STABILITY OF THE COHERENT TRANSVERSE MOTION OF A COASTNG BEAM FOR REALISTIC DISTRIBUTION FUNCTIONS AND ANY GIVEN COUPLING WITH ITS ENVIRONMENT

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1. Introduction

This paper deals with the stability criterion for transverse coherent oscillations of a coasting beam in a circular accelerator. The stability criterion is given in terms of U and V, which are parameters proportional to the out-of-phase and in-phase component of the electromagnetic force produced by the oscillating beam. Our definition of U and V agrees with ref.[1] except that the U, used in this paper takes into account the whole out-of-phase component, thus including the reactance due to the skin effect. Approximate stability criteria and growth rates in the limits of $U \approx V$ and $U \gg V$ were given in ref.[1] for some distribution functions. The spread in betatron amplitude and in energy was treated separately.

Our approach to the exact solution of the dispersion relation derived in ref.[1] is similar to that used in ref. [2, 3] for longitudinal oscillations. We include in our analysis not only the spread in energy but also the spread in amplitude. Special attention is directed to distribution functions resulting from beam stacking by R. F. acceleration.

The solution of the dispersion relation is presented in the form of a mapping of the complex frequency plane $\omega = \omega_1 + i\omega_2$ onto the U, V plane. We find the curve corresponding to infinite growth time encloses a region near the origin which is not covered by the mapping. No solution with an exponential time dependence is possible for values of U and V inside this region, which we call the stable region. Values of U and V characteristic for a particular accelerator then can be plotted in the diagram with the real frequency as running parameter. If a part of this $U(\omega_1)$, $V(\omega_1)$ curve happens to lie outside the stable region we can

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predict the growth rate of the unstable modes from the mapping. A suitable normalization is applied to obtain universal graphs.

In the evaluation of U and V the whole environment of the beam has to be taken into account [1, 4]. It was shown in ref.[5] that also the distribution functions have an influence on U and V.

2. Normalization

We use the dispersion relation ref.[1]

$$1=\Omega_0\Omega_0(U+iV)\int\int\frac{g'(a)\, a^2da\, f(W)dW}{[(\omega-n\Omega)^2-Q^2\Omega^2]} \quad (1)$$

a is the betatron amplitude of the particles in the beam, W equals $2\pi(P-P_0)$, where P_0 is the average canonical angular momentum. The number of betatron wavelengths per revolution, Q, and the revolution frequency $\Omega/2\pi$, depend upon a and W. The distribution functions are normalized in the following way

$$\int_0^\infty g(a)ada=1 \qquad \int_{-\infty}^{+\infty} f(W)dW=1$$

The time dependence of the coherent oscillations is $e^{-i\omega t}$. Linearising Q(a, W) and Ω(a, W) yields

$$Q=Q_0+a^2\frac{\partial Q}{\partial a^2}+W\frac{\partial Q}{\partial W}$$

$$\Omega=\Omega_0+a^2\frac{\partial \Omega}{\partial a^2}+W\frac{\partial \Omega}{\partial W}$$

Furthermore, we define

$$\psi_a{}^\pm=(n\pm Q_0)\frac{\partial \Omega}{\partial a^2}\pm\Omega_0\frac{\partial Q}{\partial a^2}$$

$$\psi_w{}^\pm=(n\pm Q_0)\frac{\partial \Omega}{\partial W}\pm\Omega_0\frac{\partial Q}{\partial W}$$

Since $Q_0/Q\approx 1$ and $\Omega_0/\Omega\approx 1$, we can replace (1) by

$$1=(U+iV)\,(I_+-I_-), \quad (2)$$

$I_\pm$ is given by

$$I_\pm=\frac{1}{\delta_w\psi_w{}^\pm}\int_{-\infty}^{+}F(x)\,dx\int_0^\infty\frac{G'(y)\,y\,dy}{x_1{}^\pm-y\varepsilon^\pm-x}, \quad (3)$$

where

$$x=W/\delta_w, \quad y=a^2/\delta_a^2 \quad (4a, b)$$

We define δ_w by $f(\pm\delta_w)=f_{max}/2$ and δ_a^2 by $g(\delta_a^2)=g(0)/2$.

The normalized complex frequency shift

$$x_1^{\pm}=[\omega-(n\pm Q_0)\Omega_0]/\delta_w\psi_w^{\pm}, \qquad (5)$$

replaces ω. We denote the ratio of the spread in amplitude to the spread in energy by

$$\varepsilon^{\pm}=\delta_a^2\psi_a^{\pm}/\delta_w\psi_w^{\pm}$$

The normalization of the new distribution function is:

$$\int_0^{\infty} G(y)dy=1 \quad \text{and} \quad \int_{-\infty}^{+\infty} F(x)\,dx=1 \qquad (6a, b)$$

As long as the spread in the quantity $(n-Q)\Omega$ does not become equal to the difference between the slow and the fast wave, we can neglect either I_+ or I_- depending upon whether x_1^+ or x_1^- is small. In both cases we get the same contours in the U, V plane. Thus it is sufficient to consider the slow wave.

We normalize U and V in the following way

$$U'=U/\delta_w|\psi_w^-| \qquad V_1=V/\delta_w|\psi_w^-| \qquad (7)$$

3. Results

For simplicity, we will first discuss the case $\varepsilon<<1$ which will reveal all basic features of the solution. Equation (3) becomes

$$1=(U'+iV')\frac{|\psi_w^-|}{\psi_w^-}\int\frac{F(x)dx}{x-x_1^-}$$

Splitting the integral in its principal value and in its imaginary part yields

$$\int_{-}^{+\infty}\frac{F(x)\,ds}{x-x_1^-}=P\int_{-\infty}^{+\infty}\frac{F(x)\,dx}{x-x_1^-}\pm i\pi F(x_1^-), \qquad (9)$$

where the sign of the imaginary part equals the sign of the vanishing imaginary part of x_1^-. It is apparent from (8) and (9) that there will be a region, disposed symmetrically around the U′ axis which is not covered by the mapping of the complex x_1^- plane onto the U′, V′ plane. Thus no solution for x_1^- with the exponential time dependence can exist if U′, V′ fall into this region. This does not exclude the existence of damped solutions not obeying an exponential law [6]. Outside the stable area we always find two solutions, one for the slow and one for the fast wave. They are damped or growing depending upon the sign of $V\psi^{\pm}$, which turns out to be equal to the sign of the imaginary part of ω. The symmetry of the mapping with respect to the U′ axis means that care has to be taken in designing an active feedback circuit, since an excessive negative V can give rise to growing waves[7].

As a first example we consider the distribution function F(x) shown in Fig. 1a, which represents adequately the distribution in energy after the R. F, stacking process. The flanks are parabolae of 4th order. Assuming $\varepsilon = 0$ the integral in (8) becomes

$$\int = \left(\frac{8}{15\alpha} (\delta_1 + \delta_2) + 2\xi \right) \sum_{s=1,2} \left\{ \frac{A_s}{3} + A_s(A_s^2 - 2) - (-1)^s \left[\frac{A_s^2}{2} - (A_s^2 - 1)^2 \ln \frac{A_s}{A_s + (-1)^s} - \ln A_s \delta_s \right] \right\}$$

where

$$A_s = \frac{X_1^- - (-1)^s \xi}{\alpha \delta_s} \quad \text{and} \quad \alpha = \sqrt{1 - 1/\sqrt{2}}$$

The mapping of the lower half-plane of x_1^- onto the U′, V′ plane is shown in Fig. 2. This half plane corresponds to growing solutions according to (5) because ψ_w^- is assumed to be negative.

Since the ratio of U/V is about 1 for the ISR, we gather from Fig. 2 and (7b) that stability is guaranteed if

$$V < \delta_w |\psi_w^-| \, 0{,}3 \qquad (10)$$

holds. This becomes for $(n - Q)\frac{\partial \Omega}{\partial W} \ll \frac{\partial Q}{\partial W} \Omega_0$

$$V < \frac{\partial Q}{\partial W} \delta_w \Omega_0 \, 0{,}3.$$

For high values of n we get

$$V < n \frac{\partial \Omega}{\partial W} \delta_w \, 0{,}3.$$

One should keep in mind that δ_w is the half spread at half height. Having performed the mapping for a variety of distribution functions we can state that (10) is a good criterion, as long as $U/V \approx 1$ holds.

In the case of $U \ll V$ we find, that

$$V < 0{,}6 \, \delta_w |\psi_w^-|$$

has to be fulfilled in order to ensure stability. If $V \ll U$ an inspection of the mapping is inevitable. This is apparent from Fig. 3a and Fig. 3b which show the influence of increasing flat top and asymmetry. The rectangular disrribution gives as a limiting case a circle in the U′, V′ plane.

In a second example we consider the distribution functions shown in Fig. 1b, c. Fig. 4 reveals that the stabilization by the energy spread and the stabilization by the amplitude spread are not additive. Their combined influence depends on the ratio U/V and, not very strongly, on the distribution functions.

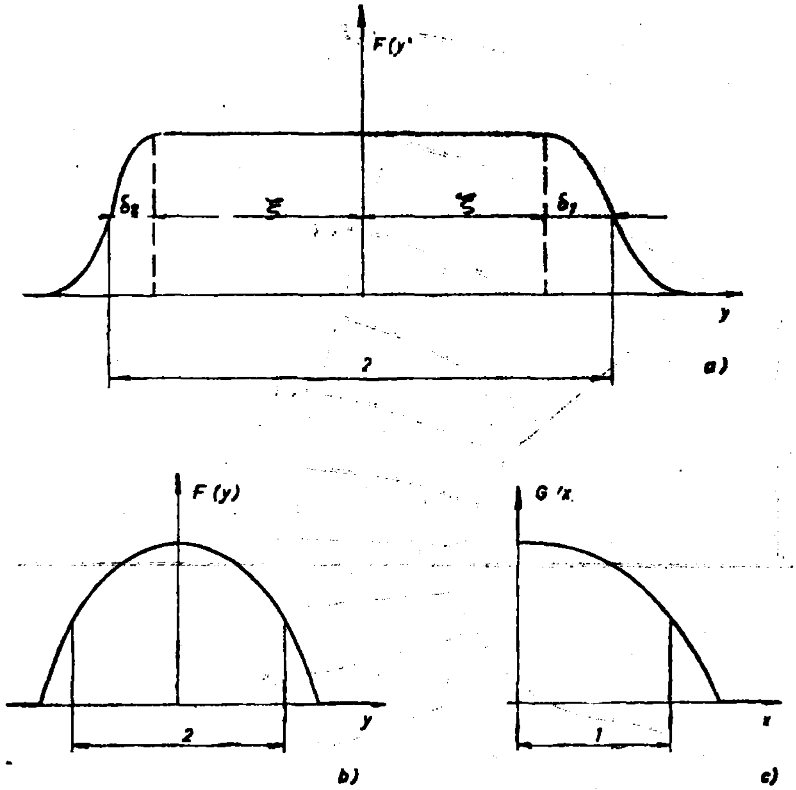

Fig. 1 Normalized distribution functions.

a) Distribution function in energy with flat top.

b) Distribution function in energy, $F(x) = \frac{3}{4\sqrt{2}}\left(1 - \frac{x^2}{2}\right)$

c) Distribution function in betatron amplitude. $G(y) = \frac{3}{2\sqrt{2}}\left(1 - \frac{y^2}{2}\right)$

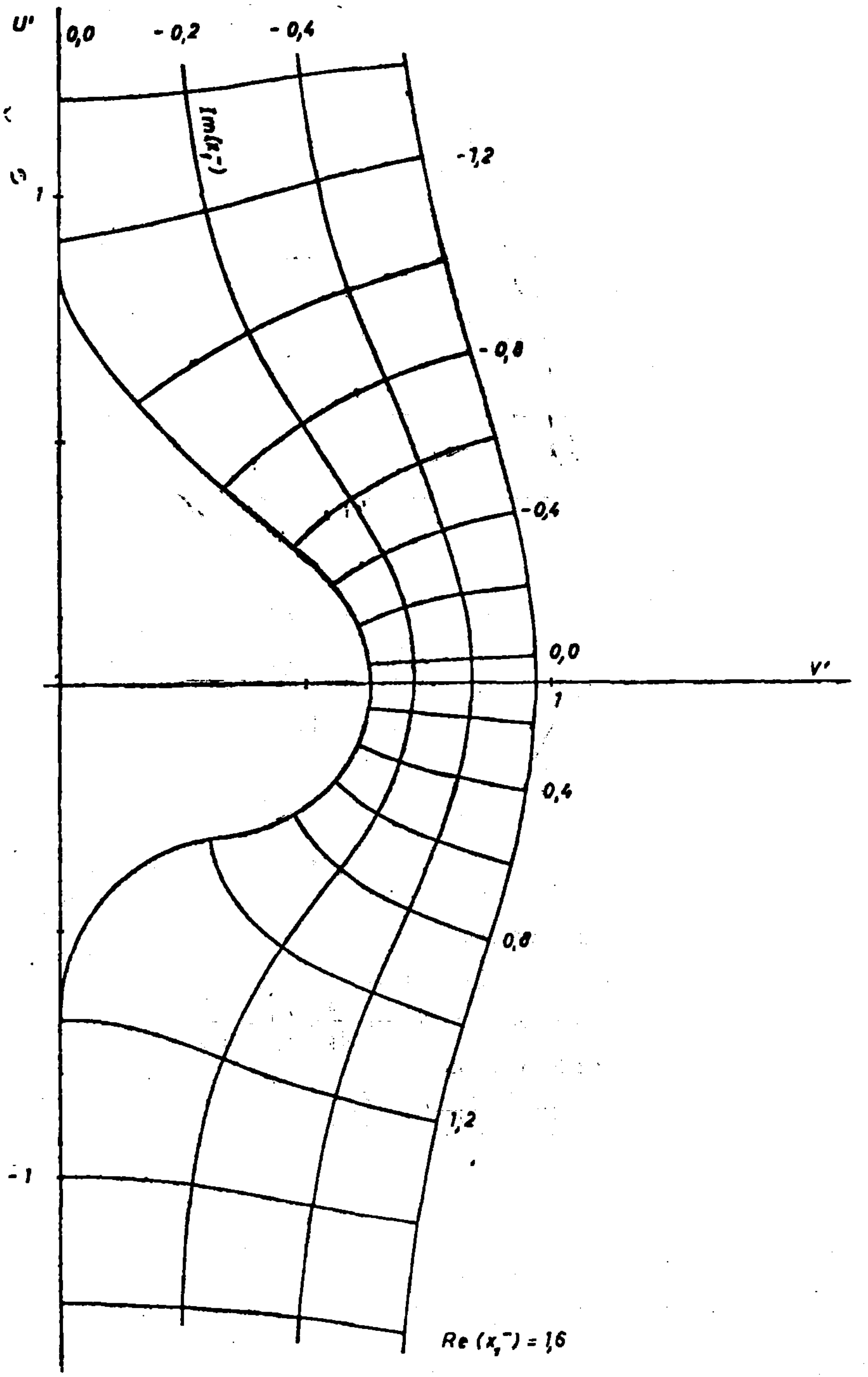

Fig. 2. Mappi g of the lower half-plane of the normalized complex frequency shift x_1^- Distribution function shown in Fig. Ia $\delta_1/\delta_2=3.8$ and $2\xi/\delta_1+\delta_2=4$. No amplitude distribution, $\varepsilon=0$.

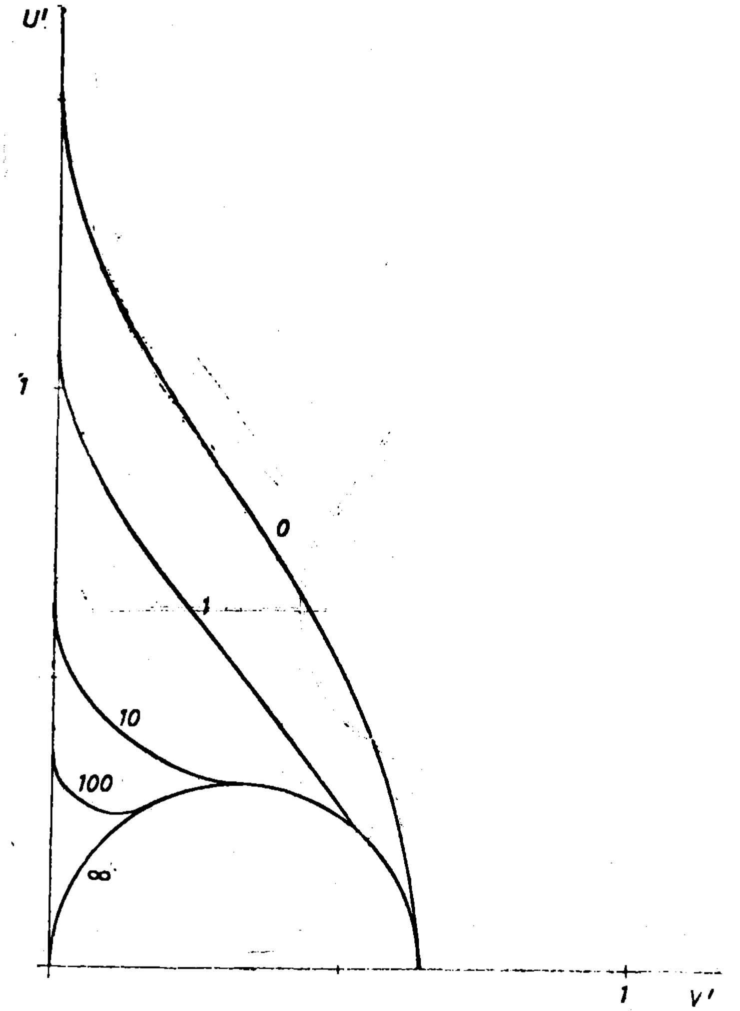

Fig. 3a. The stability limit $\mathrm{Im}(x\bar{i})=0$ for various values of $2\xi/(\delta_1+\delta)$ for $\delta_2/\delta_1=1$ and $\varepsilon \cdot 0$. Distribution function shown in Fig. 1a.

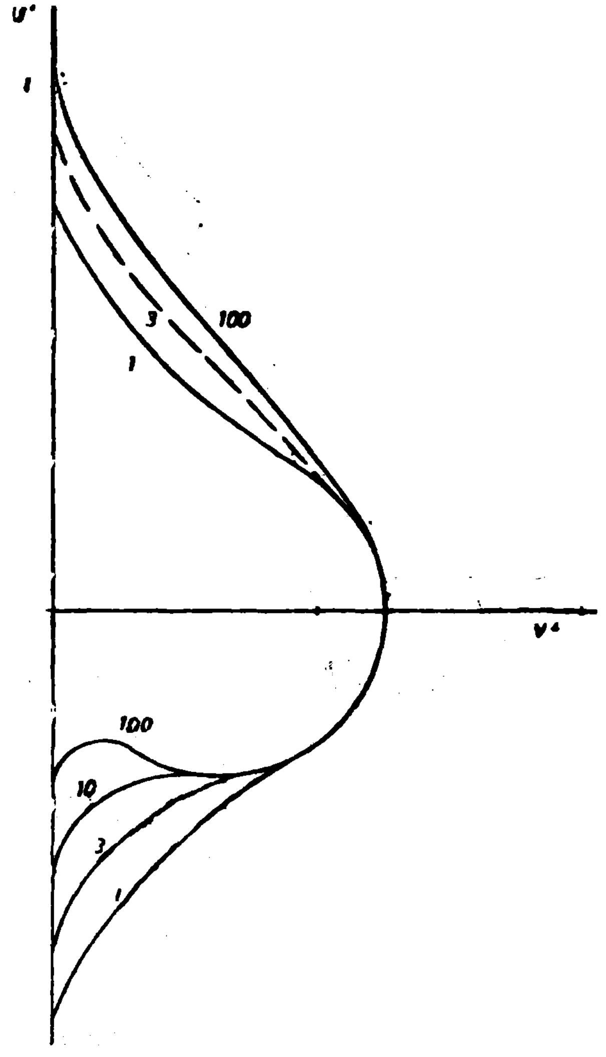

Fig. 3b. The same as Fig. 3a except that $2\xi/(\delta_1+\delta_2)=3$ and δ_2/δ_1 is varied.

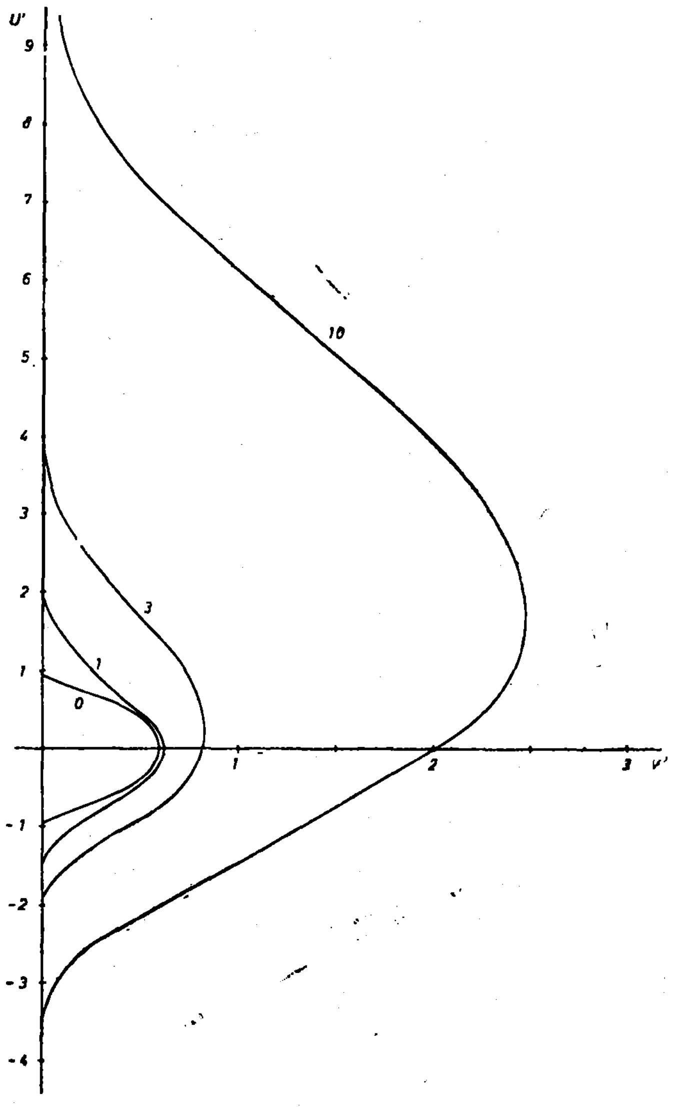

Fig. 4. The influence of the amplitude spread on the stability limit. Distribution functions shown in Fig. lb and c. The quantity ε is the ratio between amplitude and energy spread.

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