early stages of acceleration. Various possible field laws and F.M. programmes are analysed in order to determine the system which gives the best duty cycle consistent with practicable tolerances on field and frequency.

A brief analysis is given of the problem of producing spiral-ridge type fields by means of ridged magnet pole-pieces. It is shown that in general rather small magnet gaps are necessary. Parameters appropriate to a C.W. machine of $\sim$ 240 Mev (conversion of Harwell 110-inch synchro-cyclotron) and to a F.M. machine of $\sim$ 3 Gev are given by way of illustration. The use of pole-face windings is also mentioned.

In Appendix I, the possibility of using a 'hybrid' cyclotron-synchrotron machine for energies $\sim$ 6 Gev is discussed very tentatively.

2. Particle dynamics

2.1 Magnet field law

The angular velocity ω ($= v/r$) of a particle in a cyclotron is given by the well known equation

$$\omega = \frac{eB}{m}$$

where $m = m_0/(1 - \beta^2)^{1/2}$ and $\beta = v/c = \omega r/c$

For constant frequency (C.W.) operation the field law required is therefore

$$B = B_0(1 - \beta^2)^{-1/2} = B_0 \left[1 - \left(\frac{r}{r_\infty}\right)^2\right]^{-1/2} \qquad (1)$$

where $eB_0 = m_0\omega$ and $r_\infty = c/\omega$. The corresponding value of the field index 'n' is

$$n = -\frac{r}{B}\frac{\partial B}{\partial r} = -\left[\left(\frac{B}{B_0}\right)^2 - 1\right] = -\left[\left(\frac{E}{E_0}\right)^2 - 1\right]$$

The vertical motion is therfore unstable, and for the (stable) radial motion the number of free oscillations per orbit revolution is given by

$$Q_r \simeq \frac{E}{E_0} \tag{2}$$

Kerst, Symon et al. have shown that the vertical instability may be overcome by application of spiral ridge fields, and that, to a first order, the radial focusing is unaffected. Consequently there will be dangerous linear resonances at energies equal to integral multiples of half the rest energy. In addition, there are the well known coupled and non-linear resonances.

2.2 Radial and vertical stability - linear theory

The Lorentz equations of motion for a particle in a time-invariant magnetic field are, in cylindrical coordinates:

$$\begin{aligned} \frac{d}{dt}(m\dot{r}) - mr\dot{\theta}^2 &= er\dot{\theta}\,Bz - e\dot{z}B_\theta \\ \frac{d}{dt}(m\dot{z}) &= e\dot{r}B_\theta - er\dot{\theta}B_r \end{aligned} \tag{3}$$

Also, for particles with the same energy

$$dz^2 + r^2 d\theta^2 + dr^2 = v^2 dt^2 \tag{4}$$

Since time does not enter explicitly in these equations, it is convenient to use θ as the independent variable, giving

$$\begin{aligned} \frac{d^2r}{d\theta^2} - r &= \frac{er}{mv}\left[r\,B_z - \frac{dz}{d\theta}B_\theta\right] \\ \frac{d^2z}{d\theta^2} &= \frac{e}{mv}\left[\frac{dr}{d\theta}\,r\,B_\theta - r^2B_r\right] \end{aligned} \tag{5}$$

in which it is assumed that

$$\frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2, \frac{1}{r^2}\left(\frac{dz}{d\theta}\right)^2 \ll 1$$

Following Kerst, Symon et al. we assume the field in the median plane of the magnet to be

$$B_z = B_1\left(\frac{r_1}{r}\right)^k\left[1 + \delta\sin\left(N\,\theta - K\,ln\,\frac{r}{r_1}\right)\right] \tag{6}$$

For constant k and K, this field leads to "scaled dynamics", i.e. Q_r and Q_v are independent of particle momentum. In applying the field to the spiral ridge cyclotron, both k and K will vary with momentum, being determined by the type of operation (C.W. or F.M.) and the degree of vertical focusing required.

Substituting (6) in (5) the equations of motion become

$$\begin{aligned} \frac{d^2x}{d\theta^2} - x &= -x^{2+k}[1 + \delta\sin(N\,\theta - K\,ln\,x)] \\ \frac{d^2\eta}{d\theta^2} &= -x^k\,\delta\,N\cos(N\,\theta - K\,ln\,x)\frac{dx}{d\theta} \\ &\quad + x^2\frac{\partial}{\partial x}\left\{x^k[1 + \delta\sin(N\,\theta - K\,ln\,x)]\right\} \end{aligned} \tag{7}$$

where r_1 is now defined in terms of the momentum of the particle by

$$e\,r_1\,B_1 = -mv \tag{8}$$

and x, η are the normalised parameters r/r_1, z/r_1 respectively.

<u>The closed orbit</u>

The closed orbit can now be obtained by expanding the radial equation about the normalised radius.

$$\text{Substituting } x = 1 + \xi \tag{9}$$

and retaining only linear terms, this leads to the inhomogeneous equation

$$\ddot{\xi} + [(1 + k) + \delta\,A\sin(N\,\theta - \chi)]\,\xi = -\delta\sin N\,\theta \tag{10}$$

where

$$A^2 = (2 + k)^2 + K^2, \quad \tan\chi = \frac{K}{(2 + k)} \tag{11}$$

An accurate series solution can be found for the periodic solution of this equation by a variety of methods. For the machines at present under consideration only the first few terms are important and can readily be obtained by iteration.

Rewriting (10) as

$$\ddot{\xi} + (1 + k)\,\xi = -\delta\sin N\,\theta - [\delta\,A\sin(N\theta - \chi)]\,\xi$$

and ignoring the second term on the RHS gives

$$\xi_1 = \frac{\delta\sin N\,\theta}{N^2 - (1 + k)}$$

Substitution of this in the second term of the RHS, and solving again for the periodic solution, gives

$$\xi_2 = \frac{\delta\sin N\,\theta}{N^2 - (1 + k)} - \frac{\delta^2\,A}{N^2 - (1 + k)}\cdot\frac{1}{2}\left[\frac{\cos(2N\theta - \chi)}{4\,N^2 - (1 + k)} + \frac{\cos\chi}{1 + k}\right] \tag{12}$$

In this expression for the closed orbit, the first term predominates and is a sufficiently good approximation for most purposes in obtaining design data. The third term is independent of θ and therefore indicates a shift in the mean radius of the closed orbit from $r = r_1$. This means that the angular frequency of a particle of given momentum is slightly higher than it would be if there were no flutter, and a slight correction of the mean field value will be required.

Radial stability

Substituting

$$x = 1 + \xi(\theta) + \rho \tag{13}$$

in (7), and retaining only linear terms, gives

$$\ddot{\rho} + \left[\begin{array}{l}(1+k) + (2+k)\delta \sin N\theta - \delta K \cos N\theta \\ \quad + \xi \left\{\begin{array}{l}(1+k)\{(2+k)(1+\delta \sin N\theta) \\ - \delta K \cos N\theta\} - (2+k)\delta K \cos N\theta \\ - \delta K^2 \sin N\theta\end{array}\right\}\end{array}\right]\rho = 0 \tag{14}$$

In this approximation we have retained a term in $\xi\rho$ which has a significant effect on the focusing. A term in ξ^2 has been ignored since its principal effect is to alter the mean orbit radius slightly.

Examination of the order of magnitude of the terms in (14) suggests that, for preliminary design work, many can be ignored. Using the first term in (12) for the closed orbit, equation (14) reduces to

$$\ddot{\rho} + \left\{(1+k) - \frac{\delta^2 K^2}{2[N^2 - (1+k)]} - \delta K \cos N\theta\right\}\rho = 0 \tag{15a}$$

for small angle spiral $(K/N \to \infty)$, or

$$\ddot{\rho} + \left\{(1+k) + \frac{(1+k)(2+k)\delta^2}{2[N^2 - (1+k)]} + \frac{N^2(2+k)}{[N^2 - (1+k)]} \times \delta \sin N\theta\right\}\rho = 0 \tag{15b}$$

for radial ridges $(K = 0)$. These equations may both be transformed into the general Mathieu form

$$\frac{d^2\rho}{d\psi^2} + (a + 2q \cos 2\psi)\rho = 0 \tag{16}$$

from which the mode number μ is obtained to a good approximation by

$$\cos \mu = \cos(\pi\sqrt{a}) - \frac{\pi q^2}{4\sqrt{a}(1-a)} \sin(\pi\sqrt{a}) \tag{17}$$

where $\mu = 2\pi Q/N$

When $\mu \ll \pi$ the above approximation reduces to the 'smooth approximation' used by other workers.[3] For small angle spirals

$$\begin{aligned} Q_r &\simeq \frac{N^2}{4}\left[a + \frac{q^2}{2(1-a)}\right] \\ &= (1+k)\left(1 + \frac{3\delta^2 K^2}{2N^4}\right) \\ &\to 1 + k \text{ for large } N \end{aligned} \tag{18}$$

where $(1+k) \simeq (E/E_0)^2$ in a C.W. cyclotron.

When $\mu = \pi$, the accurate formula relating a to q is

$$a = 1 - q - \frac{q^2}{8} + \ldots . \tag{19}$$

Comparing this with the approximate solution obtained from (17) indicates that (17) is a good approximation for $q < 1.0$ for values of $\mu < \pi$.

For the case of Hill's equation

$$\frac{d^r\rho}{d\psi^2} + [a + 2qf(\psi)]\rho = 0 \tag{20}$$

it can be shown that the general form of (16) is

$$\cos \mu = \cos \pi\sqrt{a} + \frac{q^2}{a}\int_0^{\pi}\int_0^{\psi} f(\psi_1) f(\psi_2) \times \cos\left\{\sqrt{a}(\pi - 2\psi_1 - 2\psi_2)\right\} d\psi_2 d\psi_1 \tag{21}$$

where $f(\psi)$ is periodic in the interval π and has zero mean value.

Vertical stability

The equation of vertical motion is obtained by expanding about the closed orbit $x = 1 + \xi$, and retaining only linear terms.

Equation (7) then reduces to

$$\ddot{\eta} + \left[\begin{array}{l}(\delta N \cos N\theta)\frac{d\xi}{d\theta} - k(1 + \delta \sin N\theta) + \delta K \cos N\theta \\ + \xi\left\{\begin{array}{l}K^2\delta \sin N\theta + (2k+1)K\delta \cos N\theta \\ \quad - k(k+1)(1 + \delta \sin N\theta)\end{array}\right\}\end{array}\right]\eta = 0 \tag{22}$$

When (12) is substituted for the closed orbit, this reduces to a Hill equation and can be solved by the methods of the previous section. Again many terms can be ignored. Retaining the most important, and using the first order term for the closed orbit, reduces (22) to

$$\ddot{\eta} + \left[-k + \frac{\delta^2 N^2}{2[N^2 - (1+k)]} + \frac{\delta^2 K^2}{2[N^2 - (1+k)]} + \delta K \cos N\theta\right]\eta = 0 \tag{23}$$

For the "smooth approximation" we have

$$Q_v^2 = -k + \frac{\delta^2(N^2 + K^2)}{2[N^2 - (1+k)]} + \frac{\delta^2 K^2}{2N^2} \tag{24}$$

This is sufficiently accurate for most purposes, because $Q_v = 0$ near the centre of the cyclotron and cannot become greater than $Q_v = 1.0$ elsewhere in order to avoid disastrous integral resonances.

2.3 Linear and non-linear resonances

These have been studied experimentally by other workers in connection with A. G. synchrotrons. The cyclotron differs in that the 'working point' moves during the acceleration cycle, and one or more resonances may have to be crossed. Reference has already been made to the half-integral radial resonance at $Q_r = 1.5$, and to the integral resonance at $Q_r = 2$. It may be shown (see Appendices II and III) that the $Q_r = 1.5$ resonance may be passed by careful design and construction but that the $Q_r = 2$ resonance would be disastrous.

The $Q_r = 1.5$ resonance is caused, in general, by a third harmonic error in the field-gradient (note that when $N = 2Q_r = 3$ the resonance coincides with the intrinsic $\mu = \pi$ stop-band). Similarly the $Q_r = 2$ resonance is associated with a second harmonic error in field amplitude (coinciding with $\mu = \pi$ when $N = 4$). It is equally important that Q_v should not cross half-integral or integral resonances. Since Q_v is fixed by the choice of flutter depth and is zero at the machine centre, the field tolerance must be such that

$$0 < Q_v < 1$$

at least, and preferably $Q_v < 0.5$.

The working point must therefore lie in the region

$$1 < Q_r < 1.5$$
$$0 < Q_v < 0.5$$

if half-integral resonances cannot be crossed, or, since it is almost certain that integral resonances cannot be crossed, in the extreme case

$$1 < Q_r < 2$$
$$0 < Q_v < 1$$

Inside this region there are also the linear coupled resonances at

$$Q_r + Q_v = 2, \qquad Q_r - Q_v = 1,$$

of which the former produces a stop band, and in the case of synchrotrons is considered to be more serious than the latter. For the cyclotron, the vertical aperture is small and the vertical focusing weak, and it is likely that both these resonances are serious.

There are also non-linear resonances. Because of the non-linear nature of the fields, some are intrinsic and exist with or without machine errors. The quadratic non-linear resonance occurs at $\mu = 2\pi/3$, or $Q = N/3$, and the coupled resonances occur when

$$\begin{cases} 2Q_v - Q_r = O \\ 2Q_v + Q_r = N \end{cases} \qquad \begin{cases} Q_v + 2Q_r = N \\ Q_v - 2Q_r = O \end{cases}$$

The second pair does not occur when the magnet is symmetrical about the median plane.

The cubic non-linear resonances occur at $\mu = 2\pi/4$, or $Q = N/4$, and the coupled resonances when

$$\begin{cases} 3Q_v - Q_r = O, N \\ 3Q_v + Q_r = N \end{cases} \begin{cases} 3Q_r - Q_v = O, N \\ 3Q_r + Q_v = N \end{cases} \begin{cases} 2Q_v - 2Q_r = O \\ 2Q_v + 2Q_r = N \end{cases}$$

Only the last pair occurs in a symetrical magnet.

If there are machine errors, non-linear resonances will also occur whenever N in the above formulae is an integer. In the absence of errors, N = number of periods round the machine.

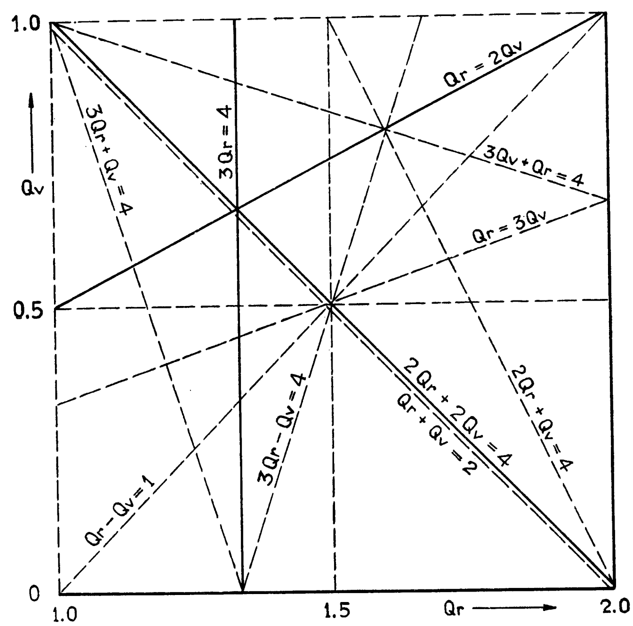


Fig. 1a. 4 Ridge magnet

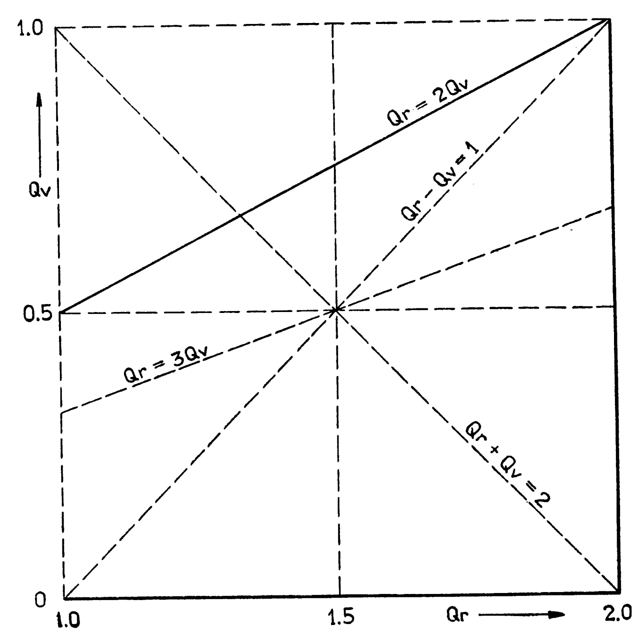


Fig. 1b. 8 Ridge magnet

Figure 1(a) gives a (Q_v, Q_r) diagram of the above-mentioned resonances for the case N = number of periods = 4 (a 4 ridge magnet), in so far as they lie within the extreme limits $0 < Q_v < 1$, $1 < Q_v < 2$. Figure 1(b) gives similar data for an 8-ridge machine. Intrinsic resonances are shown as full lines, 'error' resonances as dotted lines.

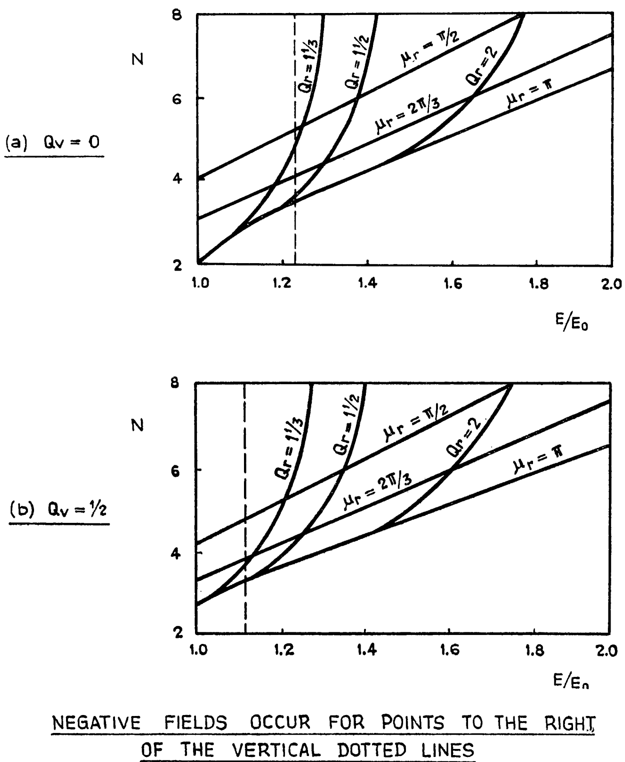


Fig. 2. Radial ridges

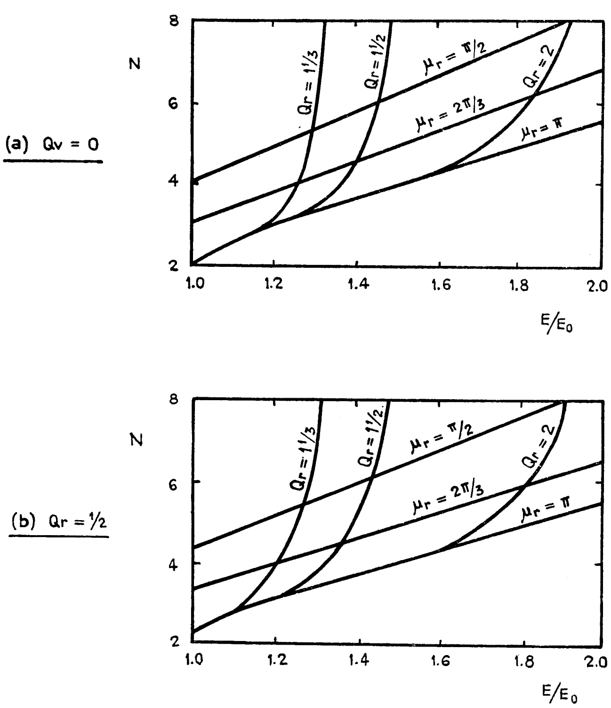


Fig. 3. Spiral ridges

3. Design data for C.W. cyclotrons

Figures 2 and 3 show radial mode number, μ_r ($= 2\pi Q_r/N$), and Q_r as functions of energy and ridge number for $Q_v = 0, \frac{1}{2}$. More accurate formulae were used than those given above, but the difference is not significant and is likely to be of the same order of magnitude as some non-linear terms which have been ignored. Figure 2 is plotted for radial ridges and figure 3 for ridges of flutter $\delta = 0.1$ (approximating to the limit of 'tight spirals' at high energies). Figure 4 gives more extensive data for $\delta = 0.1$.

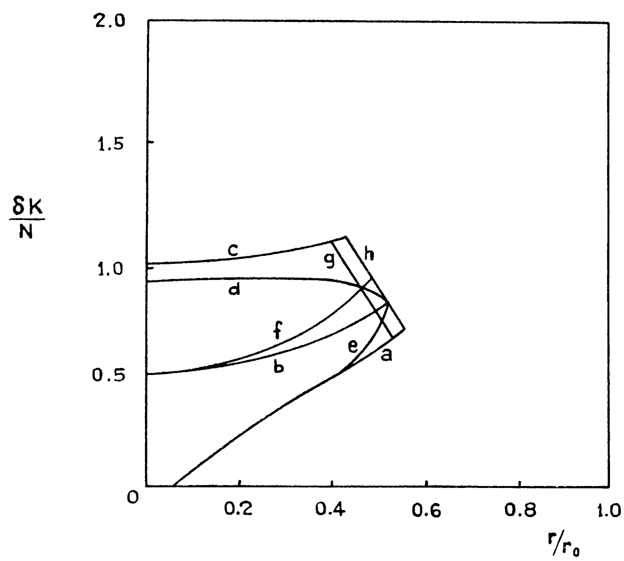


Fig. 4a. N = 3

The graphs are mostly self-explanatory, but attention should be drawn to the following features :

(*a*) For radial ridges (K = 0) we have

$$Q_v^2 \simeq -k + \frac{\delta^2}{2} = 1 - \left(\frac{E}{E_0}\right)^2 + \frac{\delta^2}{2}$$

For limitting vertical focusing ($Q_v = 0$)

$$\delta^2 = 2\left[\left(\frac{E^2}{E_0} - 1\right)\right]$$

That is, $\delta = 1.0$ for $(E/E_0)^2 = 1.5$: above this energy, negative fields are required in alternate sectors. Similarly, for $Q_v = \frac{1}{2}$, we have $\delta = 1.0$ for $(E/E_0)^2 = 1.25$.

(*b*) For large N, and $\mu_r \ll \pi$

$$Q_r \simeq (1 + k)^{\frac{1}{2}} = \frac{E}{E_0}$$

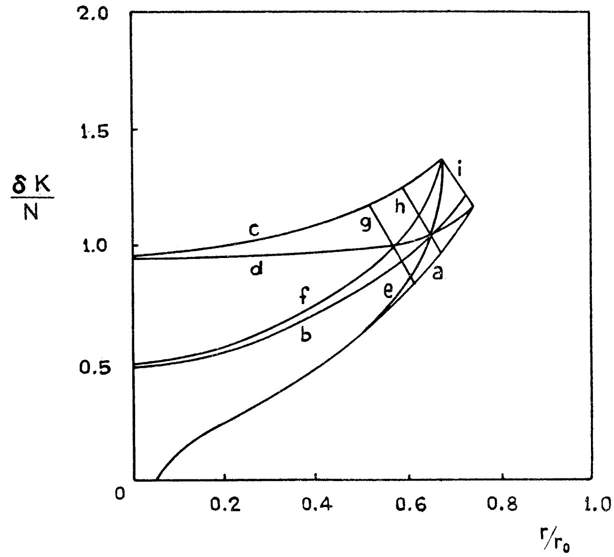


Fig. 4b. N = 4

As π-mode is approached, this simple formula underestimates Q_r. For example, at π-mode, $Q_r = N/2$ and hence, according to the simple formulae, the corresponding energy is $NE_0/2$. In actual fact the π-mode limit is reached very much earlier than this, as the curves show.

(*c*) At the centre of the machine, $Q_r = 1$ and increases with energy, the first half-integral resonance being reached at $Q_r = 1.5$, and the first integral resonance at $Q_r = 2$. As already mentioned, it is shown in Appendices II and III that the half-integral resonance may be passed with careful design and construction, but that the integral resonance would be disastrous. For $N > 4$, this $Q^r = 2$ limit falls below π-mode. Note that the limiting energies are less with radial ridges than with tight spirals.

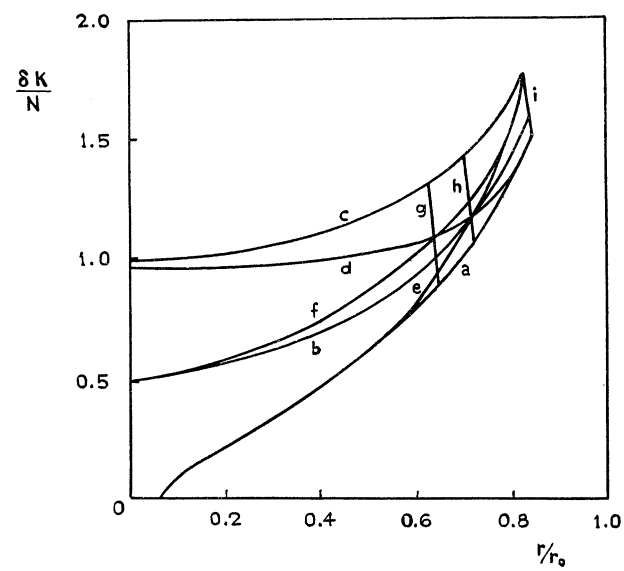


Fig. 4c. N = 6

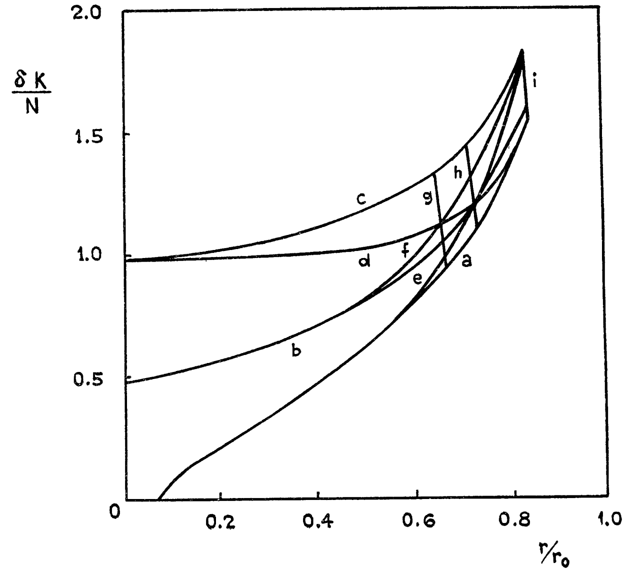
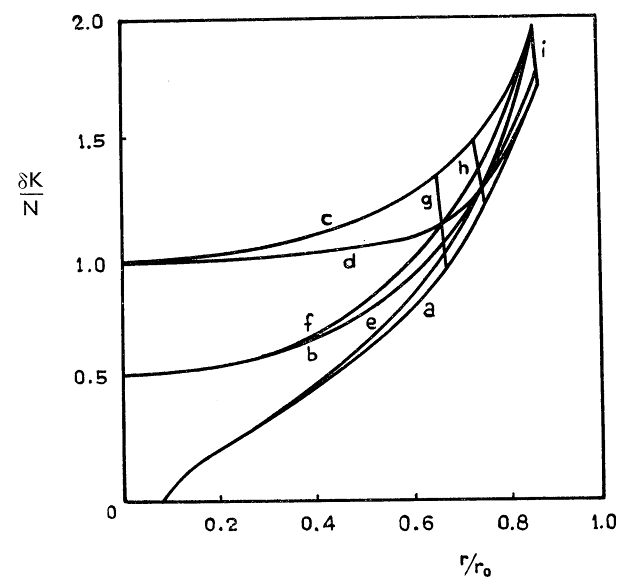


Fig. 4d. N = 8

(*d*) For N = 3, $Q_r = 1.5$ coincides with the π-mode limit, and falls well below $E/E_0 = 1.5$. Computations on the N.P.L. digital computer have confirmed that the maximum energy obtainable with N = 3 is (200 $\pm$ 10) Mev for protons.

(*e*) For N = 4 it is important to notice that an intrinsic non-linear resonance occurs at $Q_r = 4/3$. Thus the 'working point' should start at $Q_r = 1.0$, $Q_v = 0$ and finish at $Q_r = 4/3$, $Q_v \simeq 1/2$. Computations on the digital computer confirm that this limits the energy to about 250 Mev



a $Q_v = 0$ d $Q_r + Q_v = 2$ g $Q_r = 1\frac{1}{3}$
b $Q_v = \frac{1}{2}$ e $Q_r = 1 + Q_v$ h $Q_r = 1\frac{1}{2}$
c $Q_v = 1$ f $Q_r = 2Q_v$ i $Q_r = 2$

Fig. 4e. N = ∞

for protons, but the limit may extend down to 240 Mev for large amplitudes ($\rho \simeq 0.01$) of radial oscillation. More computations are required to confirm this.*

(*f*) If N is increased to 8, $Q = 2$ corresponds to $\mu = \pi/2$, and all intrinsic non-linear resonances lie outside the possible working region (see figures 1(*b*) and 4(*d*)). This would suggest that $N = 8$ is the best choice for a high energy machine. However, as the ridges converge towards the centre of the machine it becomes increasingly difficult, in practice, to produce the necessary flutter fields, and this trouble is more serious for large N. One solution might be to use 4 ridges at the centre, increasing to 8 at some intermediate radius before Q_r reaches the intrinsic resonance at $Q_r = {}^4/_3$.

4. Frequency and field tolerances in C.W. cyclotrons

The above discussion has how shown how resonance phenomena limit the maximum energy obtainable in a C.W. cyclotron. In practice the problem of R.F. phase slip may prove to be the limiting factor in some cases, since a strictly fixed frequency machine has no phase stability.

The phase equation for a particle in a C.W. cyclotron is

$$\frac{dE}{d\theta} = \frac{eV_0}{2\pi} \sin(\theta - \omega_f t) \tag{25}$$

which can be written as

$$\frac{d}{dt}\left(mr^2\dot{\theta} - e\int rBdr\right) = \frac{eV_0}{2\pi} \sin(\theta - \omega_f t) \tag{26}$$

For an FFAG cyclotron, B refers to the mean field and r to the mean orbit radius of the particle.

Multiplying the second equation by ω_f and integrating with respect to $(\theta - \omega_f t)$ gives

$$E - \omega_f\left(e\, r^2 B - e\int rBdr\right) = -\frac{eV_0}{2\pi}\cos\varphi + \text{const.} \tag{27}$$

where we have put φ for $(\theta - \omega_f t)$.

This is an exact equation relating phase to radial position (i.e. particle energy) applicable to any C.W. cyclotron.

4.1 *Analysis neglecting effects of flutter fields*

For the C.W. field law (1), and with $\omega_f = \omega_o = c/r_o$, the phase of the particle remains constant. If the frequency ω_f is not equal to the cyclotron frequency then for the above field law

$$E\left[1 - \frac{\omega_f}{\omega_o}\right] = -\frac{eV_0}{2\pi}\cos\varphi + \text{const.} \tag{28}$$

For a phase slip of 180°, the tolerance on frequency is, therefore,

$$\frac{\Delta\omega}{\omega} = \frac{eV_0}{\pi}(E - E_0) = \frac{1}{R\pi} \tag{29}$$

where R is now the number of turns. For example, for an energy of 250 Mev with 250 Kev per turn,

$$\frac{\Delta\omega}{\omega} < \frac{1}{10^3\pi}$$

If, on the other hand, the frequency is correct but there is a uniform fractional error in the field law, i.e.

$$B = B_0 \frac{(1 + \varepsilon)}{[1 - (r/r_\infty)^2]^{1/2}} \tag{30}$$

then substituting in (27) we get

$$E_0\left[1 + \frac{r^2}{r_\infty^2}\frac{B^2}{B_o^2}\right]^{1/2} - \omega_f\left[e\, r^2 B - e\int r\, Bdr\right] = -\frac{eV_0}{2\pi}\cos\varphi + \text{const.} \tag{31}$$

To first order this gives

$$E\left(1 - \frac{\omega_f}{\omega_0}\right) - E_0\sqrt{1 - \beta^2}\,\varepsilon = -\frac{eV_0}{2\pi}\cos\varphi + \text{const.} \tag{32}$$

where $\omega_0 = c/r_0$. Thus for a phase slip of less than π, with $\omega_f = \omega_0$,

$$\varepsilon < \frac{eV_0}{\pi E_0}\frac{1}{1 - \sqrt{1 - \beta^2}}$$

For example, for $\beta = 0.6$, $V = 250$ Kev/turn

$$\varepsilon < \frac{1}{4000\pi}\frac{1}{0.2} = \frac{1}{800\pi}$$

If the frequency is adjusted so that $\Delta\omega/\omega = \varepsilon$, then for a phase slip less than π

$$\varepsilon < \frac{2eV_0}{\pi E_0^2}\frac{1}{\beta^4}$$

For the example quoted, $\varepsilon < \dfrac{1}{260\pi}$

4.2 *Phase slip due to field flutter*

It was shown earlier (see equation (12)) that the mean orbit radius depends on the flutter field. For a sinusoidal

* Further computations have shown that the $Q_r = 4/3$ resonance limits the amplitude of radial stability to very small values over a considerable range in energy. For example, at 200 Mev with a flutter factor of $\pm 20\%$, the radial motion becomes unstable for $\rho > 0.015$.

field the incremental change in mean orbit radius is

$$\frac{\Delta r}{r_1} = - \frac{\delta^2}{[N_2 - (1 + k)]} \frac{2 + k}{2(1+k)} \qquad (33)$$

For example, for $k < \frac{1}{2}$ (250 Mev approx.) $N = 4$, $\delta = 0.1$, $\frac{\Delta r}{r_1} \simeq - \frac{1}{2000}$

Thus, if the usual mean field law is taken, then according to the previous section this error would cause serious phase slip. It can be corrected, however, by changing the applied frequency.

Defining r_1 as before by $m\,v = e\,r_1\,B_1$, where B_1 is the mean field, the particle frequency is

$$\omega = \frac{v}{r_1 + \Delta r} \qquad (34)$$

For C.W. conditions,

$$B_1 = \frac{mv}{er_1} = m\omega\,(1 + \frac{\Delta r}{r_1})$$

$$= \frac{m_0\omega}{\left(1 - \frac{\omega^2 r_1^2}{c^2}\right)^{\frac{1}{2}}} \left[1 + \frac{\Delta r}{r_1} \frac{1}{\left(1 - \frac{\omega^2 r_1^2}{c^2}\right)}\right] \qquad (35)$$

where it has been assumed that $\Delta r/r_1 \ll 1$, and $\Delta\, r/r_1 \ll \frac{1 - \beta^2}{2\beta^2}$

Since $\frac{\Delta r}{r_1} \cdot \frac{1}{1 - \beta^2}$ is almost constant for $\beta < 0.6$, then the phase slip can be corrected by a fractional change in the magnetic field or frequency.

5. Frequency modulation

We have seen that with fixed frequency operation, although it may be possible with care to traverse the $Q_r = 1.5$ resonance (at $\sim$ 450 Mev proton energy for large N), the integral resonance at $Q_r = 2$ ($\sim$ 900 Mev) appears to be an impassable barrier. Therefore it is necessary for high energies to use a frequency modulated accelerating system. Furthermore the problem of phase slip may call for a degree of frequency modulation well below the energy at which radial resonance difficulties occur. We can consider a variety of possible high energy machines :

(*i*) fixed frequency and C.W. field law to $Q_r = 1.5$, or 2.0, followed by a frequency modulation programme appropriate to a field with constant "k" to maintain $Q_r =$ 1.5 or 2.

(*ii*) a slight variation of the C.W. field law, with a corresponding frequency modulation, up to $Q_r = 1.5$ or 2, followed by a constant "k" field as before.

The problem is to decide which particular arrangement gives the best beam duty cycle, consistent with realisable tolerances on field and frequency.

5.1 *Phase equation, trapping range, and duty cycle*

The analysis of Bohm and Foldy[4)] can be used initially. Phase grouping in the initial turns ensures that the starting phase for a machine with frequency modulation applied from the centre is 90° (the accelerating voltage being defined as $V_o \sin\varphi$). A parameter K is defined by

$$- K \frac{\Delta E}{E_s} = \frac{\Delta\omega}{\omega_s} \qquad (36)$$

where the subscript s refers to the synchronous particle. The resulting form of the phase equation is

$$\frac{d}{dt}\left(\frac{E\,\dot{\varphi}}{\omega_s^2\,K}\right) = - \frac{eV}{2\pi} (\sin\varphi - \sin\varphi_s) \qquad (37)$$

or

$$\ddot{\varphi} + \left(\frac{eV_o}{2\pi} \frac{\omega_s^2\,K}{E_s} \cos\varphi_s\right) (\varphi - \varphi_s) = 0 \qquad (38)$$

for small deviations from φ_s, and for $(E_s/\omega_s^2 K)$ effectively constant during a period of oscillation. From (38) we obtain, for small amplitudes of oscillation,

$$\text{frequency of phase oscillation } \omega_p = \omega_s \left(\frac{eVK\cos\varphi_s}{2\pi E_s}\right)^{\frac{1}{2}} \qquad (39)$$

adiabatic variation of amplitude

$$\Phi = \varphi - \varphi_s \sim \left[\frac{\omega_s^2\,K}{V_o \cos\varphi_s\,E^s}\right]^{\frac{1}{4}} \qquad (40)$$

Useful further definitions of K are :

$$K = 1 - \frac{k}{(1 + k)\,\beta^2} = \frac{1}{\beta^2} \left(\frac{1}{1 + k} - \frac{E_0^2}{E^2}\right)$$

For a conventional synchro-cyclotron, the field near the centre may be defined as

$$B = B_0 \left(1 - \frac{hr^2}{2}\right), K \simeq 1 + \frac{hc^2}{\omega_s^2}$$

K is virtually constant, and might be of the order of 2 near the centre. In a spiral ridge cyclotron, where $Q_r > 1$, the C.W. field law has $K = 0$, and is parabolic near the centre with negative 'h'.

Bohm and Foldy give the range of acceptance in terms of a frequency range

$$\Delta\omega_s = 2\left[\frac{eV_o K \omega_s^2}{\pi E_s} F_1(\varphi_0, \varphi_s)\right]^{\frac{1}{2}} \qquad (41)$$

where $F_1(\varphi_0, \varphi_s) = \cos\varphi_0 + \cos\varphi_s - (\pi - \varphi_s - \varphi_0)\sin\varphi_s$, and where $\varphi_0 =$ starting phase.

For a machine with frequency modulation from the centre, the condition must also be applied that particles do not return to the source :

$$\Delta\omega_s = 2\left[\frac{eV_0K\omega_s^2}{\pi E_s}\right]^{1/2} L \qquad (42)$$

where L cannot be expressed simply : for the standard condition of $\varphi_0 = 90°$ it has a maximum value of 0.57 for $\varphi_s \sim 30°$.

In usual synchro-cyclotron practice, matters are complicated by lack of exact matching of the frequency law to the field law, and for a substantially linear law of frequency against time $\Delta\omega_s$ has some immediate but possibly misleading meaning. In these considerations however $\Delta\omega_s$ has no immediate significance and it is much more valuable to define the trapping range in terms of total energy.

$$\Delta E = 2\left[\frac{eV_0E_s}{\pi K} F_1(\varphi_0, \varphi_s)\right]^{1/2} \qquad (43)$$

$$\Delta E = 2\left[\frac{eV_0E_s}{\pi K}\right]^{1/2} L \qquad (44)$$

We define the duty cycle as $\frac{\Delta T}{T}$ where

ΔT = time during which particles are trapped

T = total accelerating time.

If the trapping range is sufficiently small for $\frac{d\omega}{dt}$ to be effectively constant, then

$$\Delta T = \frac{dt}{d\omega}\Delta\omega = \frac{dt}{dE_s}\Delta E = \frac{2\pi}{\omega_s}\frac{\Delta E}{(eV_0 \sin \varphi_s)} \qquad (45)$$

If ω does not change appreciably during acceleration, then

$\frac{\Delta T}{T} \sim \frac{\Delta E}{E_m - E_0}$, where E_m = final total energy.

The trapping range is proportional to $V_0^{1/2}$, so that if V_0 is fixed throughout acceleration the final mean current is also proportional to $V_0^{1/2}$. If it is possible to increase V_0 at larger radii, where the majority of the acceleration occurs, then there will be a linear improvement in mean current. However, if the ion source is "space charge" rather than "temperature" limited, increased extraction can be expected with higher dee voltage. There is some evidence, at least from the Harwell and Liverpool synchro-cyclotrons, that the space charge limited law of $I \propto V_0^{3/2}$ may apply for voltages up to 5 or 10 Kv. In this case, then, the final mean current could rise as rapidly as V_0^2.

Magnitude of trapping range in a synchrocyclotron

For an ideal synchro-cyclotron producing, say, 200 Mev, with

eV_0 = 10 Kev (i.e.5 kV dee voltage, e $V_0 \sin \varphi_s$ = 5 Kev)

K = 1 (Uniform field)

then $\Delta E \sim 2$ Mev

In practice, with the field falling with radius, K is of the order of 2, in which case $\Delta E \sim 1.4$ Mev. Under these conditions, a mean current of about 1μa is obtained. For a spiral ridge machine of 2 Gev or more, it is necessary to improve the current extracted from the source or to reduce K in ordre to obtain the 1μa or so which is desired.

5.2 Minimum K values in spiral-ridge machines

We shall refer to two limiting machines, each of which has the C.W. field law (1) up to some radius r_1, and a power law field (constant k) thereafter. It is convenient to express these field laws in terms of $E/E_0 = \gamma$, giving

$B = B_0\gamma$ for $\gamma \leqslant \gamma_1$

$$\left.\begin{array}{l} B = B_0\gamma\left(\frac{\gamma^2 - 1}{\gamma_1^2 - 1}\right)^{\frac{k}{2(1+k)}} \\ k = \gamma_1^2 - 1 \end{array}\right\} \text{for } \gamma_1 \leqslant \gamma \leqslant \gamma_m$$

The two limiting machines are designated M_1 and M_2 :

M_1 $B = B_0\gamma$ up to $\gamma = \gamma_1 = 1.5$, k = 1.25, $Q_r = 1.5$

M_2 $B = B_0\gamma$ up to $\gamma = \gamma_1 = 2.0$, k = 3, $Q_r = 2.0$

Clearly, from the point of view of magnet economics, and depth of frequency modulation, we should work as close as possible to either M_1 or M_2 depending upon our confidence of success in passing the Q_r = 1.5 resonance.

For M_1 and M_2, K is zero up to $\gamma = 1.5$ and $\gamma = 2$, respectively, and then rises according to

$$K = 1 - \frac{k}{1+k}\frac{\gamma^2}{\gamma^2 - 1}$$

Determination of the acceptance conditions of these limiting machines, where the transition from aperiodic to periodic phase motion is rapid, would be most suitably attempted with the help of an analogue computer. However, a simple approach can be made if frequency modulation is applied from the beginning.

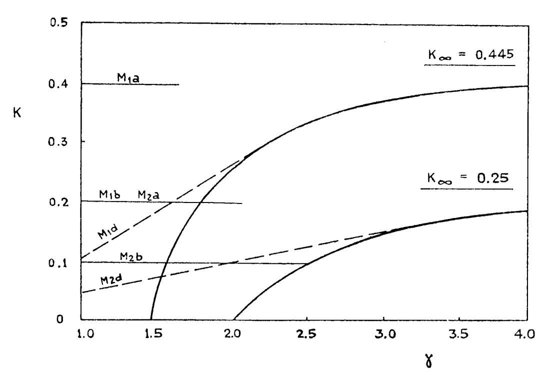


Fig. 5. K vs γ for M_1 and M_2 type machines

A number of possible cases can be treated analytically; we use figure 5 for illustration. K must not fall below the limiting curves of M_1 or M_2 if the limiting resonance is to be avoided.

The machines considered are for

$\gamma_m = 4$ (final energy $\sim$ 3 Gev).

(*a*) K constant throughout. $M_1(a)$ K must be $\geqslant 0.4$
$M_2(a)$ K must be $\geqslant 0.2$

Phase damping will occur throughout.

(*b*) K constant but non-zero until the limiting K curve is reached. A minimum value of K can be chosen such that the phase damping over the constant K region is just sufficient to cancel the subsequent growth of oscillation over the limiting K curve.

$M_1(b)$ $K_o \sim 0.2$
$M_2(b)$ $K_o \sim 0.1$

(*c*) K varying so as to achieve adiabatically a constant trapping range in energy (Δ E).
From (43), $K = K_o\gamma$ for V_o and F_1 fixed.

(*d*) K varying so as to achieve adiabatically a constant amplitude of phase oscillation.

From (40), $K = K_o\gamma\left(\frac{\omega_o}{\omega_s}\right)^2$ for constant $V_0 \cos \varphi_s$.

(The subscript "o" refers to the centre of the machine.)

Since any field law which we choose must either follow or fall below the C.W. law we know that $\left(\frac{\omega_o}{\omega_s}\right)^2$ increases with γ. Then $\frac{\partial^2 K}{\partial \gamma^2}$ must be positive, so that a minimum value of K_o can be chosen such that the resulting (K, γ) curve is tangential to the limiting (K, γ) curve.

5.3 *Computed data*

The frequency and field laws and the approximate trapping ranges for the various possible machines can be calculated from the specified values of K, by integration using (36), since in all the interesting cases K is a separable function of ω and E.

The trapping ranges for the "d" types machines are:

M_{1d} $K_o = 0.104$ $\Delta E \sim 20$ Mev
M_{2d} $K_o = 0.043$ $\Delta E \sim 30$ Mev

for a dee voltage of 50 Kv (50 Kev per turn with $\varphi_s = 30°$).

Thus the duty cycles for such machines (for $\sim$ 3 Gev) are comparable with that of the "idealised" synchro-cyclotron, but whereas the synchro-cyclotron has a small amount of phase damping, the M_{1d} and M_{2d} machines have none until $\gamma = 2.08$ and 3.02 respectively.

The depths of frequency modulation required for 3 Gev are:

M_1	34%	M_2	9.5%
M_{1d}	44%	M_{2d}	16%
M_{1b}	47%	M_{2b}	19%
M_{1a}	74%	M_{2a}	32%

There is clearly much at stake for the R.F. system in passing the $Q_r = 1.5$ resonance.

Figures 6 and 7 show $\frac{B}{B_m}$ versus $\frac{r}{r_m}$, where B_m and r_m are the common values of field and radius for $\gamma = 4$. The field plots illustrate the saving in total flux, in machines for $\sim$ 3 Gev, achieved passing $Q_r = 1.5$.

Machine	Description	Trapping region	Field tolerances
M_1a	K constant = 0.4	10.2 Mev	0.4%
M_1b	K constant = 0.2 UP to ½ integral res.	14.4 Mev	0.28%
M_1d	K_o = ·103 constant phase osc$_n$ amplitude	$\sim$ 20 Mev	0.2%

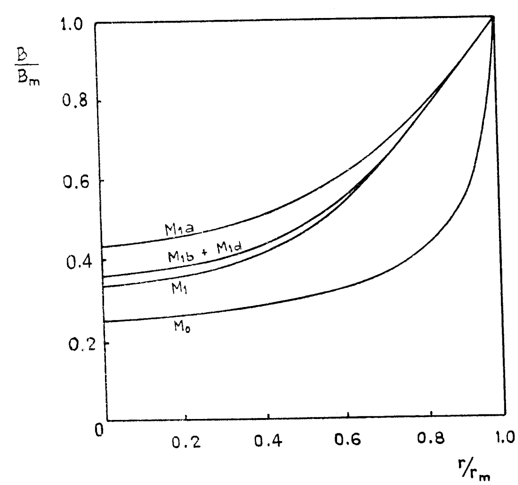


Fig. 6. Field vs radius for M_1 type machines

5.4 *Field tolerances*

The use of small K values demands great attention both to tolerances and to fundamental design problems such as that of calculated and true orbital times for the scalloped closed orbits.

As an illustration of the magnitude of the problem, for a machine with $K_o = 0.1$, at 20 Mev which is the acceptance energy range and about half the energy for a complete phase oscillation,

$$\frac{\Delta B}{B_o} = 0.018$$

For a machine with K = 0,

$$\text{at 20 Mev} \quad \frac{\Delta B}{B_o} = 0.020$$

Thus at 20 Mev the difference between a fixed frequency machine and one which is frequency modulated is 0.2% in terms of the central field.

Machine	Description	Trapping region	Field tolerances
M_2a	K constant = 0.2	14.42 Mev	0.28%
M_2b	K constant = 0.1 UP to k = 3	20.4 Mev	0.2%
M_2d	$K_0 = 0.43$. constant phase oc^{n} amplitude to integral resonance	30.3 Mev	0.13%

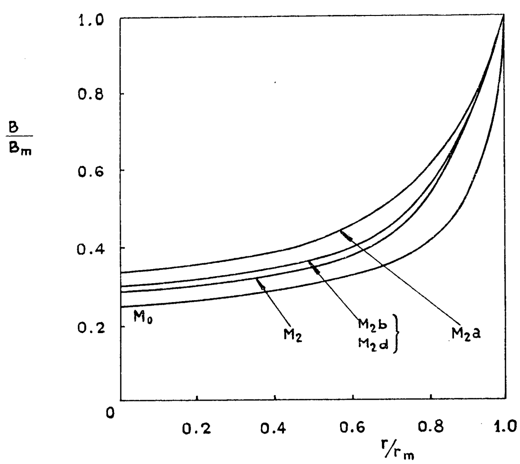


Fig. 7. Field vs radius for M_2 type machines

5.5 Machines with fixed frequency centres

Although phase slip errors due to errors in field distribution are serious and demand high accelerating voltages there are many reasons for pursuing machines M_1 and M_2.

In machines M_{1d} and M_{2d}:

$$K = K_0 \left(\frac{\omega_o}{\omega_s}\right)^2 \gamma$$

$\frac{d\omega}{dt}$ is a constant.

Looking at $\Delta\omega$ and ΔE as numerical quantities (excluding F_1 or L) we find that $\Delta\omega$ is independent of γ whilst ΔE is proportional only to $\frac{\omega_s}{\omega_0}$.

Hence
$$\left.\begin{aligned} \Delta T &= \frac{dt}{d\omega}\Delta\omega \\ &= \frac{2\pi}{\omega_s}\frac{\Delta E}{(eV_o \sin\varphi_s} \end{aligned}\right\} \text{ independent of } \gamma.$$

It therefore seems highly probable that machines such as M_1 and M_2 would initially try to trap a large range of particles as K rises from zero, but would continuously spill particles from the potential well as K increases until a range $\Delta\omega$ or ΔE consistent with K_o is reached. Since particles with "large" amplitudes of phase oscillation are most likely to be lost due to machine errors we are not very much concerned about whether we choose machines of type c or d for this comparison. We initially choose type d rather than c because of the greater simplicity of its field and frequency laws. It has constant amplitude of "small" phase oscillations making use of the variation $\omega_s^{1/2}$. This damping term due to decreasing frequency of phase oscillation clearly does not apply for the limiting "large" amplitudes and consequently machines M_{1c} and M_{2c} might be a better choice. In machines M_1 and M_2 however the potential well is filled not at ω_o for M_{1d} and M_{2d} but at a new $\omega_o = \omega_s$ corresponding to $\gamma = 1.5$ or 2, with a consequent increase in trapping time. It can therefore be argued that for machines M_1 and M_2 one should use machines of type d for this trapping range analogy.

The behaviour of F_1 or L introduces complications. If the varying frequency is applied to the whole of the R.F. system the initial phase for trapping purposes is no longer fixed at 90°, and is subject to phase slip except for the reference (or centre of trapping range) particle. Consequently the factor F_1 varies over the trapping range. If the R.F. system is physically divided into fixed and variable sections again F will vary over the trapping range since the relative phase of the two sections can only be fixed for the reference particle. Fortunately or unfortunately the trapping ranges are likely to be sufficiently small for this effect to be unimportant but for $\Delta E \sim$ 50-100 Mev it would be serious. Some improvement in trapping can be achieved by bringing φ_0 to φ_s (in theory a factor $\sqrt{2}$ for any value of φ_s) either by phase slip or phase difference, depending upon the choice of R.F. system.

More specific calculations of overall trapping efficiency demand more detailed specification of machine parameters, but it appears that the duty cycle might not differ very much from that of an all—F.M. machine using the minimum permissible value of K at the centre.

5.6 Effect of phase motion on passage through resonances

From the figures presented in Appendix III it appears that the $Q_r = 1.5$ resonance may be expected to produce a stop band occupying about 100 turns for 50 Kev energy gain per turn, and that it might be traversed with careful shimming. However, if the frequency modulated part of the acceleration embraces this region, a particle executing phase oscillation can spend a very much longer time in the resonance region than a synchronous particle.

In machine M_{2d} for example, at $Q_r = 1.5$, a complete phase oscillation occupies a time equivalent to 1700 revolutions. The worst case would obviously occur for a particle having $\varphi = \varphi = 0$ at the centre of the stop band. It would spend about 300 turns within the stop band with disastrous consequences.

More detailed analysis is required for specific machines but sufficient has already been done to indicate yet again the desirability of a machine with fixed frequency operation until $Q_r = 1.5$ has been passed or at least until φ_0 can be slipped from 90° to Φ_s before starting the F.M. cycle.

6. Notes on magnet design

6.1 Production of flutter fields by pole-face ridges

As a first attempt to assess this problem, we make three major simplifications :

(i) reduce the problem to two dimensions,
(ii) assume iron of infinite permeability,
(iii) consider a single harmonic—i.e. sinusoidal flutter.

In Cartesian co-ordinates, we consider the field

$$B = B_0 \left[1 + \delta \sin\left(\frac{2\pi x}{\lambda}\right)\right] \tag{46}$$

in the plane $z = 0$.
Or, in terms of a magnetic potential U,

$$\frac{1}{B_0}\frac{\partial U}{\partial z} = 1 + \delta \sin\left(\frac{2\pi x}{\lambda}\right) \tag{47}$$

The appropriate solution of $\nabla^2 U = 0$ is

$$\frac{1}{B_0}\frac{\partial U}{\partial z} = 1 + \delta \sin\left(\frac{2\pi x}{\lambda}\right) \cosh\left(\frac{2\pi z}{\lambda}\right) \tag{48}$$

$$\frac{U}{B_0} = z + \frac{\lambda}{2\pi}\,\delta \sin\left(\frac{2\pi x}{\lambda}\right) \sinh\left(\frac{2\pi z}{\lambda}\right) \tag{49}$$

Plotting equipotentials from (49) for a sufficiently small value of $U = U_1$ gives results of the form illustrated by the full-line curves in figure 8. The lower branch of the curves (the continuous curve) represents a possible magnet pole profile. The equipotential $U = -U_1$ (not shown) defines the corresponding magnet pole at negative z. The effective mean gap, Gm between these two poles is

$$G = \frac{2U_1}{B_0} \tag{50}$$

Plotting equipotentials from (49) for a sufficiently large value of $U = U_2$ gives results of the form illustrated by the

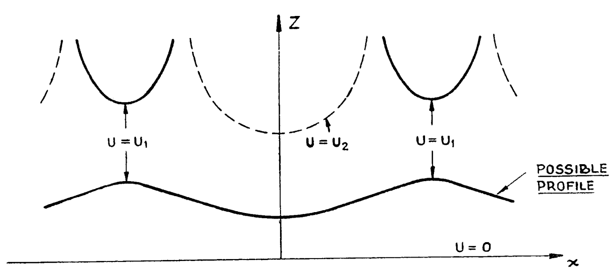


Fig. 8. Equipotentials

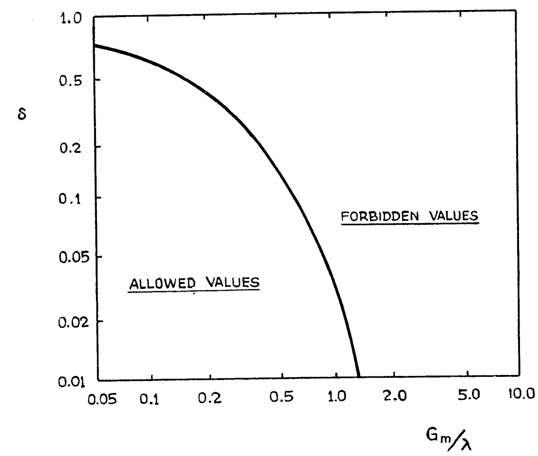


Fig. 9. Limit of flutters produced by pole-face ridges

dotted curves in figure 8; these clearly do not represent suitable pole profiles.

It is of practical interest to determine the greatest value of G which yields continuous equipotentials of the $U = U_1$ type. This is readily shown to be

$$G_{max} = \frac{\lambda}{\pi}\left[\cosh^{-1}\left(\frac{1}{\delta}\right) - \sqrt{1-\delta^2}\right] \tag{51}$$

which is plotted in figure 9, where points to the right of the curve cannot be realised by pole-face ridging.

The "available gap", g, between the lips of the magnet ridges, is also of practical importance; it is related to the mean gap G by

$$\frac{G}{g} = 1 + \frac{\delta \sinh(\pi g/\lambda)}{(\pi g/\lambda)} \tag{52}$$

For $g \sim \lambda/\pi$ this approximates to the obvious form $G/g \simeq 1 + \delta$ (53)

6.2 Limitations of pole-face ridges in cyclotrons

The above considerations imply severe limitations on the possible magnet gap and ridge parameters of a spiral ridge cyclotron.

An approximation of the expression for Q_v, given as (24) in Section 2.2 of this paper, may be written, when $N_2 \gg k + 1$,

$$Q_v^2 \simeq -k + \frac{\delta^2}{2} + \frac{\delta^2 K^2}{N^2}$$

Putting $\delta_0^2 = 2(k + Q_v^2)$, and expressing $\frac{K}{N}$ in terms of the ridge wavelengths λ, λ_0 as defined in figure 10, we may write

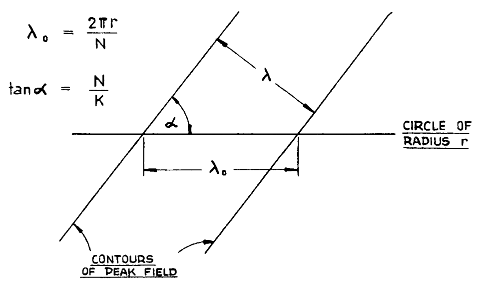


Fig. 10. Definition of spiral parameters

$$\frac{\delta}{\lambda} \simeq \frac{\delta_o}{\lambda_o}\left[\frac{1+\left(\frac{\delta}{\delta_o}\right)^2}{2}\right]^{\frac{1}{2}}$$

$$\text{or } \frac{\delta G}{\lambda} \simeq \delta_o \frac{G}{\lambda_o}\left[\frac{1+\left(\frac{\delta}{\delta_o}\right)^2}{2}\right]^{\frac{1}{2}} \qquad (54)$$

For given δ_o we may superpose on figure 9 a series of "vertical focusing" curves computed from (54) for various ratios G/λ_o. This procedure is illustrated in figure 11, where the dotted curves correspond to the special case $\delta_o = 1$.

Remembering that points to the right of the curve marked G_m/λ cannot be realised, figure 11 demonstrates how, for given δ_o, an upper limit for G/λ_o may be determined. Thus for $\delta_o = 1$, we find $G/\lambda_o \leqslant 0.12$. If G is made significantly less than this upper limit, a range of practicable values of δ and λ becomes available; generally a small δ is to be referred in order to keep the available

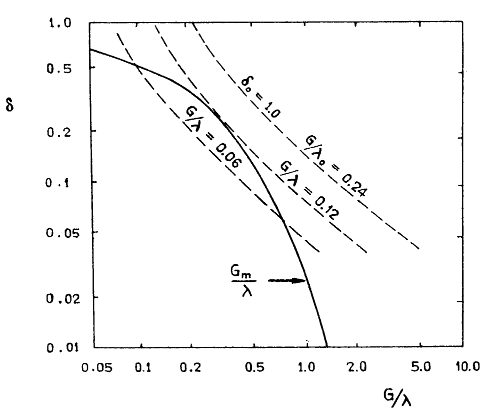


Fig. 11. Comparison of focusing requirements with ridge limitations

gap g as large as possible. Detailed computations of permissible limits and optimum values of δ have been made, but the results will not be given here.

The main results of the above considerations are summarised in figure 12. The maximum permissible G for given focusing conditions δ_o and λ_o may be deduced from the curve which gives $(\delta_o\, G/\lambda_o)_{max}$ as a function of δ_o; note that $(\delta_o G/\lambda_o)_{max} \rightarrow 0.124$ as $\delta_o \rightarrow \infty$. The corresponding unique values of δ are also plotted, and $\rightarrow 0.286$ as $\delta_o \rightarrow \infty$. For $\delta_o \leqslant 0.1$, the unique δ value is δ_o, i.e. radial ridges permit the greatest G.

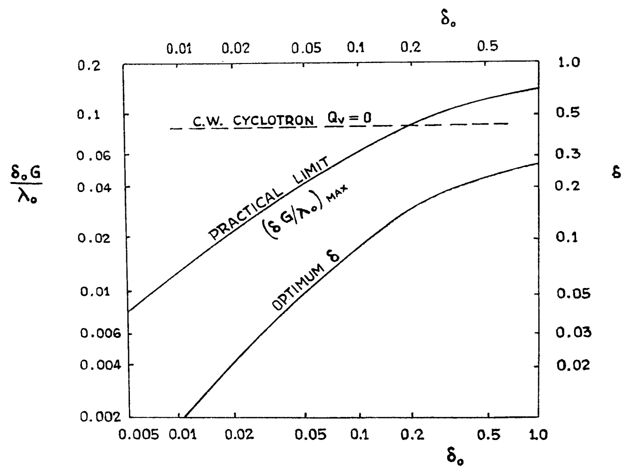


Fig. 12. Limiting ridge parameters for cyclotrons

It is of interest to note that if we assume that the mean field B, at any radius r in a cyclotron, is inversely proportional to the mean gap G, then for a C.W. cyclotron with $Q_v = 0$,

$\delta_o G/\lambda_o$ = constant.

6.3 Experimental programme

There is no apparent reason for insisting on sinusoidal flutter. At the limit of rectangular flutter waveform, the effective "δ^2" is increased by a factor 2 as compared with a sinusoidal flutter waveform of the same peak to peak amplitude.

Experiments have been initiated on rectangular ridges; no detailed results will be presented here, but the main conclusion so far is that it appears possible to achieve values of mean gap G $\sim$ 25 % greater than those predicted by the simple theory described above. This holds true for fields at least up to 19 kilogauss mean. The corresponding improvement in available gap "g", between the ridge peaks, diminishes with increasing B_o, but is still significant at 19 kilogauss mean.

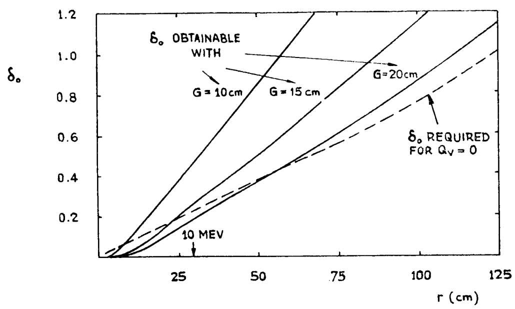


Fig. 13. Harwell Cyclotron - C.W. conversion N = 4

6.4 Pole face windings

The production of flutter fields entirely by pole face windings has been studied theoretically; details will not be reported here. The powers required in typical applications (see next section) are generally very high.

7. Applications

7.1 Conversion of the Harwell 110-inch synchro-cyclotron

We are working on the possibility of converting the present synchro-cyclotron to C.W. operation using spiral ridges. The present energy of the machine is 175 Mev (protons) but we wish to increase this to perhaps 240 Mev by increasing the field at the outer edge to over 19,000 gauss.

From the data given in section 3 it follows that N = 4 is the best choice for this machine. Assuming that the flutter fields are to be produced by pole-face ridges, the permissible magnet gap has been investigated on the basis of section 6.2 above. Values of the "δ_0" required at any radius r may be derived from relations already given :

$$\delta_0^2 = 2(k + Q_v^2),\ k = \beta^2 (1 - \beta^2)^{-1},\ r = \beta r_\infty$$

Values corresponding to $Q_v = 0$ are shown by the dotted curve in figure 13.

Values of the maximum δ_0 obtainable at any radius r may be deduced from the $(\delta_0 G/\lambda_0)_{max}$ curve of figure 12. The full-line curves in figure 13 are labelled according to the assumed values of mean gap G (in cm) at r = 125 cm; it is assumed that, elsewhere, G varies as 1/B.

Comparison of the "required" and "obtainable" δ_0 curves in figure 13 shows that the required focusing conditions are most readily achieved at large radii. There is always a central region where vertical defocusing is unavoidable if the C.W. field variation (1) is followed. For example, for G = 15 cm. at r = 125 cm. this region extends to r ~ 25 cm (~ 7 Mev). This region can be reduced only by reducing G. It should be emphasized that the 2-dimensional theory on which the curves are based is particularly optimistic at small radii.

Production of the flutter fields by pole-face windings, assuming similar values, would require at least 200-400 kW of power per pole face. Again a central region of defocusing would remain. We have concluded that, for our purposes, iron ridges are to be preferred. However, pole-face windings to correct (and possibly to increase) the δ values will be incorporated.

Further investigation, both theoretical and experimental, of the problem of the machine centre is planned. Preliminary theoretical estimates indicate that a uniform mean field should not cause serious phase slip up to 25 turns (or 5 Mev at 250 Kv per turn), and this field does not cause defocusing. An exact C.W. mean field, with no flutter, produces disastrous defocusing in 25 turns.

7.2 A 3 Gev spiral ridge cyclotron

Possible parameters for a 3 Gev proton cyclotron are:
N = 8, possibly reducing to 4 at small radii.
Proton energy 3 Gev (kinetic) at r_m = 8 metres, B_m = 16,000 gauss.
C.W., or near C.W., operation for r < 6 metres, constant 'k' operation thereafter.

Magnet steel weight ~ 12,000 tons.

Figure 14 gives the appropriate curves of δ_0 'required' and obtainable, derived from section 6.2 above. Each δ_0 'obtainable' curve is labelled according to the value of the product N × (G at r = r_m). For N = 8 it appears that G must be less than ~ 11 cm. at r = r_m. For N = 8, G = 11 cm. the central region of defocusing extends to r ~ 2 metres (~ 40 Mev), but this region may be reduced by reducing G or N. Reduction of N to 4 for r ⩽ 4.4 metres is permissible (see section 3), the defocusing region then being confined to r ~ 50 cm (~ 2 Mev).

Possible methods of increasing the magnet gap significantly must be considered. First, by accepting a smaller value B_m and making r_m > 8 metres, G may be increased in proportion to r_m. The limit to this process clearly lies in the size and weight of the magnet structure. Production of the flutter fields by pole-face windings, assuming similar gaps to be left available for 'dee' structures as in the case of pole-face ridges, would require ~ a few MW of power per pole face (independently of the choice of B_m and r_m). Going to larger gaps, the power required increases (initially) roughly as $(\text{gap})^2$.

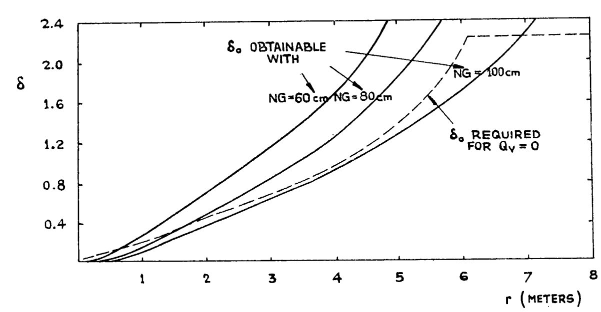


Fig. 14. 3 Gev cyclotron

The above remarks serve to indicate the orders of magnitude involved in a 3 Gev cyclotron design. Much detailed work remains to be done before the most practical and economical design can be specified.

8. Conclusions

From considerations of particle dynamics, magnet problems and frequency modulation, as summarised in this paper, it appears likely that a spiral ridge cyclotron for $\sim$ 3 Gev would be feasible. By accepting close tolerances on field and frequency laws, a mean current of a few microamperes might be achieved as in existing synchro-cyclotrons operating at a few hundred Mev, on account of the possibility of extending the effective trapping range. A high dee voltage would be an advantage, in extracting more ions from the source, in driving quickly through the $Q_r =$ 1.5 resonance, and, in the case of a machine with a fixed frequency centre, in reducing random phase slip due to field errors. Certainly it would be desirable to traverse the $Q_r = 1.5$ resonance in a machine for $\sim$ 3 Gev, on account of the great saving in magnet steel which could be realised.

Much detailed work remains to be done before a complete practical design can be specified; some of the problems are fundamental, and can best be tackled with the help of electron analogue accelerators. Other problems are mechanical, such as the problems of assembling the large magnet and accelerating electrodes.

However, many of the most severe problems are avoided in a C.W. machine for 200-250 Mev such as the proposed conversion of cour 110-inch synchro-cyclotron.

Appendix I : Hybrid machines for high energies

For our purposes it appears that the economic limit for a cyclotron would be reached at about 3-3.5 Gev. A 6 Gev cyclotron would require at least 30,000 tons of steel ! An alternative possibly would be a 'hybrid' machine, using

(*a*) A spiral ridge cyclotron for, say, 2 Gev, but using relatively low magnetic fields so that its extreme radius is roughly that of the Bevatron (i.e. $\sim$ 50 feet).

(*b*) Surrounding this, and concentric with it, an A.C. annular-gap magnet into which particles can pass freely around the whole 360° of their orbits. This magnet to be pulsed up to $\sim$ 15 kilogauss.

General features

1. Continuity of particle dynamics

Q_r, Q_v must remain constant, or nearly constant, during the transition between the cyclotron and a synchrotron portions, otherwise disastrous resonances will occur. Therefore spiral ridges must be used on the A.C. magnet poles, and the machine must have a fixed working point during and beyond the transition. (It is pointed out in the main paper that a spiral ridge cyclotron should have a fixed working point beyond 800 Mev or so.)

2. There would be severe field-matching problems at the cross-over.

3. The magnet and vacuum chamber would be very large, with severe mechanical problems.

4. The intensity would be high—good injection conditions.

5. It would be possible, in principle, to make a machine of this kind with C.G. focusing in both parts. However, in order to avoid serious resonances in the cyclotron part it would be necessary to work with $0 < n < 0.2$, and continuity demands the same conditions in the synchrotron part. The particle dynamics would then be simple, but the requirement of small n would make for severe tolerances on field gradient in the synchrotron portion for a 6 Gev machine. Moreover, it is doubtful whether the synchrotron oscillations could be confined in a reasonable aperture in the synchrotron portion.

Table of possible parameters for 6 Gev.

Magnet weight (cyclotron portion)	15,000 tons
Magnet weight (synchrotron portion)	$<$ 1,000 tons
Stored energy (synchrotron portion)	$\sim$ 2.5 Megajoules
Pulse repetition rate	30 p.p.s.
Mean beam current	$\sim$ 0.5 μa.

Appendix II: Passage through an integral resonance

Integral resonances are caused by errors in field amplitude, which for A.G. synchrotrons are usually analysed in terms of misalignments of magnet sections. For cyclotrons, a more convenient way is to analyse the errors in terms of their field harmonics. Resonant build-up occurs when one of these harmonics has the same periodicity as the free-oscillation. The effects of the other harmonics are relatively unimpotant and can be ignored. For example, at the $Q = 2$ resonance only the second harmonic error in field amplitude is important. However, since this harmonic is of a lower order than the periodicity of the machine, a "smoothed" approximation can be used to represent the particle motion. Passage through an integral resonance can then be studied by solving

$$\ddot{\rho} + Q^2\rho = \varepsilon \sin S\theta \qquad (1)$$

in which ε is the relative amplitude $\Delta B/B$ of the harmonic error and Q is assumed to vary with particle energy. Resonance occurs when $Q = S$. If Q is fixed at this value the amplitude of oscillation builds up linearly according to

$$\rho = -\frac{\varepsilon}{2s}\,\theta \cos Q\theta \qquad (2)$$

that is, the build-up per revolution is

$$\pi\varepsilon\, r/Q$$

where r is the mean orbit radius. Since the $Q = 2$ resonance will occur at a radius of a few metres, this means that the build-up is of the order of several centimetres in 10 turns for $\varepsilon = 10^{-3}$.

The amplitude of the forced oscillation of resonance is

$$p = \frac{\varepsilon}{Q^2 - S^2} \sin S\theta \simeq \frac{\varepsilon}{2S\Delta Q} \sin S\theta \qquad (3)$$

with a "build-up time" (number of turns) of $1/_2\Delta Q$. Serious build-up will therefore occur if $\frac{1}{2\Delta Q} \geqslant 10$ turns. For a C.W. machine, $Q_r = E/E_0$, and hence the increment of Q per turn is eV/E_0 where V is the voltage gain per turn. Equating the transit time to the build-up time, we get

$$\frac{2\Delta Q}{eV/E_0} = \frac{1}{2\Delta Q}$$

$$\text{or} \begin{cases} \Delta Q & = \frac{1}{2}(eV/E_0)^{1/2} \\ \text{No. of turns} & = (E_0/eV)^{1/2} \end{cases} \qquad (4)$$

In order to cross the resonance in ~ 10 turns it would therefore be necessary to accelerate at the rate of several Mev per turn.

We shall now carry out a more accurate analysis to confirm this conclusion. Since the width of the resonance is confined to a narrow region near Q = S, we assume that Q varies linearly in this region and is represented by

$$Q = S + \zeta\,\theta \qquad (5)$$

$$\text{where } \zeta = \frac{dQ}{d\theta} \qquad (6)$$

Substituting in (1), we have to find a particular integral of the inhomogeneous equation

$$\frac{d^2\rho}{d\theta^2} + Q^2\rho = \varepsilon \sin S\,\theta \qquad (7)$$

Applying Green's theorem, we get

$$\rho = \frac{\varepsilon}{Q}\int^{\theta} \sin S\theta_1 \sin\left\{\int Q\,d\,\theta - \int Q\,d\,\theta_1\right\} d\theta_1 \qquad (8)$$

Since the build-up occurs near $\theta = 0$, the total build-up can be obtained by taking infinite limits for the range of integration. The above solution will then represent an oscillation of zero amplitude initially, building up to some finite amplitude in passing through the resonance.

Substituting (5) for Q,

$$\rho = C \sin\int Q\,d\,\theta - D\cos\int Q\,d\,\theta \qquad (9)$$

$$\text{where } C = \frac{\varepsilon}{Q}\int_{-\infty}^{\infty} \sin S\,\theta \cos(S\,\theta + \tfrac{1}{2}\zeta\,\theta^2)\,d\,\theta$$

$$= -\frac{\varepsilon}{2Q}\sqrt{\frac{2\pi}{\zeta}}\left[\sin\frac{\pi}{4} - \sin\left(\frac{\pi}{4} - \frac{2S^2}{\zeta}\right)\right] \qquad (10)$$

$$D = \frac{\varepsilon}{Q}\int_{-\infty}^{\infty} \sin S\,\theta \sin(S\,\theta + \tfrac{1}{2}\zeta\,\theta^2)\,d\,\theta$$

$$= \frac{\varepsilon}{2Q}\sqrt{\frac{2\pi}{\zeta}}\left[\cos\frac{\pi}{4} - \cos\left(\frac{\pi}{4} - \frac{2S^2}{\zeta}\right)\right] \qquad (11)$$

The amplitude of the oscillation is therefore

$$(C^2 + D^2)^{\frac{1}{2}} = \frac{\varepsilon}{2Q}\sqrt{\frac{2\pi}{\zeta}}\left[2 - 2\cos\frac{2S^2}{\zeta}\right]^{\frac{1}{2}} \qquad (12)$$

The cosine term depends critically on ζ, since this is a small quantity. The mean value of the amplitude, averaging over ζ, is

$$\frac{\varepsilon}{Q}\left(\frac{\pi}{\zeta}\right)^{\frac{1}{2}} r_1 = \frac{\varepsilon\pi}{Q}\left(\frac{2E_0}{eV}\right)^{\frac{1}{2}} r_1 \qquad (13)$$

Thus the build-up is of the same order of magnitude as predicted crudely above.

For example, with Q = 2, r_1 = 300 cms, E_0 = 103 Mev, V = 200 kV, $\varepsilon = 10^{-3}$, the induced amplitude is 47 cms.

Appendix III: Passage through half-integral resonances

A technique similar to that of Appendix II can be used. We now take

$$\ddot{\rho} + [Q^2 + \varepsilon \sin S\theta]\,p = 0 \qquad (1)$$

in which the inperturbed motion is represented by a "smoothed" approximation, whereas the perturbation $\varepsilon \sin S\theta$ corresponds to a harmonic error in field gradient, of lower order than the magnet periodicity. In particular, we are interested in the resonance at $Q = 1\frac{1}{2}$, S = 3, (Q = S/2).

From the theory of Mathieu equations, the mode number for constant Q and small ε near the resonance is

$$\nu = \frac{S}{2} + i\left[\left(\frac{\varepsilon}{2S}\right)^2 - (\Delta Q)^2\right]^{\frac{1}{2}} \qquad (2)$$

where ν corresponds to Q when ε is zero. That is, there is a stop band of width $\Delta Q = \pm\,\varepsilon/2S$, in the region where the phase change of the oscillation per period of the harmonic perturbation is π. It should be noted that a stop band would also occur at $Q = 1\frac{1}{2}$, S = 1 where the phase change per period is 3π. However, the width is of lower order than that for S = 3, and provided the first harmonic is reasonably small it should not cause serious build up.

For S = 3, the exponential blow-up is

$$\exp\int\left[\left(\frac{\varepsilon}{2S}\right)^2 - (\Delta Q)^2\right]^{\frac{1}{2}} d\theta \qquad (3)$$

integrated through the stop band. For a C.W. machine, $Q_r = E/E_o$ so that

$$\frac{dQ_r}{d\theta} = \frac{eV}{2\pi E_o}$$

where V is the voltage gain per turn. Substituting in (3) gives

$$\exp \frac{\pi\varepsilon^2}{2S^2}\frac{E_o}{eV}\int_{-1}^{1}(1-t^2)^{1/2}\,dt$$

$$= \exp\left\{\frac{\pi^2}{4}\frac{\varepsilon^2}{S^2}\frac{E_o}{eV}\right\} \qquad (4)$$

In order to avoid serious build-up

$$\frac{\pi^2}{4}\frac{\varepsilon^2}{S^2}\frac{E_0}{eV} \ll 1 \qquad (5)$$

For example, for V = 50 Kv/turn, E_o = 103 Mev, S = 3

$$\varepsilon \ll 1.4\%$$

where ε is to be interpreted as the amplitude of a third harmonic error in the field gradient.

The above theory underestimates the blow-up, since it does not take into account the beat phenomena outside the stop band. At the edge of the stop band, for example, the perturbation is exactly in resonance with the oscillation, and would cause a linear increase in amplitude if the particle remained there. To take this into account it is necessary to carry out a second order perturbation theory with varying Q. This has not yet been done, but we shall now carry out the perturbation theory to first order.

We write (1) in the form

$$\ddot{\rho} + Q^2\rho = -\varepsilon\rho \sin S\theta \qquad (6)$$

and regard the R.H.S. as a small order perturbation. In the absence of the perturbation the oscillation is

$$\rho = C \sin\left[\int Q d\theta + \sigma\right] \qquad (7)$$

Substituting this in the R.H.S. of (6) we get

$$\ddot{\rho} + Q^2\rho = -\frac{\varepsilon C}{2}\left[\cos\left\{S\theta - \int Q d\theta - \sigma\right\} - \cos\left\{S\theta + \int Q d\theta - \sigma\right\}\right] \qquad (8)$$

where the first term in the R.H.S. is in resonance with the oscillation at Q = S/2, and the second term at Q = 0. Near the resonances of interest here, Q = S/2, only the first term is of importance. Applying Green's theorem as before the incremental change produced by this perturbation is

$$\Delta C = -\frac{\varepsilon C}{2Q}\int_{-\infty}^{\infty}\cos\left\{S\theta_1 - \int Q d\theta_1 - \sigma\right\} \times \sin\left\{\int Q d\theta - \int Q d\theta_1\right\} d\theta_1 \qquad (9)$$

where $Q = S/2 + \zeta\theta$

Solving and substituting $\zeta = \frac{dQ}{d\theta} = \frac{eV}{2\pi E_0}$, we find for the amplitude of the induced oscillation:

$$\frac{\Delta C}{C} = \frac{\varepsilon\pi}{4Q}\left(\frac{2E_0}{eV}\right)^{\frac{1}{2}} \qquad (10)$$

In order to avoid serious build-up,

$$\varepsilon \ll \frac{2S}{\pi}\left(\frac{eV}{2E_0}\right)^{\frac{1}{2}} \qquad (11)$$

Comparison with (5) shows that this is of the same order of magnitude. Since this theory takes into account the resonance phenomenon but, to this approximation, would not predict a stop band it is likely that the above formulae give roughly the correct order of magnitude for the tolerance.

Appendix IV: Passage through vertical linear resonances.

This is likely to present the same order of difficulty as passage through radial resonances. The formulae developed above still apply, provided ζ is taken to mean

$$\frac{dQv}{d\theta}.$$

We have $\zeta = \frac{dQv}{d\theta} = \frac{dQv}{dr}\cdot\frac{dr}{d\theta}$

Also $Q_r^2 \simeq \left(\frac{E}{E_0}\right)^2 = \frac{1}{1-\left(\frac{r}{r_0}\right)^2}$

and hence $\frac{dr}{d\theta} = \frac{dr}{dQ_r}\frac{dQ_r}{d\theta} = \frac{r_\infty}{\beta}\left(\frac{E_0}{E}\right)^3\frac{eV}{2\pi E_0}$

Substituting, $\zeta = \frac{dQv}{d\left(\frac{r}{r_\infty}\right)}\frac{1}{\beta}\left(\frac{E_0}{E}\right)^3\frac{eV}{2\pi E_0}$

The quantity ζ has now been related to the machine parameters for vertical motion, and the previous formulae can be used. However, the variation of Q_v with radius must be known, and at the time of writing we have no values for this.

II. 6-7 GEV MACHINES

1. Introduction

Many problems remain to be solved before a 3 Gev cyclotron, as outlined in Part I of this paper, can be designed in detail. We have studied several other possible accelerators with the object of achieving a more modest improvement in intensity in a shorter time. We thought it wise, in view of the smaller intensity, to aim for a somewhat higher energy.

2. A.G. synchrotrons

An A.G. proton synchrotron for an energy about 6 Gev, with similar tolerances to those of the machines under construction by CERN and Brookhaven, would have a peak magnetic stored energy of about 2 megajoules. Although the capacitor bank would be a major enterprise, it would be possible to resonate the magnet windings for operation at 10-30 p.p.s. A machine of this kind was tentatively proposed by H. L. Anderson, with the object of obtaining a mean current higher than that of the Bevatron. We have considered the overall particle dynamics of such a machine, and have concluded that the synchrotron oscillations could be accomodated within the vacuum chamber even at 50 p.p.s. operation with acceleration on the 6th harmonic. The maximum amplitude would be about 4 cm. and the radial aperture 12 cm. with the magnet biased so that injection occurs at about 6° after the minimum magnetic field. However, it seems likely that only a few turns could be accommodated at injection, and in any case multiturn injection is limited in time since B varies so rapidly around 6°. The mean intensity would then be perhaps two or three times the present yield of the Bevatron, for the same injector current. There would be severe R.F. problems in achieving high repetition rates; the energy gain per unit length of donut for 20 p.p.s. would be 10 times that in the CERN machine. Tracking at this high rate would also be difficult.

We conclude that this machine does not offer the facilities which we require, unless we assume that a radically improved system of injection can be devised, applicable to a machine of high repetition rate.

3. Adaptations of conventional machines for 6-7 Gev

The object is to preserve the long injection time, over many revolutions, of machines of the Cosmotron and Bevatron type, but to reduce the steel weight and peak stored energy of the magnet. A higher repetition rate is then possible. We have considered two methods of approach, namely a "double synchrotron" and a spiral ridge synchrotron.

3.1 Double synchrotron *

Two A.C. magnets are used. The first has an aperture roughly the same as the Bevatron, but protons are accelerated only to ~ 1 Gev (~ 3 kilogauss) so that the stored energy is only ~ 4 % of that of the Bevatron. The particles are then made to pass into a second magnet, adjacent and concentric, around the whole 360° of their orbits, and are accelerated to full energy at about 15 kilogauss. The space consumed by the beam has been reduced by the damping of the oscillations, and the aperture of the second magnet may be made smaller than the first. The stored energy of the second magnet might be ~ 10 % of that of the Bevatron. The energy (and field) at which cross-over occurs may be chosen to minimise the total stored energy of the two magnets. Since the betatron oscillations are damped (as $B^{-1/2}$) from approximately 4 times the injection energy, and the synchrotron oscillations are damped from the beginning of the acceleration as β^{-1} and antidamped as $(B)^{1/4}$, a choice of the optimum system involves a choice of injection energy and initial rate of rise of the magnetic field. The choice of operating conditions is also complicated by the effects of gas scattering.

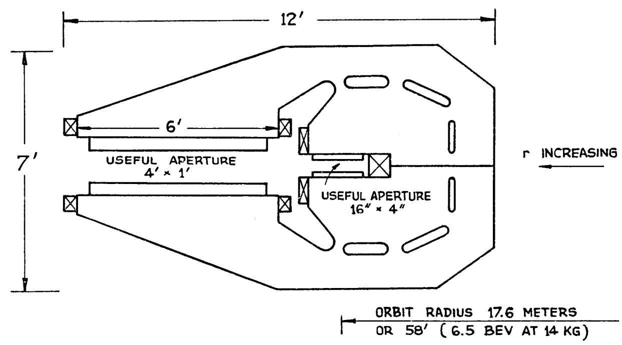


Fig. 1. Double synchrotron magnet profile

Magnet

Figure 1 is a diagrammatic outline of a possible magnet structure. The rough outside dimensions shown are considered to be maxima; they might be considerably reduced in practice. The steel weight would be below 5,000 tons. A critical point is the amount of mutual coupling which can be tolerated between the two magnet systems. The effects are

(i) Coupled voltages in the two power supplies. These are thought to be not very serious.

(ii) The coupling is such that one magnet coil sends negative flux through the gap of the other. The consequent mutual influence on remanent fields could be troublesome.

(iii) Increase in stored energies.

Figure 1 shows no special arrangements for assisting the cross-over problem; it is necessary, of course, to preserve the field configuration between the stability limits during

* Dr. G. Salvini has drawn our attention to his paper suggesting essentially the same idea [5].

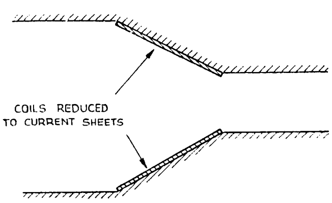


Fig. 2a. Field matching by an ideal distribution of coils and magnet poles

cross-over (e.g. 0.8 > n > 0.5). Two highly idealised arrangements are indicated in Figure 2, (*a*) and (*b*). The first of these aggravates the mutual coupling problem; both would reduce the amount of correction needed from correcting windings.

Injection energy

There might be two advantages in reducing the injection energy to 5 Mev :

(i) Since the betatron damping would start from a lower value of the magnetic field, it would be possible to choose a magnet combination with lower total stored energy.

(ii) Since the 3.5 Mev Van de Graaff injector at Brookhaven has performed reliably at a peak current of about 3 ma, whereas the 10 Mev linear accelerator at Berkeley has, so far, given no more than 300 μa, it may be possible to obtain a higher intensity than the present Berkeley value by using 5 Mev injection, which is within the range of feasible Van de Graaff generators. Moreover, the focusing of the Van de Graaff beam is better, and gives a smaller initial betatron amplitude.

However, from other points of view 10 Mev is a better injection energy. Gas scattering is obviously less serious at a given working pressure, the required range of frequency modulation is less and the effects of remanent fields are smaller. In a new linear accelerator, grid focusing could be discarded in favour of A.G. lenses, certainly if a lower accelerating rate or longer wavelength were used, and this would increase the injected current. A final choice would depend upon the degree of confidence with which injector currents could be specified for each type of machine, and upon the lowest gas pressure attainable reliably in the vacuum system.

There is evidence from Berkeley that a much longer effective injection time can be achieved with the Bevatron by reducing the initial rate of rise of magnetic field. The effects of such a reduction of B upon the gas scattering loss must be considered, and this again is influenced by injection energy.

Gas scattering

Although in practice the output of the three existing proton synchrotrons appears to vary with pressure in a manner consistent with multiple scattering theory, the observed pressure for a loss of about 10% seem to be too low a factor between two and three when the scattering is assumed to be entirely due to air. The fault does not appear to lie in the theory, although loss from single large angle scattering is not negligible when the total loss is small; for the Cosmotron and the Bevatron, when the loss from multiple scattering is 10% the additional loss from single scattering is about 3%.

Plots of beam intensity versus vertical aperture are given in figure 3, for the Bevatron and the Birmingham machine. The pressure is corrected by the factor two or three mentioned above. The agreement between theory and experiment is then tolerable, except for the sharpness of the cut-off. However, for conditions leading to very large loss, the probability distribution of amplitudes due to multiple scattering will have its maximum very close to the wall of the vacuum chamber, and consequently at this point the probability of further loss by mean square multiple scattering may be less than that of a final single scattering straight to the wall. This phenomenon should cause a sharper cut-off, as is observed.

It appears from these considerations that the observed minimum useful vertical aperture in the existing machines may well be explained mainly by gas scattering, with a minor contribution from median plane and similar errors. Riddiford published similar arguments in 1951.

The discrepancy of a factor 2-3 between theory and experiment may be due to the presence of additional gases and vapours. Pump oil vapours contain molecules with as many as 40 carbon atoms, and the term $\Sigma(Z^2N)$ in the gas scattering formula is important; a small partial pressure due to heavy organic molecules could contribute largely to this term.

If we may assume that the minimum vertical aperture of machines like the Cosmotron and the Bevatron is mainly determined by gas scattering then in considering future machines we may consider the following expression to be virtually invariant, for a given gas pressure :

$$\frac{Q^2A^2\left(1-\frac{a}{A}\right)^2 T_i\ (\mathrm{eV})}{R_0^3\,(1+\alpha)^3}$$

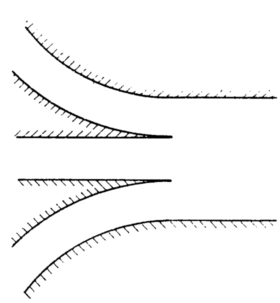
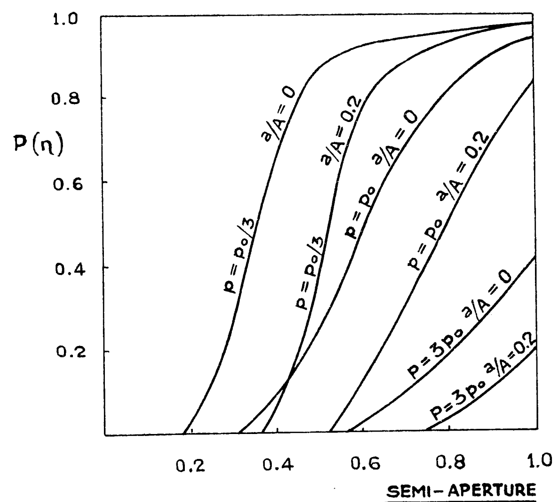


Fig. 2b. Field matching by an ideal configuration of magnet poles

a Theoretical



b Experimental

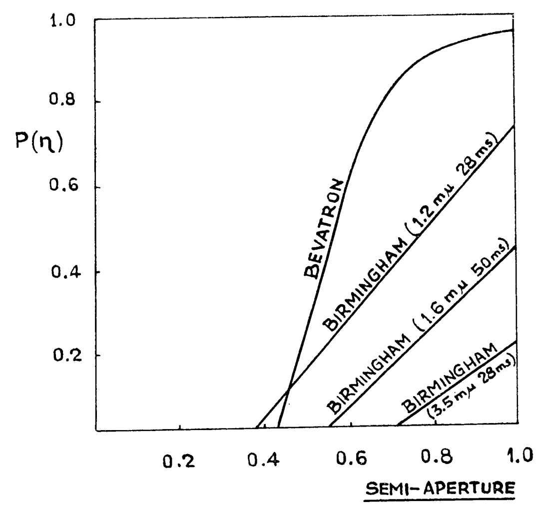


Fig. 3a. $P(\eta)$ is calculated using multiple scatt. theory but includes a correction for single scattering

Fig. 3b. The Birmingham figures are for three different values of pressure and time after injection

where Q = number of vertical oscillations/turn
A = vertical aperture
a = initial vertical amplitude, due to divergence of the injected beam
T_i = injection energy
eV = energy gain per turn
R_0 = equilibrium sector radius

$$\alpha = \frac{\text{total length of straight sections}}{2\pi R_0}$$

Repetition rate

With the stored energy envisaged for a double machine, it should be possible to achieve a repetition rate of 1-1.5 pulses per second. The power supplied would have to deliver a total of about 70 MVA to the two sections of magnet, assuming a rise time of ~ 0.2 second in each section. The acceleration rate would then be lower in the first (low energy) section of the magnet, in about the ratio 1:7. A large range of B would be required in the low energy section, to preserve a long injection time.

Intensity

The object would be to achieve an increase over the *present* mean intensity of the Bevatron by a factor 100. In the example considered, this factor would be made up of a factor 10 in repetition rate and a factor 10 in injected current. The factor 10 in injected current could come from the use of a Van de Graaff generator or an improved linear accelerator with, possibly, the use of a slower rate of rise of magnetic field at injection.

Problems

The main problems are associated with

(i) field matching at cross-over, and the corresponding acceleration programme.

(ii) magnet power supplies with varying B in one or possibly both, sections. The use of capacitors for energy storage is not ruled out, at least for the low energy section. A varying B could then be realised by D.C. biassing. With alternator-ignitron supplies, it would be necessary to consider saturable reactors and/or voltage control by ignitron timing to give the desired B programme.

3.2 *Spiral ridge synchrotron*

In a machine with spiral ridge focusing, the frequency of the radial oscillations is almost unaffected by the ridges whereas the frequency of the vertical oscillations is increased. In principle, therefore, we can design a synchrotron which has the same radial aperture and injection conditions as the Bevatron, but with stronger vertical focusing and a smaller vertical aperture. The stored energy in the magnet is proportionately reduced, and we may operate at a higher repetition rate.

We have begun studies of such a system, and have found that for a field which has the same power law index as the Bevatron ($k = -0.6$, or $n = 0.6$) the radial oscillations have the same frequency as in the Bevatron, and are stable for amplitudes up to 2 feet, whereas the strength of vertical focusing has been increased by a factor 4. The machine considered has 24 ridges at an angle of $^1/_{12}$ radian to a circle concentric with the magnet units, and the depth of flutter is $\pm 25\%$. The radial oscillations extend over four spiral pitches.

Computations

The equations of motion are represented by

$$\frac{d^2x}{d\theta^2} - x = -x^{(2+k)}\left[1 + \delta \sin(N\theta - K\, ln\, x)\right]$$

$$\frac{d^2\eta}{d\theta^2} + x^{(k+1)}\left[-k\left\{1 + \delta \sin(N\theta - K\, ln\, x)\right\} + \right.$$

$$\left. + K\delta \cos(N\theta - K\, ln\, x)\right]\eta = 0$$

where x is the normlised radius and η is the normalised vertical displacement; only linear terms in η are included in the second equation.

Orbits were computed for $N = 12$, $\delta = 1/4$, $k = 0.6$ and $K/N = 12$. For a mean radius of 50 feet a displacement of 0.04 from the mean radius $x = 1$ represents an amplitude of oscillation of 2 feet. Violent vertical instability was evident for this case, caused by coupling between radial and vertical oscillation. For small radial oscillations,

$$Q^2_V = -k + \frac{\delta^2 K^2}{N^2}, \text{ or } Q_V \simeq 3 \text{ in this case, and}$$

$$\mu = \frac{2\pi Q}{N} = \frac{\pi}{2}$$

However, for large radial oscillations, the particle crosses the ridges at appreciably smaller angles on its outward excursions, and with the parameters chosen the mode number μ increases beyond π causing exponential instability.

When N was increased to 24, keeping the other parameters as before, both radial and vertical oscillations were found to be stable and all features of the orbits have been explained by an approximate theory.

Theory

For the small k value chosen the radial equation can be simplified to

$$\ddot{\rho} + (1 + k)\rho = -\delta \sin (N\theta - K\rho) \tag{1}$$

in which $x = 1 + \rho$

Certain small order non-linear terms have been ignored in the approximation but these do not affect the major conclusions. An approximate solution of (1) is

$$\rho = A \cos (Q_r\theta + \varepsilon) + \frac{\delta}{\Omega^2} \sin \int \Omega d\theta \tag{2}$$

$$\text{where} \begin{cases} \Omega = N - K\dot{\rho} \\ \simeq N + KA\, Q_r \sin (Q_r\theta + \varepsilon) \\ Q_r = \sqrt{1 + k} \end{cases}$$

The mode number is $\mu = \frac{2\pi Q_r}{N}$. For $N = 12$ this is $\pi/9$, and for $N = 24$ it is $\pi/18$.

This solution proved bad for a computed case in which $Q_r = 1.0$, $A > 0.02$, $N = 12$, $\delta = 1/4$, $k = 0$. However, by changing k to -0.6, i.e. $Q_r = 2/3$, the above solution was a good approximation for both $N = 12$ and $N = 24$ for $A < 0.04$.

There is a high frequency modulation, in both amplitude and frequency, because of the mean variation in the orbit.

The equation for the vertical oscillation can also be simplified to

$$\ddot{\eta} + [-k + \delta K \cos (N\theta - K\rho) \eta = 0 \tag{3}$$

Substituting for ρ from (2), we have

$$\cos (N\theta - K\rho) = \cos \left[\int \Omega d\theta - \frac{K\delta}{\Omega^2} \sin \int \Omega d\theta \right] \simeq$$

$$\simeq \cos \int \Omega d\theta + \frac{K\delta}{\Omega^2} \sin^2 \int \Omega d\theta$$

where it is assumed that the high frequency ripple is small.

Equation (3), thus becomes

$$\ddot{\eta} + \left[-k + \frac{\delta^2 K^2}{\Omega^2} \sin^2 \int \Omega d\theta + \delta K \cos \int \Omega d\theta \right] \eta = 0$$

Since the vertical oscillation frequency is high compared with the radial oscillation, then according to the smooth approximation the vertical frequency is given by

$$\nu^2 = -k + \frac{\delta^2 K^2}{\Omega^2}$$

in which Ω varies with the radial oscillation, and

$$\eta = \bar{\eta} \left[1 + \frac{\delta K}{\Omega^2} \cos \int \Omega d\theta \right]$$

with $\ddot{\bar{\eta}} + \nu^2 \bar{\eta} = 0$.

That is, from adiabatic theory,

$$\eta \simeq \frac{\beta \cos \int \nu d\theta}{\nu^{1/2}} \left[1 + \frac{\delta K}{\Omega^2} \cos \int \Omega d\theta \right] \tag{4}$$

Comparison with computed orbits shows that this is a good representation of the dynamics. The mean motion varies in frequency according to ν; for $N = 24$, $A = 0.04$, $\delta = 1/4$, $k = -0.6$, $K/N = 12$, the extreme limits of ν are from 2.4 to 4.5, or $\mu_V = \frac{2\pi\nu}{\Omega}$ varies from 0.15π to 0.55π.

For the case where $N = 12$, the variation takes μ_V beyond π and explains the vertical blow-up observed in the computed orbits for this case.

The modulation in amplitude of the mean motion is predicted by the term in $\nu^{-1/2}$. The final term in (4) agrees very well with the high frequency vertical ripple observed in the computed orbits. Like the radial ripple, there is again both frequency and amplitude modulation; the radial ripple is in quadrature with the vertical ripple.

Resonances

In the theory advantage was taken of the fact that the vertical oscillation frequency was high compared with the radial frequency. Thus Ω can be regarded as an effective N value which varies slowly with the radial oscillation. With $\Omega < N$ (increasing radial motion) the instantaneous value of μ_V increases, and there is danger of the π-mode being reached. This occurs when

$$a = 1 - q$$

$$a = \frac{4}{\Omega^2}\left[- k + \frac{\delta^2 K^2}{2\Omega^2}\right]$$

$$q = \frac{2}{\Omega^2}\,\delta K$$

That is,

$$\frac{4}{\Omega^2}\left[- k + \frac{\delta^2 K^2}{2\Omega^2}\right] = 1 - 2\,\frac{\delta K}{\Omega^2}$$

For $k \simeq 0$, $\frac{\delta K}{\Omega^2} = 0.366$

$$\Omega = N - KAQ_r$$

$$\therefore \frac{\delta K}{N^2} < 0.366\left[1 - \frac{KAQ_r}{N}\right]^2$$

For example, $\frac{\delta K}{N} = 3$, $\delta = 1/4$, $\frac{K}{N} = 12$

$Q_r = 2/3$, $k = -0.6$ $A = 0.04$

$\therefore$ N must be > 17 to avoid reaching π-mode during part of the radial oscillation.

For $N = 24$, the maximum value of μ in the above example is $\pi/2$.

Although the π-mode blow-up is disastrous from a practical point of view it should not be supposed that the motion is unbounded. Unbounded motion will occur only if stop bands are predicted when the correct value of Ω is used. For $N = 24$, we can simplify by using the smooth approximation, i.e. we regard

$$\ddot{\eta} + \left[- k + \frac{\delta^2 K^2}{\Omega^2}\right]\eta = 0$$

as a Hill equation periodic in the interval $0 < Q_r\theta < 2\pi$, where

$$\Omega = N + KAQ_r \sin (Q_r\theta + \varepsilon).$$

Expanding Ω to first order we have

$$\ddot{\eta} + \left[- k + \frac{\delta^2 K^2}{N^2}\left\{1 - \frac{2KAQ_r}{N}\sin (Q_r\theta + \varepsilon)\right\}\right]\eta = 0$$

Resonances occur for small A when

$$\left\{- k + \frac{\delta^2 K^2}{N^2}\right\}^{\frac{1}{2}} = \frac{S}{2}\,Q_r$$

For the examples quoted above, $Q_r = 2/3$ and hence stop-bands will be placed at 1/3 intervals in Q_v. When $\delta \simeq 0$ and $\frac{\delta K}{N} = 3$ the closest stop-band occurs for $S = 9$ The effect of this on the dynamics has not yet been examined, but it is a high-order Mathieu function stop-band and may be less serious than linear resonances.

Practical Aspects

Figure 4 is a diagrammatic section of a possible magnet The effective aperture would be 5″ × 48″, and the stored energy at 15 kilogauss would be about 20 megajoules for a 6-7 Gev machine. It might be possible to reduce the stored energy somewhat, by using saturable pole tips as shown diagrammatically in this figure, on taking advantage of the damping of the radial oscillations. The magnet weight, corresponding to the full 20 megajoules stored energy and no saturable pole tips, would be about 5,000 tons; a lower weight would result from the use of saturable pole tips.

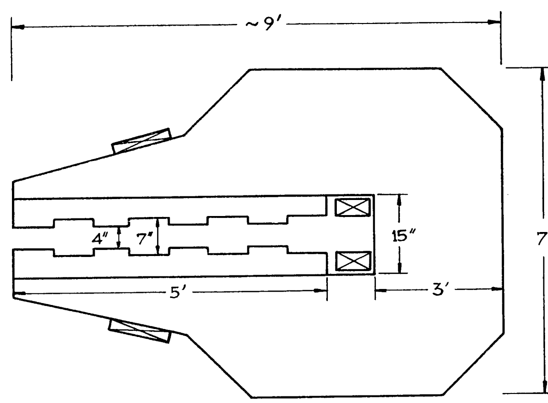


Fig. 4. Spiral ridge synchrotron magnet profile

With 20 megajoules, a power supply of 50 MVA would give a rise time of about 0.8 second. The repetition rate of such a machine would be about 35 ppm., or, with some contribution from saturable pole tips, perhaps 60 ppm.

Other features, such as injection conditions and programming of B, would be rather similar to the double synchrotron.

In contrast to the double synchrotron, it should be noted that the spiral ridge synchrotron does not depend for its operation upon the damping of the vertical oscillations.

In both machines, the main aim is to reduce considerably the stored energy in the magnet without introducing very tight tolerances. This saving can be used either to cheapen the power supplies or to obtain a higher repetition rate.

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