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Unitarity-Cuts and Berry's Phase

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Abstract Elaborating on the observation that two-particle unitarity-cuts of scattering amplitudes can be computed by applying Stokes' theorem, we relate the optical theorem to the Berry phase, showing how the imaginary part of arbitrary one-loop Feynman amplitudes can be interpreted as the flux of a complex 2-form.

Keywords Feynman diagrams,, unitarity,, scattering amplitudes,, optical theorem,
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Mathematics Subject Classification (2000) 81T18, 81Q70, 81U20

1 Introduction

Unitarity and geometric phases are two ubiquitous properties of physical systems.

The Berry phase is the phase acquired by a system when it is subjected to a cyclic evolution, resulting only from the geometrical properties of the path traversed in the parameter space because of anholonomy (1; 2).

Unitarity represents the probability conservation in particle scattering processes described by the unitary *scattering operator*, S. The relation, $S = 1 + iT$, between the S-operator and the *transition operator*, T, leads to the optical theorem,

$$-i(T - T^{\dagger}) = T^{\dagger}T. \quad (1)$$

The matrix elements of this equation between initial and final states are expressed, in perturbation theory, in terms of Feynman diagrams. The evaluation of the right

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hand side requires the insertion of a complete set of intermediate states. Therefore, since $-i(T-T^{\dagger}) = 2\ \mathrm{Im}T$, Equation (1) yields the computation of the imaginary part of Feynman integrals from a sum of contributions from all possible intermediate states. A Feynman diagram is thus responsible for an imaginary part of the scattering amplitudes when the intermediate, virtual particles go on-shell.

The Cutkosky–Veltman rules, implementing the unitarity conditions, allow the calculation of the discontinuity across a branch cut of an arbitrary Feynman amplitude, which corresponds to its imaginary part (3; 4; 5; 6; 7; 8; 9). Accordingly, the imaginary part of a given Feynman integral can be computed by evaluating the phase–space integral obtained by cutting two internal particles, which amounts to applying the on-shell conditions and replacing their propagators by the corresponding δ-function, $(p^2-m^2+i0)^{-1} \to (2\pi i)\ \delta^{(+)}(p^2-m^2)$.

In later studies the problem of finding the discontinuity of a Feynman integral associated to a singularity was addressed in the language of homology theory and differential forms (10).

More recently multi-particle cuts have been combined with the use of complex momenta (11) for on-shell internal particles into very efficient techniques, by-now known as unitarity-based methods, to compute scattering amplitudes for arbitrary processes (see (12; 13) for a comprehensive list of references).

In this letter we establish an explicit relation between Unitarity and Berry's phase, by showing that the imaginary part of a general *one-loop* Feynman amplitude, computed by applying the optical theorem, can be interpreted as a Berry phase, resulting from the curved geometry in effective momentum space experienced by the two on-shell particles going around the loop.

In a recent work (14) it has been shown that double-cuts of one-loop scattering amplitudes can be efficiently evaluated by using the well-known *Generalised Cauchy Formula*, also known as *Cauchy-Pompeiu Formula*, or *Cauchy-Green Formula* as well (17). In the case of double-cuts, the 4-dimensional loop-momentum can be decomposed in terms of an ad hoc basis of four massless vectors whose coefficients depend on two complex-conjugated variables, left over as free components after imposing the two on-shell cut-constraints. Therefore, the double-cut phase–space integral is written as a twofold integration over these two variables.

The integration is finally carried out by using Generalised Cauchy Formula as an application of Stokes' Theorem for rational function of two complex-conjugated variables. As such, the result of the phase–space integration can be naturally interpreted as the flux of a 2-form that is given by the product of the two tree-level amplitudes sewn along the cut.

2 Double-Cut

The two-particle Lorentz invariant phase–space (LIPS) in the K^2-channel is defined as,

$$\int d^4\Phi = \int d^4\ell_1\ \delta^{(+)}(\ell_1^2-m_1^2)\ \delta^{(+)}((\ell_1-K)^2-m_2^2), \tag{2}$$

where K^μ is the total momentum across the cut. We introduce a suitable parametrization for ℓ_1^μ (14; 15; 16), in terms of four massless momenta, which is a solution of the two on-shell conditions, $\ell_1^2 = m_1^2$ and $(\ell_1 - K)^2 = m_2^2$,

$$\ell_1^\mu = \frac{1-2\rho}{1+z\bar{z}} \left(p^\mu + z\bar{z}\, q^\mu + z\varepsilon_+^\mu + \bar{z}\varepsilon_-^\mu\right) + \rho K^\mu, \tag{3}$$

where p_μ and q_μ are two massless momenta with the requirements,

$$\begin{aligned} &p_\mu + q_\mu = K_\mu, \\ &p^2 = q^2 = 0, \\ &2p\cdot q = 2p\cdot K = 2q\cdot K \equiv K^2; \end{aligned} \tag{4}$$

the vectors ε_+^μ and ε_-^μ are orthogonal to both p^μ and q^μ, with the following properties,[1]

$$\varepsilon_+^2 = \varepsilon_-^2 = 0 = \varepsilon_\pm \cdot p = \varepsilon_\pm \cdot q, \tag{5}$$

$$2\varepsilon_+ \cdot \varepsilon_- = -K^2. \tag{6}$$

The parameter ρ is the pseudo-threshold,

$$\rho = \frac{K^2 + m_1^2 - m_2^2 - \sqrt{\lambda(K^2, m_1^2, m_2^2)}}{2K^2}, \tag{7}$$

with the Källen function defined as,

$$\lambda(K^2, m_1^2, m_2^2) = (K^2)^2 + (m_1^2)^2 + (m_2^2)^2 - 2K^2 m_1^2 - 2K^2 m_2^2 - 2m_1^2 m_2^2, \tag{8}$$

and depends only on the kinematics.

The complex conjugated variables z and $\bar{z}$ parametrize the degrees of freedom left over by the cut-conditions.

Analogously to the massless case (14), corresponding to the $\rho \to 0$ limit, because of (3), the LIPS in (2) reduces to the remarkable expression,

$$\int d^4\Phi = (1-2\rho) \iint \frac{\mathrm{d}z \wedge \mathrm{d}\bar{z}}{(1+z\bar{z})^2}. \tag{9}$$

The double-cut of a generic n-point amplitude in the K^2-channel is defined as

$$\Delta \equiv \int d^4\Phi \, A_L^{\text{tree}}(\ell_1) \, A_R^{\text{tree}}(\ell_1), \tag{10}$$

where $A_{L,R}^{\text{tree}}$ are the tree-level amplitudes sitting at the two sides of the cut (see Figure 1). By using (9) for the LIPS, and (3) for the loop-momentum ℓ_1^μ, one has,

$$\Delta = (1-2\rho) \iint \mathrm{d}z \wedge \mathrm{d}\bar{z} \, \frac{A_L^{\text{tree}}(\rho, z, \bar{z}) \, A_R^{\text{tree}}(\rho, z, \bar{z})}{(1+z\bar{z})^2}, \tag{11}$$

[1] In terms of spinor variables that are associated to massless momenta, we can define $p^\mu = (1/2)p.\gamma^\mu.p$ and $q^\mu = (1/2)q.\gamma^\mu.q$, hence $\varepsilon_+^\mu = (1/2)q.\gamma^\mu.p$ and $\varepsilon_-^\mu = (1/2)p.\gamma^\mu.q$.

Fig. 1 Double-cut of one-loop amplitude in the K^2-channel.

where the tree-amplitudes $A_L^{\rm tree}$ and $A_R^{\rm tree}$ are rational in z and $\bar{z}$. Notice that ρ is independent of z and $\bar{z}$, therefore its presence in the integrand does not affect the integration algorithm. For ease of notation, we give the ρ-dependence of the integrand as understood.

In (14), we aimed at proposing an efficient method for computing the double-cut of one-loop scattering amplitudes. Accordingly, by applying a special version of the so called *Generalised Cauchy Formula* also known as the *Cauchy-Pompeiu Formula* (17), one can write the twofold integration in z- and $\bar{z}$-variables appearing in Equation (11) simply as a convolution of an unbounded $\bar{z}$-integral and a contour z-integral,[2]

$$\Delta = (1-2\rho) \oint \mathrm{d}z \int \mathrm{d}\bar{z} \, \frac{A_L^{\rm tree}(z,\bar{z}) \, A_R^{\rm tree}(z,\bar{z})}{(1+z\bar{z})^2} \, , \tag{12}$$

where the integration contour has to be chosen as enclosing all the complex z-poles.

In this letter we rather want to focus on what links Equations (11) and (12), namely Stokes' Theorem (14), and on the geometrical interpretation of its consequence: the double-cut Δ in Equation (11) is the flux of a 2-form. It corresponds to an integral over the complex tangent bundle of the Riemann sphere, where the curvature 2-form, Ω, is defined as,[3]

$$\Omega = \frac{\mathrm{d}z \wedge \mathrm{d}\bar{z}}{(1+|z|^2)^2} . \tag{13}$$

The product $A_L^{\rm tree} A_R^{\rm tree}$ is a rational function of z and $\bar{z}$, hence it can be written as ratio of two polynomials, P and Q,

$$A_L^{\rm tree}(z,\bar{z}) \, A_R^{\rm tree}(z,\bar{z}) = \frac{P(z,\bar{z})}{Q(z,\bar{z})}, \tag{14}$$

with the following relations among their degrees,

$$\deg_z Q = \deg_z P, \quad \deg_{\bar{z}} Q = \deg_{\bar{z}} P. \tag{15}$$

2 The roles of z and $\bar{z}$ can be equivalently exchanged.

3 In (14) it has been shown that the double-cut of the scalar 2-point function, $\Delta I_2 = \int d^4\Phi$ amounts to the integral $\iint \Omega = -2\pi i$. This result corresponds to the integration of the first Chern class, $(i/\pi) \iint \Omega = 2$.

3 Optical Theorem

In the double-cut integral (11), we did not make any assumptions on the tree-level amplitudes sewn along the cut, thus providing a general framework to the integration method developed in (14). If we now choose $A_L^{\rm tree} = A_{m\to 2}^{*,{\rm tree}}$, that is the conjugate scattering amplitude of a process $m \to 2$, and $A_R^{\rm tree} = A_{n\to 2}^{\rm tree}$, that is the amplitude of a process $n \to 2$, then Δ reads,

$$\Delta = \int d^4\Phi \, A_{m\to 2}^{*,{\rm tree}} \, A_{n\to 2}^{\rm tree} = -i \left[A_{n\to m}^{\rm one-loop} - A_{m\to n}^{*,{\rm one-loop}} \right] = 2 \, {\rm Im} \left\{ A_{n\to m}^{\rm one-loop} \right\}, \tag{16}$$

which is the definition of the two-particle discontinuity of the one-loop amplitude $A_{n\to m}^{\rm one-loop}$ across the branch cut in the K^2-channel, corresponding to the field-theoretic version of the optical theorem (1) for one-loop Feynman amplitudes.

On the other side, because of Stokes' Theorem in Equations (11) and (12), one has,

$$\Delta = (1-2\rho) \int\!\!\int {\rm d}z \wedge {\rm d}\bar{z} \, \frac{A_{m\to 2}^{*,{\rm tree}} \, A_{n\to 2}^{\rm tree}}{(1+z\bar{z})^2} = (1-2\rho) \oint {\rm d}z \int {\rm d}\bar{z} \, \frac{A_{m\to 2}^{*,{\rm tree}} \, A_{n\to 2}^{\rm tree}}{(1+z\bar{z})^2}, \tag{17}$$

which provides a geometrical interpretation of the imaginary part of one-loop scattering amplitudes, as a flux of a complex 2-form through a surface bounded by the contour of the z-integral (the contour should enclose all the poles in z exposed in the integrand after the integration in $\bar{z}$ (14)).

Given the equivalence of (16) and (17), a correspondence between the imaginary part of scattering amplitudes and the anholonomy of Berry's phase does emerge, since the latter is indeed defined as the flux of a 2-form in presence of curved space (1; 2). In this context, one could establish a parallel description between the Aharonov–Böhm (AB) effect and the double-cut of one-loop Feynman integrals.

In the AB-effect (18), an electron-beam splits with half passing by either side of a long solenoid, before being recombined. Although the beams are kept away from the solenoid, so they encounter no magnetic field ($\mathbf{B} = 0$), they arrive at the recombination with a phase-difference that is proportional to the magnetic flux through a surface encircled by their paths. The non-trivial anholonomy in this case is a consequence of Stokes' Theorem, where the 2-form Berry curvature is written as the differential of the 1-form vector potential ($\nabla \times \mathbf{A}$).

In the case of the double-cut of one-loop Feynman integrals, we could describe the evolution of the system depicted in Figure 1, from the left to the right. The two particles produced in the A_L-scattering, going around the loop and initiating the A_R-process, at the A_R-interaction point would experience a phase-shift due to the non-trivial geometry in effective momentum space induced by the on-shell conditions. As in the AB-effect, the anholonomy phase-shift is a consequence of Stokes' Theorem, and here it corresponds to the imaginary part of the one-loop Feynman amplitude.

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