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More on $\mathcal{N} = 8$ attractors

Anna Ceresole,[1,*] Sergio Ferrara,[2,3,4,†] Alessandra Gnecchi,[5,‡] and Alessio Marrani[6,§]

[1]*INFN, Sezione di Torino, and Dipartimento di Fisica Teorica, Università di Torino, Via Pietro Giuria, 1, I-10125 Torino, Italy*
[2]*Theory Division-CERN, CH 1211, Geneva 23, Switzerland*
[3]*INFN-LNF, Via Enrico Fermi 40, I-00044 Frascati, Italy*
[4]*Department of Physics and Astronomy, University of California, Los Angeles, California USA*
[5]*Dipartimento di Fisica "Galileo Galilei," and INFN, Sezione di Padova, Via Marzolo 8, I-35131 Padova, Italy*
[6]*Stanford Institute for Theoretical Physics, Department of Physics, 382 Via Pueblo Mall, Varian Lab, Stanford University, Stanford, California 94305-4060, USA*

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We examine a few simple extremal black hole configurations of $\mathcal{N} = 8$, $d = 4$ supergravity. We first elucidate the relation between the BPS Reissner-Nördstrom black hole and the non-BPS Kaluza-Klein dyonic black hole. Their classical entropy, given by the Bekenstein-Hawking formula, can be reproduced via the attractor mechanism by suitable choices of symplectic frame. Then, we display the embedding of the axion-dilaton black hole into $\mathcal{N} = 8$ supergravity.

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I. INTRODUCTION

It has been known for some time [1] that extremal BPS black hole (BH) states coming from string and M theory compactifications to four and five dimensions, preserving various fractions of the original $\mathcal{N} = 8$ supersymmetry, can be invariantly classified in terms of orbits of the fundamental representations of the exceptional groups $E_{7(7)}$ and $E_{6(6)}$. These are the duality groups of the low energy actions, whose discrete subgroups appear as symmetries of the nonperturbative spectrum of BPS states [2]. These orbits, which have been further studied in ([3–5], correspond to well-defined categories of allowed entropies of extremal BHs in $d = 5$ and in $d = 4$, given in terms of the cubic $E_{6(6)}$ invariant I_3 ([1,4,6]) and the quartic $E_{7(7)}$ invariant I_4 ([7–9]). There are three types of orbits depending on whether the BH background preserves $1/2$, $1/4$, or $1/8$ of the original supersymmetry. Only $1/8$ BPS states have nonvanishing entropy and regular horizons, while $1/4$ and $1/2$ BPS configurations lead to vanishing classical entropy.

The $\mathcal{N} = 8$ attractors have been explored in [9] by solving the criticality condition for the suitable BH effective potential and extending the lore of $\mathcal{N} = 2$ special Kähler geometry [10].

In this paper we focus on some specific simple configurations in $\mathcal{N} = 8$, $d = 4$ supergravity which capture some representatives of the regular (sometimes called "*large*"), i.e. with nonvanishing classical entropy, extremal BPS and non-BPS BH charge orbits of the theory. One is the Reissner-Nördstrom (RN) dyonic BH, with electric and magnetic charge e and m respectively, and Bekenstein-Hawking entropy (in unit of Planck mass) [11]

$$S_{\mathrm{RN}} = \pi(e^2 + m^2). \tag{1.1}$$

Another one is the Kaluza-Klein (KK) dyonic BH, with a KK monopole charge p and a KK momentum q, which is dual to a $D0 - D6$ brane configuration in Type II A supergravity. Its Bekenstein-Hawking entropy reads

$$S_{\mathrm{KK}} = \pi|pq|. \tag{1.2}$$

One more interesting example is the extremal axion-dilaton BH, a subsector of pure $\mathcal{N} = 4$ supergravity in $d = 4$ which was considered in the past in [12,13].

Our aim is to show how the entropies of these BHs can be obtained in the context of $\mathcal{N} = 8$, $d = 4$ supergravity by exploiting the attractor mechanism [10,14–16] for extremal BPS and non-BPS BHs. Earlier studies for some specific cases were examined in [17,18].

It is in fact known that while the BH charge configuration with entropy given by (1.1) is $1/8$ BPS [11], the entropy (1.2) is related to a non-BPS one. Indeed, the $E_{7(7)}$ quartic invariant I_4 on these configurations reduces to

$$\sqrt{I_4^{\mathrm{RN}}} = e^2 + m^2; \tag{1.3}$$

$$\sqrt{-I_4^{\mathrm{KK}}} = |pq|. \tag{1.4}$$

In particular we note that, if the magnetic (or electric) charge is switched off, the RN BH remains regular, whereas the KK BH reaches zero entropy ($I_4 = 0$) and becomes $1/2$ BPS [3].

The simplest way to obtain these configurations is to observe that the BPS and non-BPS charge orbits with $I_4 \neq 0$ in $\mathcal{N} = 8$, $d = 4$ supergravity are given by [1]

$$\mathcal{O}_{1/8\text{-BPS}}\colon \frac{E_{7(7)}}{E_{6(2)}}, \qquad I_4 > 0; \tag{1.5}$$

*ceresole@to.infn.it
†sergio.ferrara@cern.ch
‡alessandra.gnecchi@pd.infn.it
§marrani@lnf.infn.it

$$\mathcal{O}_{\text{non-BPS}}: \frac{E_{7(7)}}{E_{6(6)}}, \qquad I_4 < 0. \tag{1.6}$$

The moduli spaces corresponding to the above disjoint orbits are [19]

$$\mathcal{M}_{1/8\text{-BPS}} = \frac{E_{6(2)}}{SU(6)\times SU(2)}, \qquad \mathcal{M}_{\text{non-BPS}} = \frac{E_{6(6)}}{USp(8)}. \tag{1.7}$$

Hence, a convenient representative of these orbits is given by the (unique) E_6 singlets in the decomposition of the fundamental representation **56** of $E_{7(7)}$ into the two relevant noncompact real forms of E_6:

RN

$$\mathcal{O}_{1/8\text{-BPS}}: \begin{cases} E_{7(7)} \longrightarrow E_{6(2)} \times U(1); \\ \mathbf{56} \longrightarrow (\mathbf{27}, 1) + (\mathbf{1}, 3) + (\overline{\mathbf{27}}, -1) + (\mathbf{1}, -3); \end{cases} \tag{1.8}$$

KK

$$\mathcal{O}_{\text{non-BPS}}: \begin{cases} E_{7(7)} \longrightarrow E_{6(6)} \times SO(1,1); \\ \mathbf{56} \longrightarrow (\mathbf{27}, 1) + (\mathbf{1}, 3) + (\mathbf{27}', -1) + (\mathbf{1}', -3), \end{cases} \tag{1.9}$$

where the $U(1)$ charges and $SO(1,1)$ weights are indicated, and the prime denotes the contravariant representations. Notice that, consistently with the group factors $U(1)$ and $SO(1,1)$, **27** is complex for $E_{6(2)}$, whereas it is real for $E_{6(6)}$. Both $E_{6(2)} \times U(1)$ and $E_{6(6)} \times SO(1,1)$ are maximal noncompact subgroups of $E_{7(7)}$, with symmetric embedding.

Our result is simply stated as follows.

The two extremal BH charge configurations determining the embedding of RN and KK extremal BHs into $\mathcal{N} = 8$, $d = 4$ supergravity with entropies (1.1) and (1.2), are given by the two E_6 singlets in the decompositions (1.8) and (1.9).

The two situations can be efficiently associated to two different parametrizations of the real symmetric scalar manifold $\frac{E_{7(7)}}{SU(8)}$ ($\dim_{\mathbb{R}} = 70$, rank $= 7$) of $\mathcal{N} = 8$, $d = 4$ supergravity.

For the branching (1.8), pertaining to the RN extremal BH, the relevant parametrization is the $SU(8)$-covariant one. This corresponds to the Cartan's decomposition basis, where the coset coordinates ϕ_{ijkl} ($i = 1, \ldots 8$) sit in the four-fold antisymmetric self-real irrep **70** of $SU(8)$. The attractor mechanism implies that at the horizon

$$\phi_{ijkl,H} = 0, \tag{1.10}$$

i.e. the scalar configuration at the event horizon of the 1/8-BPS extremal BH is given by the origin of $\frac{E_{7(7)}}{SU(8)}$. Some care should be taken with regard to "*flat*" directions [8,19]. Because of the existence of the moduli space $\frac{E_{6(2)}}{SU(6)\times SU(2)}$ ($\dim_{\mathbb{R}} = 40$, rank $= 4$) of the $\frac{1}{8}$-BPS attractor solutions, strictly speaking 40 scalar degrees of freedom out of 70 are actually undetermined at the event horizon of the given $\frac{1}{8}$-BPS RN extremal BH. In other words, 40 real scalar degrees of freedom, spanning the quaternionic symmetric coset $\frac{E_{6(2)}}{SU(6)\times SU(2)}$ (which is the *c map* [20] of the vector multiplets' scalar manifold of $\mathcal{N} = 2$, $d = 4$ "*magic*" supergravity based on $J_3^{\mathbb{C}}$), can be set to any real value, without affecting the RN BH entropy (1.1).

It should be noticed that, consistent with the Gaillard-Zumino formulation of electric-magnetic duality in presence of scalar fields [21], the solution (1.10) to the attractor equations is the only one allowed in the presence of a *compact* underlying symmetry [in this case $U(1)$].

On the other hand, the best parametrization for the branching (1.9), pertaining to the KK extremal BH, is given by the KK radius

$$r_{\text{KK}} \equiv \mathcal{V}^{1/3} \equiv e^{2\varphi}, \tag{1.11}$$

by the 42 real scalars ψ_{ijkl} ($i = 1, \ldots 8$) sitting in the **42** of $USp(8)$, and by the 27 real *axions* a^I ($I = 1, \ldots, 27$) sitting in the **27** of $USp(8)$ [or equivalently, in the **27** of $E_{6(6)}$].

In virtue of the attractor mechanism, the KK radius is stabilized as follows [22]:

$$r^3_{\text{KK},H} \equiv \mathcal{V}_H \equiv e^{6\varphi_H} = 4\left|\frac{q}{p}\right|, \tag{1.12}$$

while all *axions* vanish:

$$a^I_H = 0. \tag{1.13}$$

The 42 real scalars ψ_{ijkl} are actually undetermined at the event horizon of the non-BPS KK BH, without affecting its entropy (1.2). Indeed, they span the moduli space $\frac{E_{6(6)}}{USp(8)}$ ($\dim_{\mathbb{R}} = 42$, rank $= 6$) of the non-BPS attractor solutions, which is the real symmetric scalar manifold of $\mathcal{N} = 8$, $d = 5$ supergravity [19].

It should be clear from our discussion that the possibility of having a nonvanishing scalar stabilized at the horizon of the KK extremal BH is related to the presence of a singlet in the relevant decomposition of the 70 scalars. This in turn is related to the existence of an underlying noncompact symmetry [$SO(1,1)$ in the present case], admitting no compact subsymmetry.

An alternative way to obtain Eqs. (1.1) and (1.2) is to use appropriate truncations for the bare charges in the corresponding expression of the quartic invariant I_4, which is known to be related to the Bekenstein-Hawking entropy by the formula

$$S = \sqrt{|I_4|}. \tag{1.14}$$

The manifestly $SU(8)$-invariant expression of I_4 reads as follows:

$$\mathcal{I}_4 = \mathrm{Tr}(ZZ^\dagger)^2 - \tfrac{1}{4}\mathrm{Tr}^2(ZZ^\dagger) + 8\,\mathrm{Re}Pf(Z), \qquad (1.15)$$

where $Z \equiv Z_{AB}(\phi)$ is the central charge 8×8 skew-symmetric matrix. Since (1.15) is moduli-independent, it can be evaluated at $\phi = 0$ without loss of generality, and in such a case Z_{AB} is replaced by Q_{AB}, the *bare* charge matrix in the $SU(8)$ basis.

Considering the RN black hole, we will see that a suitable truncation of the $\mathcal{N} = 8$ *bare* charge matrix Q_{AB} ($A, B = 1, \ldots 8$), reduces it to the form

$$Q_{AB}^{\mathrm{RN}} \to (z\epsilon_{ab}, 0), \qquad z \equiv e + im, \qquad (1.16)$$

where $a, b = 1, 2$ and $\epsilon^T = -\epsilon$. Thus one obtains

$$\mathcal{I}_4 = |z|^4 = (e^2 + m^2)^2, \qquad (1.17)$$

which is nothing but Eq. (1.3) and it is also the same result as in *pure* $\mathcal{N} = 2$, $d = 4$ supergravity, which has a $U(1)$ global $\mathcal{R}$ symmetry [11].

On the other hand, the manifestly $E_{6(6)}$-invariant expression of $\mathcal{I}_4$ in terms of the cubic invariant $\mathcal{I}_3$, as function of the bare electric and magnetic charges is given by [1,5,23]

$$\mathcal{I}_4 = -(p^0q_0 + p^iq_i)^2 + 4[q_0\mathcal{I}_3(p) - p^0\mathcal{I}_3(q) + \{\mathcal{I}_3(p), \mathcal{I}_3(q)\}]. \qquad (1.18)$$

By truncating the fluxes in such a way that

$$p^i = 0 = q_i, \qquad (1.19)$$

one obtains ($p^0 \equiv p$, $q_0 \equiv q$)

$$\mathcal{I}_4 = -(pq)^2, \qquad (1.20)$$

which now coincides with Eq. (1.4).

We will show that there is yet another way to obtain the two entropies for RN and KK black holes (1.1) and (1.2). This consists of using the attractor equations for the effective black hole potential $\frac{\partial V_{\mathrm{BH}}}{\partial \phi} = 0$ and the expression of the entropy as the value of such potential at the critical point (crit)

$$S = \pi V_{\mathrm{BH}}|_{\mathrm{crit}}. \qquad (1.21)$$

The plan of this paper is as follows.

In Sec. II we consider various bases of $\mathcal{N} = 8$, $d = 4$ supergravity, namely, the $SL(8,\mathbb{R})$-, $SU(8)$-, and $USp(8)$-covariant ones, exploiting the relevant branchings of the U-duality group $E_{7(7)}$. Then, Sec. III is devoted to the computation of the fundamental quantities for the geometry of the scalar manifold $\frac{E_{7(7)}}{SU(8)}$ in the $SL(8,\mathbb{R})$-covariant basis. Then, Sec. IV analyzes the $E_{6(6)}$-covariant basis, with the goal of exhibiting the connection with $\mathcal{N} = 8$, $d = 5$ supergravity: the $d = 4$ effective BH potential is recast in a manifestly $d = 5$ covariant form. Moreover, the charge configurations of this potential leading to vanishing axion fields are studied along with the corresponding attractor solutions. In Sec. V the embedding of the axion-dilaton extremal BH in $\mathcal{N} = 8$, $d = 4$ supergravity, through an intermediate embedding into $\mathcal{N} = 4$, $d = 4$ theory with 6 vector multiplets, is analyzed. Finally, Sec. VI contains an outlook, as well as some concluding comments and remarks. The paper also contains in an appendix the embedding of the $d = 5$ uplift of the *stu* model [the so-called $(SO(1,1))^2$ model] into $d = 5$ maximal supergravity.

II. SYMPLECTIC FRAMES

The de Wit-Nicolai [24] formulation of $\mathcal{N} = 8$, $d = 4$ supergravity is based on a symplectic frame where the maximal noncompact symmetry of the Lagrangian is $SL(8,\mathbb{R})$ [25], according to the decomposition

$$E_{7(7)} \to SL(8,\mathbb{R}), \qquad \mathbf{56} \to \mathbf{28} + \mathbf{28}', \qquad (2.1)$$

where $SL(8,\mathbb{R})$ is a maximal noncompact subgroup of $E_{7(7)}$, and $\mathbf{28}$ is its two-fold antisymmetric irreducible representation. Since the theory is pure, the $\mathcal{R}$ symmetry, namely $SU(8)$, is the stabilizer of the scalar manifold. It is not a symmetry of the Lagrangian, but only of the equations of motion. The maximal compact symmetry of the Lagrangian is the intersection of $SL(8,\mathbb{R})$ with $SU(8)$, which is $SO(8)$ [the maximal compact subgroup of $SL(8,\mathbb{R})$ itself].

Another symplectic frame corresponds to the decomposition (1.9). In this case, the maximal noncompact symmetry of the Lagrangian is $E_{6(6)} \times SO(1,1) \otimes_s T_{27}$, with "$\otimes_s$" denoting the semidirect group product and T_{27} standing for the 27-dimensional Abelian subgroup of $E_{7(7)}$. The maximal compact symmetry is now $USp(8)$, which is also the maximal compact symmetry of the Lagrangian. Note that all terms in the Lagrangian are $SU(8)$ invariant, with the exception of the vector kinetic terms, which are $SU(8)$ invariant only on shell.

Let us decompose $E_{7(7)}$ along two different maximal noncompact subgroups according to the following diagram:

$$\begin{array}{ccc} E_{7(7)} & \to & SL(8,\mathbb{R}) \\ \downarrow & & \downarrow \\ E_{6(6)} \times SO(1,1) & \to & SL(6,\mathbb{R}) \times SL(2,\mathbb{R}) \times SO(1,1). \end{array} \qquad (2.2)$$

If one goes first horizontally, the $\mathbf{56}$ of $E_{7(7)}$ decomposes as

$$\mathbf{56} \to \mathbf{28} + \mathbf{28}' \to \begin{cases} (\mathbf{15},\mathbf{1},1) + (\mathbf{6},\mathbf{2},-1) + (\mathbf{1},\mathbf{1},-3) + \\ +(\mathbf{15}',\mathbf{1},-1) + (\mathbf{6}',\mathbf{2},1) + (\mathbf{1},\mathbf{1},3). \end{cases} \qquad (2.3)$$

Alternatively, one can first go downward, and use that

$$\begin{aligned} &E_{6(6)} \to SL(6,\mathbb{R}) \times SL(2,\mathbb{R}); \\ &\mathbf{27} \to (\mathbf{15},\mathbf{1}) + (\mathbf{6}',\mathbf{2}), \\ &\mathbf{1} \to (\mathbf{1},\mathbf{1}), \end{aligned} \qquad (2.4)$$

thus obtaining

$$\begin{aligned} \mathbf{56} &\to (\mathbf{27}, 1) + (\mathbf{1}, 3) + (\mathbf{27}', -1) + (\mathbf{1}, -3) \\ &\to \begin{cases} (\mathbf{15}, \mathbf{1}, 1) + (\mathbf{6}', \mathbf{2}, 1) + (\mathbf{1}, \mathbf{1}, 3) + \\ +(\mathbf{15}', \mathbf{1}, -1) + (\mathbf{6}, \mathbf{2}, -1) + (\mathbf{1}, \mathbf{1}, -3). \end{cases} \end{aligned} \tag{2.5}$$

Therefore, either way on the diagram and irrespective of the intermediate decomposition, one obtains the same irreducible representations of $SL(6, \mathbb{R}) \times SL(2, \mathbb{R}) \times SO(1, 1)$, which enjoys a unique embedding in the U-duality group $E_{7(7)}$. In particular, one sees that the singlets are indeed the same in the two cases, and the alternative decompositions are related by the interchange of $(\mathbf{15}, \mathbf{1}, 1)$ with $(\mathbf{15}', \mathbf{1}, -1)$. Then one concludes that these two formulations, corresponding to two different symplectic frames, can be interchanged by dualizing 15 out of the 28 vector fields.

An analogous argument holds if one decomposes $E_{7(7)}$ according to two different maximal compact subgroups along the diagram

$$\begin{array}{ccc} E_{7(7)} & \to & SU(8) \\ \downarrow & & \downarrow \\ E_{6(2)} \times U(1) & \to & SU(6) \times SU(2) \times U(1). \end{array} \tag{2.6}$$

This time, going first horizontally along the diagram, the result reads

$$\begin{aligned} \mathbf{56} &\to \mathbf{28} + \overline{\mathbf{28}} \\ &\to \begin{cases} (\mathbf{15}, \mathbf{1}, 1) + (\mathbf{6}, \mathbf{2}, -1) + (\mathbf{1}, \mathbf{1}, -3) + \\ +(\overline{\mathbf{15}}, \mathbf{1}, -1) + (\bar{\mathbf{6}}, \mathbf{2}, 1) + (\mathbf{1}, \mathbf{1}, 3). \end{cases} \end{aligned} \tag{2.7}$$

Equivalently, one can first go vertically on the diagram and use

$$\begin{aligned} E_{6(2)} &\to SU(6) \times SU(2); \\ \mathbf{27} &\to (\mathbf{15}, \mathbf{1}) + (\bar{\mathbf{6}}, \mathbf{2}), \\ \mathbf{1} &\to (\mathbf{1}, \mathbf{1}), \end{aligned} \tag{2.8}$$

thus obtaining

$$\begin{aligned} \mathbf{56} &\to (\mathbf{27}, 1) + (\overline{\mathbf{27}}, -1) + (\mathbf{1}, 3) + (\mathbf{1}, -3) \\ &\to \begin{cases} (\mathbf{15}, \mathbf{1}, 1) + (\bar{\mathbf{6}}, \mathbf{2}, 1) + (\mathbf{1}, \mathbf{1}, 3) + \\ +(\overline{\mathbf{15}}, \mathbf{1}, -1) + (\mathbf{6}, \mathbf{2}, -1) + (\mathbf{1}, \mathbf{1}, -3). \end{cases} \end{aligned} \tag{2.9}$$

Again, either of the two alternative branchings in (2.6), which are related by the interchange of $(\mathbf{15}, \mathbf{1}, 1)$ with $(\overline{\mathbf{15}}, \mathbf{1}, -1)$, yield the same decomposition into irreducible representations of $SU(6) \times SU(2) \times U(1)$. Moreover, the $U(1)$ singlet which commutes with $SU(6) \times SU(2)$ is the same as the one which commutes with $E_{6(2)}$.

Let us now turn to the scalar sector. As mentioned above, the coordinate system for the scalar manifold $\frac{E_{7(7)}}{SU(8)}$ based on the Cartan decomposition are the real scalars ϕ_{ijkl} that sit in the **70** (four-fold antisymmetric and self-real irreducible representation) of $SU(8)$ with $i = 1, \ldots, 8$. The embedding of the RN extremal BH is related to the further decomposition

$$\begin{aligned} &SU(8) \to SU(6) \times SU(2) \times U(1), \\ &\mathbf{70} \to (\mathbf{20}, \mathbf{2}, 0) + (\mathbf{15}, \mathbf{1}, -2) + (\overline{\mathbf{15}}, \mathbf{1}, 2). \end{aligned} \tag{2.10}$$

On the other hand, for describing the KK extremal BH one decomposes $SU(8)$ under its maximal subgroup $USp(8)$:

$$SU(8) \to USp(8), \qquad \mathbf{70} \to \mathbf{42} + \mathbf{27} + \mathbf{1}, \tag{2.11}$$

where **42** and **27** are, respectively, the four-fold and two-fold antisymmetric irreducible representations (both skew-traceless and self-real) of $USp(8)$.

The crucial difference between (2.10) and (2.11) is that the latter decomposition contains a real singlet, whereas the first one does not. This is related to an underlying maximal compact $U(1)$ symmetry which is present for (2.10) and not for (2.11). This feature explains the different behavior of the two solutions at the attractor point: the RN solution has the behavior (1.10) while the KK solution is given by (1.12) and (1.13).

III. $SL(8, \mathbb{R})$ BASIS

In this section we aim at making contact between the symplectic formalism for extended supergravities reviewed in [26] and the original formulation of $\mathcal{N} = 8$ supergravity of [24] for some of the key geometrical objects that are relevant for the present investigation (see also [27] for recent developments).

We start by considering the coset representative for $E_{7(7)}/SU(8)$, which is parametrized as [24]

$$\mathcal{V} = \begin{pmatrix} u_{ij}^{IJ} & v_{ijKL} \\ v^{klIJ} & u^{kl}_{KL}. \end{pmatrix} \tag{3.1}$$

The submatrices u and v carry indices of both $E_{7(7)}$ and $SU(8)$ ($I = 1, \ldots, 8$, $I = 1, \ldots, 8$) but one can choose a suitable $SU(8)$ gauge for the fields, and then retain only manifest invariance with respect to the rigid diagonal subgroup of $E_{7(7)} \times SU(8)$, without distinction among the two types of indices. Comparing the notation of [24] (in particular Appendix B) with the symplectic formalism of [21,26], we can identify

$$\begin{array}{l} \phi_0 \equiv u \\ \phi_1 \equiv v \end{array} \to \begin{array}{l} u_{ij}{}^{kl} = (P^{-1/2})_{ij}{}^{kl}, \\ v^{ijkl} = -(\bar{P}^{-1/2})^{ij}{}_{mn}\bar{y}^{mnkl}, \end{array}$$

so that

$$\begin{aligned} \mathbf{f} &= \frac{1}{\sqrt{2}}(\phi_0 + \phi_1) = \frac{1}{\sqrt{2}}(u + v) \\ i\mathbf{h} &= \frac{1}{\sqrt{2}}(\phi_0 - \phi_1) = \frac{1}{\sqrt{2}}(u - v). \end{aligned} \tag{3.2}$$

Since sections are submatrices of the symplectic representation, relative to electric and magnetic subgroups, their explicit indices components are given by

$$f_{ij}{}^{kl} = \frac{1}{\sqrt{2}}((P^{-1/2})_{ij}{}^{kl} - (\bar{P}^{-1/2})^{ij}{}_{mn}\bar{y}^{mnkl}),$$
$$h_{ij,kl} = \frac{-i}{\sqrt{2}}((P^{-1/2})_{ij}{}^{kl} + (\bar{P}^{-1/2})^{ij}{}_{mn}\bar{y}^{mnkl}), \tag{3.3}$$

where, in matrix notation

$$P = 1 - YY^\dagger, \qquad Y = B\frac{\tanh\sqrt{B^\dagger B}}{\sqrt{B^\dagger B}},$$
$$B_{ij,kl} = -\frac{1}{2\sqrt{2}}\phi_{ijkl}, \tag{3.4}$$

the last definition coming from the choice of the symmetric gauge for the coset representative in Eq. (B.1) of [24]. If one defines

$$\tilde{P} = 1 - Y^\dagger Y, \tag{3.5}$$

and uses the identity

$$(\tilde{P}^{-1/2})Y^\dagger = Y^\dagger(P^{-1/2}), \tag{3.6}$$

the following simple expressions for **f** and **h** are finally achieved:

$$\mathbf{f} = \frac{1}{\sqrt{2}}[P^{-1/2} - (\tilde{P}^{-1/2})Y^\dagger] = \frac{1}{\sqrt{2}}[1 - Y^\dagger]\frac{1}{\sqrt{1 - YY^\dagger}}, \tag{3.7}$$

$$\mathbf{h} = -\frac{i}{\sqrt{2}}[P^{-1/2} + (\tilde{P}^{-1/2})Y^\dagger]$$
$$= -\frac{i}{\sqrt{2}}[1 + Y^\dagger]\frac{1}{\sqrt{1 - YY^\dagger}}. \tag{3.8}$$

The above notations are such that

$$P^{1/2} = \sqrt{1 - YY^\dagger} \to P_{ij}{}^{kl} = \delta_{ij}^{kl} - y_{ijmn}\bar{y}^{mnkl},$$
$$\bar{P}^{1/2} = \sqrt{1 - Y^\dagger Y} \to \bar{P}^{kl}{}_{ij} = \delta_{ij}^{kl} - \bar{y}^{klmn}y_{mnij}. \tag{3.9}$$

It is easily checked that the symplectic sections satisfy the usual relations

$$i(\mathbf{f}^\dagger\mathbf{h} - \mathbf{h}^\dagger\mathbf{f}) = 1, \qquad \mathbf{h}^T\mathbf{f} - \mathbf{f}^T\mathbf{h} = 0. \tag{3.10}$$

These are obtained writing the symplectic sections as in (3.7) and (3.8), and using the identity

$$Y\tilde{P}^{-1} = P^{-1}Y. \tag{3.11}$$

The kinetic matrix is given in terms of the symplectic sections by [26]

$$\mathcal{N} = \mathbf{h}\mathbf{f}^{-1}. \tag{3.12}$$

Therefore, Eqs. (3.7) and (3.8) yield

$$\mathcal{N} = i[1 + Y^\dagger]\frac{1}{\sqrt{1 - YY^\dagger}}\sqrt{1 - YY^\dagger}\frac{1}{1 - Y^\dagger}$$
$$= -i\frac{1 + Y^\dagger}{1 - Y^\dagger}$$
$$\Downarrow$$
$$\mathcal{N}_{ij|kl} = -i(\delta_{mn}^{kl} + \bar{y}^{mnkl})(\delta_{ij}^{mn} - \bar{y}^{ijmn})^{-1} \tag{3.13}$$

We now turn to the central charge function, which is defined by

$$Z_{ij} = f_{ij}{}^{kl}q_{kl} - h_{ij|kl}p^{kl}, \tag{3.14}$$

where electric and magnetic charges are in the same $SO(8)$ adjoint representation as vector fields. Using the definitions in (3.3), one obtains[1]

$$Z_{ij} = \frac{1}{\sqrt{2}}((P^{-1/2})_{ij}{}^{kl} - (\bar{P}^{-1/2})^{ij}{}_{mn}\bar{y}^{mnkl})q_{kl}$$
$$+ \frac{i}{\sqrt{2}}((P^{-1/2})_{ij}{}^{kl} + (\bar{P}^{-1/2})^{ij}{}_{mn}\bar{y}^{mnkl})p^{kl}$$
$$= (P^{-1/2})_{ij}{}^{kl}Q_{kl} - (\bar{P}^{-1/2})^{ij}{}_{mn}\bar{y}^{mnkl}\bar{Q}_{kl}$$
$$= \frac{1}{\sqrt{2}}\left[\left(\frac{1}{\sqrt{1 - Y\bar{Y}}}\right)_{ij}^{kl}Q_{kl} - \left(\frac{1}{\sqrt{1 - \bar{Y}Y}}\right)^{ij}_{mn}\bar{Y}^{mnkl}\bar{Q}_{kl}\right], \tag{3.15}$$

where the complex charges

$$Q_{ij} \equiv \frac{1}{\sqrt{2}}(q_{ij} + ip^{ij}) \tag{3.16}$$

have been introduced.

Then one can also give an expression for the BH potential, which is given by

$$V_{\mathrm{BH}} = \tfrac{1}{2}Z_{ij}\bar{Z}^{ij}$$
$$= \tfrac{1}{4}[(1 - Y\bar{Y})^{-1ijkl}Q_{kl}\bar{Q}_{ij}$$
$$+ -(\sqrt{1 - Y\bar{Y}})^{-1ab}_{ij}Q_{ab}(\sqrt{1 - \bar{Y}Y})^{-1ij}{}_{cd}Y_{cdkl}Q_{kl}$$
$$+ -(\sqrt{1 - \bar{Y}Y})^{-1ij}{}_{ab}\bar{Y}^{abkl}\bar{Q}_{kl}(\sqrt{1 - Y\bar{Y}})^{-1\,cd}_{ij}\bar{Q}_{cd}$$
$$+ (1 - \bar{Y}Y)^{-1}{}_{ijkl}\bar{Y}^{ijab}Y_{klmn}\bar{Q}_{ab}Q_{mn}]. \tag{3.17}$$

Thus, in the expansion around the zero field configuration, the BH receives contribution from the term

$$V_{\mathrm{BH}}(\phi = 0) = \frac{1}{4}Q_{ij}\bar{Q}^{ij}. \tag{3.18}$$

The linear term in the expansion of the BH potential near the point $\phi = 0$ receives contributions from the second and third row of Eq. (3.17), yielding the condition

[1]The expression with explicit indices is given by

$$\bar{P}^{ij}{}_{kl} = (\tilde{P})_{kl}{}^{ij}.$$

$$Q_{ij}\phi_{ijkl}Q_{kl} - \bar{Q}_{ij}\bar{\phi}^{ijkl}\bar{Q}_{kl} = 0, \tag{3.19}$$

$$\Downarrow$$

$$Q_{ij}Q_{kl}\delta^{mnpq}_{ijkl} - \tfrac{1}{4!}\bar{Q}_{ij}\bar{Q}_{kl}\epsilon^{ijklmnpq} = 0. \tag{3.20}$$

The configuration corresponding to charges Q_{AB} in the singlet of $SU(2)\times SU(6)$ trivially satisfies condition (3.20). Furthermore, it sets to zero the linear term for all values of ϕ, implying the $\phi = 0$ point to be an attractor point for this configuration.

IV. $E_{6(6)}$ BASIS AND RELATION TO $d = 5$

This section is aimed to establish the relation between the $\mathcal{N} = 8$, $d = 4$ theory and $\mathcal{N} = 8$, $d = 5$ supergravity ([28,29]), especially for what concerns the effective BH potential.

In our normalizations the kinetic Lagrangian for vector fields in the $\mathcal{N} = 2$ theory reads (with $\mathcal{F}_{\mu\nu} \equiv \frac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu) = \partial_{[\mu}A_{\nu]}$) [30,31]

$$\mathcal{L} = \ldots - \mathrm{Im}\mathcal{N}_{\Lambda\Sigma}\mathcal{F}^{\Lambda}\mathcal{F}^{\Sigma} - \mathrm{Re}\mathcal{N}_{\Lambda\Sigma}\mathcal{F}^{\Lambda *}\mathcal{F}^{\Sigma}, \tag{4.1}$$

where $\mathcal{N}_{\Lambda\Sigma}$ is the $d = 4$ vector kinetic matrix, with Λ, $\Sigma = 0, 1, \ldots, 27$. The effective BH potential is given by [16]

$$V_{\mathrm{BH}} = -\tfrac{1}{2}Q^T\mathcal{M}(\mathcal{N})Q, \tag{4.2}$$

where Q is the symplectic charge vector

$$Q = \begin{pmatrix} p^{\Lambda} \\ q_{\Lambda} \end{pmatrix},$$

and the matrix $\mathcal{M}$ reads [16]

$$\mathcal{M}(\mathcal{N}) = \begin{pmatrix} \mathrm{Im}\mathcal{N} + \mathrm{Re}\mathcal{N}(\mathrm{Im}N)^{-1}\mathrm{Re}\mathcal{N} & -\mathrm{Re}\mathcal{N}(\mathrm{Im}\mathcal{N})^{-1} \\ -(\mathrm{Im}\mathcal{N})^{-1}\mathrm{Re}\mathcal{N} & (\mathrm{Im}\mathcal{N})^{-1} \end{pmatrix}. \tag{4.3}$$

The $d = 5$ U-duality group $E_{6(6)}$ acts linearly on the 27 vectors $\hat{A}^I_{\hat{\mu}}$, with $\hat{\mu} = 1, \ldots, 5$ and $I = 1, \ldots, 27$. The $d = 5$ vector kinetic matrix $\hat{\mathcal{N}}_{IJ}$ is a function of the scalar fields spanning the $d = 5$ scalar manifold $\frac{E_{6(6)}}{USp(8)}$ ($\dim_{\mathbb{R}} = 42$, rank $= 6$).

According to the splitting $\Lambda = \{0, I\}$, the $d = 4$ kinetic vector matrix assumes the block form

$$\mathcal{N}_{\Lambda\Sigma} = \begin{pmatrix} \mathcal{N}_{00} & \mathcal{N}_{0J} \\ \mathcal{N}_{I0} & \mathcal{N}_{IJ} \end{pmatrix}. \tag{4.4}$$

By using to the formulas obtained in [32] which determine $\mathcal{N}_{\Lambda\Sigma}$ in terms of five-dimensional quantities, in a normalization[2] that is suitable for comparison to $\mathcal{N} = 2$, one obtains

$$\mathcal{N}_{\Lambda\Sigma} = \begin{pmatrix} \frac{1}{3}d_{IJK}a^Ia^Ja^K - i(e^{2\phi}a_{IJ}a^Ia^J + e^{6\phi}) & -\frac{1}{2}d_{IJK}a^Ia^K + ie^{2\phi}a_{KJ}a^K \\ -\frac{1}{2}d_{IKL}a^Ka^L + ie^{2\phi}a_{IK}a^K & d_{IJK}a^K - ie^{2\phi}a_{IJ} \end{pmatrix}. \tag{4.5}$$

Since the d_{IJK} tensor, the a^I fields, the $d = 5$ vector kinetic matrix a_{IJ}, and the field ϕ are real, the expressions for $\mathrm{Im}\mathcal{N}$ and $\mathrm{Re}\mathcal{N}$ are given by

$$\mathrm{Im}\,\mathcal{N}_{\Lambda\Sigma} = -e^{6\phi}\begin{pmatrix} 1 + e^{-4\phi}a_{IJ}a^Ia^J & -e^{-4\phi}a_{KJ}a^K \\ -e^{-4\phi}a_{IK}a^K & e^{-4\phi}a_{IJ} \end{pmatrix}, \tag{4.6}$$

$$\mathrm{Re}\,\mathcal{N}_{\Lambda\Sigma} = \begin{pmatrix} \frac{1}{3}d_{KLM}a^Ka^La^M & -\frac{1}{2}d_{JLM}a^La^M \\ -\frac{1}{2}d_{ILM}a^La^M & d_{IJK}a^K \end{pmatrix} = \begin{pmatrix} \frac{1}{3}d & -\frac{1}{2}d_J \\ -\frac{1}{2}d_I & d_{IJ} \end{pmatrix}, \tag{4.7}$$

where the following shorthand notation has been introduced:

$$d \equiv d_{IJK}a^Ia^Ja^K, \qquad d_I \equiv d_{IJK}a^Ja^K, \qquad d_{IJ} \equiv d_{IJK}a^K. \tag{4.8}$$

The inverse matrix $(\mathrm{Im}\mathcal{N}_{\Lambda\Sigma})^{-1} \equiv \mathrm{Im}\mathcal{N}^{\Lambda\Sigma}$ can be determined by noticing the block structure of (4.6). Then, by performing computations analogous to those of [22], one finds

$$(\mathrm{Im}\mathcal{N}^{-1})^{\Lambda\Sigma} = -e^{-6\phi}\begin{pmatrix} 1 & a^J \\ a^I & a^Ia^J + e^{4\phi}a^{IJ} \end{pmatrix}, \tag{4.9}$$

where $a^{IJ} \equiv (a_{IJ})^{-1}$. Inserting the above expressions into Eq. (4.2), the $\mathcal{N} = 8$, $d = 4$ effective BH potential can finally be rewritten in a $d = 5$ language:

[2] Compared to the notation of [32], here we use $\mathcal{N}_{\Lambda\Sigma} \to 4\mathcal{N}_{\Lambda\Sigma}$, $2\hat{\mathcal{N}}_{IJ} \to a_{IJ}$, $d_{IJK} \to -d_{IJK}/4$, and $a^I \to -a^I$.

$$V_{\rm BH}=(p^0)^2\bigg[\frac{1}{2}e^{2\phi}a_{IJ}a^Ia^J+\frac{1}{2}e^{6\phi}+\frac{1}{8}e^{-6\phi}\bigg(\frac{d^2}{9}+e^{4\phi}a^{IJ}d_Id_J\bigg)\bigg]+p^0p^I\bigg[-e^{2\phi}a_{IJ}a^J-\frac{1}{4}e^{-6\phi}\bigg(\frac{1}{3}dd_I+2e^{4\phi}a^{KJ}d_Kd_{JI}\bigg)\bigg]+p^Ip^J\bigg[\frac{1}{2}e^{2\phi}a_{IJ}+\frac{1}{8}e^{-6\phi}(d_Id_J+4e^{4\phi}a^{KL}d_{IK}d_{LJ})\bigg]+\frac{1}{6}q_0p^0e^{-6\phi}d+\frac{1}{6}q_Ip^0e^{-6\phi}[da^I+3e^{4\phi}a^{KI}d_K]-\frac{1}{2}q_0p^Ie^{-6\phi}d_I-\frac{1}{2}q_Ip^Je^{-6\phi}[d_Ja^I+2e^{4\phi}a^{KI}d_{JK}]+\frac{1}{2}(q^0)^2e^{-6\phi}+q_0q_Ie^{-6\phi}a^I+\frac{1}{2}q_Iq_Je^{-6\phi}[a^Ia^J+e^{4\phi}a^{IJ}]. \tag{4.10}$$

Notice that this formula becomes identical to the corresponding one of [22] concerning (purely cubic) $\mathcal{N}=2$ geometries [33,34], where $a_{IJ}=4e^{4\phi}g_{ij}$ and $\mathcal{V}\equiv e^{6\phi}$.

The potential (4.10), because of the definitions (4.8), can be seen to be a polynomial of a degree up to the sixth in the axion fields, whose general solutions are hard to determine. However, one can consider, in particular, attractor solutions with vanishing axion fields. These are given by specific charge configurations that solve the following attractor equations:

$$\frac{\partial V_{\rm BH}}{\partial a^I}\bigg|_{a^J=0}=-e^{2\phi}p^0p^Ka_{KI}-e^{-2\phi}q_Jp^Kd_{ILK}a^{JL}+q_0q_Ie^{-6\phi}=0. \tag{4.11}$$

Therefore, the BH charge configurations $Q=(p^0,p^I,q_0,q_I)$ supporting axion-free solutions fall into three classes:

$$\begin{aligned}&a)\ Q_e=(p^0,0,0,q_I)\ \text{Electric BH};\\&b)\ Q_m=(0,p^I,q_0,0)\ \text{Magnetic BH};\\&c)\ Q_0=(p^0,0,q_0,0)\ \text{KK charged BH}.\end{aligned} \tag{4.12}$$

In each of these classes, we now specify the BH potential by setting to zero the appropriate charge configuration in (4.10):

(a) Electric BH:

$$V_{\rm BH}(\phi,p^0,q_I)|_{a^I=0}=\tfrac{1}{2}e^{6\phi}(p^0)^2+\tfrac{1}{2}e^{-2\phi}a^{IJ}q_Iq_J. \tag{4.13}$$

(b) Magnetic BH:

$$V_{\rm BH}(\phi,q_0,p^J)|_{a^I=0}=\tfrac{1}{2}e^{-6\phi}(q_0)^2+\tfrac{1}{2}e^{2\phi}a_{IJ}p^Ip^J. \tag{4.14}$$

(c) BH charged with respect to the KK vector:

$$V_{\rm BH}(\phi,q_0,p^0)|_{a^I=0}=\tfrac{1}{2}e^{-6\phi}(q_0)^2+\tfrac{1}{2}e^{6\phi}(p^0)^2. \tag{4.15}$$

In order to recover the complete attractor solution, one also has to stabilize e^ϕ. For the KK charged BH one gets

$$\frac{\partial V^{\rm KK}_{\rm BH}(\phi,q_0,p^0)}{\partial\phi}\bigg|_{a^I=0}=0\Leftrightarrow e^{6\phi}=\bigg|\frac{q_0}{p^0}\bigg|, \tag{4.16}$$

thus yielding

$$V^{\rm KK}_{\rm BH}(q_0,p^0)|_{a^I=0}=|q_0p^0|. \tag{4.17}$$

In the electric case it holds that

$$\frac{\partial V^e_{\rm BH}}{\partial\phi}\bigg|_{a^I=0}=0\Leftrightarrow e^{2\phi}=\bigg(\frac{a^{IJ}q_Iq_J}{3(p^0)^2}\bigg)^{1/4}, \tag{4.18}$$

implying the critical value

$$V^e_{\rm BH}(q_I,p^0)|_{a^I=0}=2|p^0|^{1/2}\bigg(\frac{a^{IJ}q_Iq_J}{3}\bigg)^{3/4}. \tag{4.19}$$

Analogously, for the magnetic BH one finds

$$\frac{\partial V^m_{\rm BH}}{\partial\phi}\bigg|_{a^I=0}=0\Leftrightarrow e^{2\phi}=\bigg(\frac{a_{IJ}p^Ip^J}{3q_0^2}\bigg)^{-(1/4)}, \tag{4.20}$$

yielding

$$V^m_{\rm BH}(q_0,p^I)|_{a^I=0}=2|q_0|^{1/2}\bigg(\frac{a_{IJ}p^Ip^J}{3}\bigg)^{3/4}. \tag{4.21}$$

In virtue of the Bekenstein-Hawking entropy-area formula, the above expressions for the critical electric and magnetic BH potentials must be compared with appropriate powers of the $E_{6(6)}$ cubic invariants $\mathcal{I}_3(p)\equiv\frac{1}{3!}d_{IJK}p^Ip^Jp^K$ and $\mathcal{I}_3(q)\equiv\frac{1}{3!}d^{IJK}q_Iq_Jq_K$. Indeed, in $d=5$ it must hold that [10]

$$S\sim V^{3/4}|_{\rm crit}\sim|\mathcal{I}_3|^{1/2}. \tag{4.22}$$

Defining the electric and magnetic $d=5$ effective potentials, respectively, as

$$V^e_5=a^{IJ}q_Iq_J,\qquad V^m_5=a_{IJ}p^Ip^J, \tag{4.23}$$

one obtains

$$V^e_{\rm crit}=2|p^0|^{1/2}\bigg(\frac{V^e_5}{3}\bigg)^{3/4}\bigg|_{\rm crit}, \tag{4.24}$$

and

$$V^m_{\rm crit}=2|q^0|^{1/2}\bigg(\frac{V^m_5}{3}\bigg)^{3/4}\bigg|_{\rm crit}. \tag{4.25}$$

By comparison with $\mathcal{N}=2$ symmetric d geometries having

$$V_5^e|_{\rm crit} = |\mathcal{I}_3(q)|^{2/3} = |q_1 q_2 q_3|, \tag{4.26}$$

one obtains the expressions for the critical potential of the four dimensional electric and magnetic BHs:

$$V^e_{\rm BH\,crit}(q_I, p^0) = 2\sqrt{\frac{|p^0 d^{IJK} q_I q_J q_K|}{3!}}, \tag{4.27}$$

and

$$V^m_{\rm BH\,crit}(q_0, p^I) = 2\sqrt{\frac{|q_0 d_{IJK} p^I p^J p^K|}{3!}}. \tag{4.28}$$

More generally, these solutions can be compared with the embedding of the $\mathcal{N}=2$ purely cubic supergravities into $\mathcal{N}=8$ supergravity, and using the above critical values of the BH potential in (1.21), one finds, for the three family of configurations under examination, the correct result:

$$\frac{S_{\rm BH}}{\pi} = \sqrt{|\mathcal{I}_4|}. \tag{4.29}$$

It is interesting to remark that the KK black hole can be connected to the RN solution by performing an analytic continuation of the charges, as one can see from the redefinition

$$p^0 \to p + iq, \qquad q_0 \to p - iq,$$

which allows one to recover the RN entropy

$$S_{\rm RN} = \pi(p^2 + q^2). \tag{4.30}$$

We conclude this section by pointing out that the 70 scalars of $\mathcal{N}=8$, $d=4$ supergravity have been decomposed according to representations of $USp(8)$ [maximal compact subgroup of $E_{6(6)} \times SO(1,1)$] as follows:

$$\mathbf{70} \to \mathbf{42} + \mathbf{27} + \mathbf{1}. \tag{4.31}$$

The 42 unstabilized fields are the coordinates of the corresponding moduli space [19]. The noncompact form of the exceptional group, $E_{6(6)}$, in fact, enters in the expression of the coset

$$\frac{E_{6(6)}}{USp(8)}, \tag{4.32}$$

which is the moduli space of the $d=4$ non-BPS, $Z_{AB} \neq 0$ extremal BHs, whose orbit is precisely

$$\mathcal{O} = \frac{E_{7(7)}}{E_{6(6)}}. \tag{4.33}$$

Indeed, the KK BH is indeed a nonsupersymmetric solution (see also Sec. I).

V. EMBEDDING OF THE AXION-DILATON EXTREMAL BH

The embedding of the axion-dilaton BH in $\mathcal{N}=8$, $d=4$ supergravity can be performed by a three step supersymmetry reduction, which can be schematically indicated as

$$\begin{aligned} &\mathcal{N}=8 \to \mathcal{N}=4,\\ &n_V = 6 \to \text{pure } \mathcal{N}=4 \to \mathcal{N}=2\,\text{quadratic},\\ &n_V = 1, \end{aligned} \tag{5.1}$$

where n_V denotes the number of vector multiplets coupled to the supergravity multiplet. More precisely, the first step consists in truncating $\mathcal{N}=8$ supergravity to an $\mathcal{N}=4$ theory interacting with six matter (*vector*) multiplets. In the second step, $\mathcal{N}=4$ reduces to the pure theory, while in the last reduction one obtains $\mathcal{N}=2$ supergravity *quadratic* [35] theory with a single vector multiplet.

Let us examine more precisely each intermediate step.

(1) In the first step, the $\mathcal{N}=8$ central charge matrix Z_{AB} assumes the block form ($a, b = 1, \ldots, 4$, $i, j = 1, \ldots, 4$):

$$Z_{AB} \to \begin{pmatrix} Z_{ab} & 0 \\ 0 & i\bar{Z}_{ij} \end{pmatrix}, \tag{5.2}$$

where Z_{ab} is the $\mathcal{N}=4$ central charge matrix and Z_{ij} are the matter charges of the 6 vector multiplets [sitting in the two-fold antisymmetric of $SU(4)$, or equivalently in the vector representation of $SO(6) \sim SU(4)$].

Consequently, the $\mathcal{N}=8$ scalar manifold $\frac{E_{7(7)}}{SU(8)}$, reduces to

$$\frac{SL(2,\mathbb{R})}{U(1)} \times \frac{SO(6,6)}{SO(6)\times SO(6)} = \frac{SL(2,\mathbb{R})}{U(1)} \times \frac{SO(6,6)}{SU(4)\times SU(4)}, \tag{5.3}$$

which admits three orbits. This is the scalar manifold for $\mathcal{N}=4$ supergravity coupled to 6 vector multiplets.

(2) In the second step, the 6 vector multiplets are eliminated and $Z_{ij}=0$; this corresponds to retaining only states which are singlets with respect to the second $SU(4)$ in the stabilizer of the coset (5.3), and the theory becomes pure $=4$, with U duality $SL(2,\mathbb{R}) \times SU(4)$:

$$\begin{pmatrix} Z_{ab}\epsilon & 0 \\ 0 & i\bar{Z}_{ij\epsilon} \end{pmatrix} \to \begin{pmatrix} Z_{ab} & 0 \\ 0 & 0 \end{pmatrix}, \tag{5.4}$$

with

$$\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Accordingly, the scalar manifold reduces to $\frac{SL(2,\mathbb{R})}{U(1)}$.

Notice that, the presence of the axion-dilaton s spanning $\frac{SL(2,\mathbb{R})}{U(1)}$ in the $\mathcal{N}=4$ supergravity multiplet, only an $SU(4)$ out of the whole (local) $\mathcal{N}=4$ $\mathcal{R}$ symmetry $U(4)$ gets promoted to (global) U-duality symmetry.

(3) In the last step, 4 out of 6 graviphotons drop out, reducing the overall gauge symmetry from $U(1)^6$ to $U(1)^2$, with resulting U duality $SL(2,\mathbb{R})\times U(1)$. Thus, the framework becomes $\mathcal{N}=2$ supersymmetric, with the two skew-eigenvalues (Z_1, Z_2) of Z_{ab} related to the $\mathcal{N}=2$ central and matter charges $(Z, D_s Z)$:

$$Z_{ab}\rightarrow\begin{pmatrix} Z & 0\\ 0 & i\bar{D}_{\bar{s}}\bar{Z}\end{pmatrix}. \tag{5.5}$$

Therefore, at the $\mathcal{N}=2$ level one can have both BPS attractors ($D_s Z=0$) and the non-BPS ($Z=0$) ones [5].

On a group theoretical side, this step corresponds to performing the decomposition

$$\begin{aligned} &SU(4)\rightarrow SU(2)\times SU(2)\times U(1),\\ &\quad\mathbf{4}\rightarrow(\mathbf{2},\mathbf{1},\tfrac{1}{2})+(\mathbf{1},\mathbf{2},-\tfrac{1}{2}),\\ &\quad\mathbf{6}\rightarrow(\mathbf{2},\mathbf{2},0)+(\mathbf{1},\mathbf{1},1)+(\mathbf{1},\mathbf{1},-1),\end{aligned} \tag{5.6}$$

and to retaining only the singlets of $SU(2)\times SU(2)$.

The above three step reduction can be viewed from the point of view of the classification of *large* charge orbits [26,36]. One starts with the $\mathcal{N}=8$ scalar manifold $E_{7(7)}/SU(8)$ admitting the two regular orbits (1.5) and (1.6). The large charge orbits of $\mathcal{N}=4$, $d=4$ supergravity coupled to 6 vector multiplets are

$$\begin{aligned} &\mathcal{O}_{1/4\mathrm{BPS}}\colon SL(2,\mathbb{R})\times\frac{SO(6,6)}{SO(2)\times SO(6,4)};\\ &\mathcal{O}_{\text{non-BPS},Z_{ab}=0}\colon SL(2,\mathbb{R})\times\frac{SO(6,6)}{SO(2)\times SO(6,4)};\\ &\mathcal{O}_{\text{non-BPS},Z_{ab}\neq0}\colon SL(2,\mathbb{R})\times\frac{SO(6,6)}{SO(1,1)\times SO(5,5)},\end{aligned} \tag{5.7}$$

where the coincidence of the first two orbits is due to the symmetry between the gravity and the matter sector.

The corresponding moduli spaces for the $\mathcal{N}=4$, $n=6$ attractor solutions, exploiting the hidden symmetries of the above charge orbits, are given by

$$\begin{aligned} \mathcal{M}_{\mathrm{BPS}}&=\frac{SO(6,4)}{SU(4)\times SU(2)\times SU(2)};\\ \mathcal{M}_{\text{non-BPS},Z_{ab}=0}&=\frac{SO(6,4)}{SO(6)\times SO(4)};\\ \mathcal{M}_{\text{non-BPS},Z_{ab}\neq0}&=SO(1,1)\times\frac{SO(5,5)}{SO(5)\times SO(5)}\\ &=SO(1,1)\times\frac{SO(5,5)}{USp(4)\times USp(4)}.\end{aligned} \tag{5.8}$$

Notice that $\mathcal{M}_{1/4\mathrm{BPS}}$ (and $\mathcal{M}_{\text{non-BPS},Z_{ab}=0}$) are homogeneous symmetric quaternionic manifolds, as in the $\mathcal{N}=4\rightarrow\mathcal{N}=2$ reduction they become the hypermultiplets' scalar manifold [26].

The truncation of the $\mathcal{N}=8$ theory into $\mathcal{N}=4$ is based on the decomposition

$$E_{7(7)}\rightarrow SL(2,R)\times SO(6,6), \tag{5.9}$$

and on the following group embeddings:

$$SO(6,4)\times SO(2)\subsetneq E_{6(2)}; \tag{5.10}$$

$$SO(5,5)\times SO(1,1)\subsetneq E_{6(6)}. \tag{5.11}$$

Therefore, one can readily establish that the orbits 1/4 BPS and non-BPS, $Z_{ab}=0$ descend from the $\mathcal{N}=8$, BPS orbit $\frac{E_{7(7)}}{E_{6(2)}}$, whereas the orbit $\mathcal{O}_{\text{non-BPS},Z_{ab}\neq0}$ comes from the $\mathcal{N}=8$, non-BPS orbit $\frac{E_{7(7)}}{E_{6(6)}}$.

There is also another way to interpret the three step reduction (5.1), that is in terms of U-duality invariant representations. At the group level, the embedding of the axion-dilaton extremal BH into $\mathcal{N}=8$, $d=4$ supergravity is based on the decomposition of $E_{7(7)}\rightarrow SU(8)$ and

$$\begin{aligned} &SU(8)\rightarrow SU(4)\times SU(4)\times U(1),\\ &\quad\mathbf{8}\rightarrow(\mathbf{4},\mathbf{1},\tfrac{1}{2})+(\mathbf{1},\mathbf{4},-\tfrac{1}{2}),\\ &\quad\mathbf{28}\rightarrow(\mathbf{4},\mathbf{4},0)+(\mathbf{6},\mathbf{1},1)+(\mathbf{1},\mathbf{6},-1),\\ &\quad\overline{\mathbf{28}}\rightarrow(\bar{\mathbf{4}},\bar{\mathbf{4}},0)+(\mathbf{6},\mathbf{1},-1)+(\mathbf{1},\mathbf{6},1),\end{aligned} \tag{5.12}$$

where $SU(4)\times SU(4)\times U(1)$ is a maximal subgroup of $SU(8)$.

Then, the first truncation ($\mathcal{N}=8\rightarrow\mathcal{N}=4$, $n=6$) consists in setting

$$(\mathbf{4},\mathbf{4},0)=0=(\bar{\mathbf{4}},\bar{\mathbf{4}},0), \tag{5.13}$$

which gives rise to the decomposition (5.2).

We recall that the quartic invariant of the U-duality group $SL(2,\mathbb{R})\times SO(6,n)$ of $\mathcal{N}=4$, $d=4$ supergravity coupled to n vector multiplets is [8]

$$\mathcal{I}_4=\mathcal{S}_1^2-|\mathcal{S}_2|^2, \tag{5.14}$$

where the three $SO(6,n)$ invariants $\mathcal{S}_1$, $\mathcal{S}_2$, and $\bar{\mathcal{S}}_2$ are defined by ($a, b = 1, \ldots, 4$, $I = 1, \ldots, n$)

$$\mathcal{S}_1 \equiv \tfrac{1}{2} Z_{ab}\bar{Z}^{ab} - Z_I\bar{Z}^I; \tag{5.15}$$

$$\mathcal{S}_2 \equiv \tfrac{1}{4}\epsilon^{abcd} Z_{ab}Z_{cd} - \bar{Z}_I\bar{Z}^I. \tag{5.16}$$

The case $n = 6$ is remarkably symmetric, as the symmetry of the gravity and matter sector is the same and furthermore, due to the isomorphism $SU(4) \sim SO(6)$, the $SO(6)$ vector Z_I of matter charges can be equivalently represented as the $SU(4)$-antisymmetric tensor $i\bar{Z}_{ij}$ ($i, j = 1, \ldots, 4$). Consequently, for $n = 6$ we have

$$\mathcal{S}_{1,n=6} \equiv \tfrac{1}{2} Z_{ab}\bar{Z}^{ab} - \tfrac{1}{2}\bar{Z}_{ij}Z^{ij}; \tag{5.17}$$

$$\mathcal{S}_{2,n=6} \equiv \tfrac{1}{4}\epsilon^{abcd} Z_{ab}Z_{cd} - \tfrac{1}{4}\epsilon_{ijkl}Z^{ij}Z^{kl}. \tag{5.18}$$

Notice that $\mathcal{O}_{1/4\mathrm{BPS}}$ and $\mathcal{O}_{\mathrm{non\text{-}BPS},Z_{ab}=0}$ in Eq. (5.7) correspond to the two disconnected branches of the same manifold, classified by the sign of the real $SO(6,6)$ invariant [26] Indeed, $\mathcal{S}_{1,n=6} > 0$ for $\mathcal{O}_{1/4\mathrm{BPS}}$ and $\mathcal{S}_{1,n=6} < 0$ for $\mathcal{O}_{\mathrm{non\text{-}BPS},Z_{ab}=0}$.

By a suitable $U(1) \times SU(4) \times SU(4)$ transformation, one can reach the *normal frame* for both gravity sector and matter sector, such that the two matrices Z_{ab} and Z_{ij} are simultaneously skew-diagonalized, obtaining

$$Z_{ab} \to \begin{pmatrix} Z_1 & \\ & Z_2 \end{pmatrix} \otimes \epsilon; \tag{5.19}$$

$$Z_{ij} \to e^{i\theta}\begin{pmatrix} Z_3 & \\ & Z_4 \end{pmatrix} \otimes \epsilon, \tag{5.20}$$

where $Z_1, Z_2 \in \mathbb{R}^+$, and $Z_3Z_4 \in \mathbb{R}^+$, $\theta \in [0, 2\pi)$. Thus, in the *normal frame* one obtains

$$\mathcal{S}_{1,n=6} \equiv |Z_1|^2 + |Z_2|^2 - |Z_3|^2 - |Z_4|^2; \tag{5.21}$$

$$\mathcal{S}_{2,n=6} \equiv 2(Z_1Z_2 - \bar{Z}_3\bar{Z}_4); \tag{5.22}$$

$$\begin{aligned} I_{4,n=6} &= \mathcal{S}^2_{1,n=6} - |\mathcal{S}_{2,n=6}|^2 \\ &= \sum_{i=1}^4 |Z_i|^4 - 2\sum_{i<j=1}^4 |Z_i|^2|Z_j|^2 \\ &\quad + 4\Big(\prod_{i=1}^4 Z_i + \prod_{i=1}^4 \bar{Z}_i\Big). \end{aligned} \tag{5.23}$$

Equation (5.23) coincides with the expression of the quartic invariant of $\mathcal{N} = 8$, $d = 4$ supergravity, as given by [7] (see also [3]). Considering now the second step of the reduction, where one reaches the pure $\mathcal{N} = 4$ theory, one sets $Z_{ij} = 0$, or equivalently $Z_3 = 0 = Z_4$ in the normal frame [that is, retaining only states which are singlets with respect to the second $SU(4)$ in the stabilizer of the coset (5.3)]. Notice that, by doing so, $I_{4,n=0}$ becomes a perfect square:

$$\begin{aligned} I_{4,n=0} &= \mathcal{S}^2_{1,n=0} - |\mathcal{S}_{2,n=0}|^2 = (|Z_1|^2 - |Z_2|^2)^2 \\ &= (Z_1^2 - Z_2^2)^2. \end{aligned} \tag{5.24}$$

Equation (5.24) implies that $I_{4,n=0}$ is (weakly) positive, and as a consequence an unique class of large attractor exists, namely, the 1/4-BPS one. The (weak) positivity of $I_{4,n=0}$ is consistent with the known expression of $I_{4,n=0}$ in terms of the magnetic and electric charges (p^Λ, q_Λ) ($\Lambda = 1, \ldots, 6$):

$$I_{4,n=0} = 4[p^2q^2 - (p\cdot q)^2], \tag{5.25}$$

where here $p^2 \equiv p^\Lambda p^\Sigma \delta_{\Lambda\Sigma}$, $q^2 \equiv q_\Lambda q_\Sigma \delta^{\Lambda\Sigma}$, and $p \cdot q \equiv p^\Lambda q_\Sigma \delta^\Lambda_\Sigma$. Notice that in the basis of *bare* charges $I_{4,n=0}$, as given by Eq. (5.25), is (weakly) positive due to the *Schwarz inequality*, and not because it is a nontrivial perfect square of an expression of the bare magnetic and electric charges [37].

Notice that $\sqrt{I_{4,n=0}}$ [with $I_{4,n=0}$ given by Eq. (5.25)] must coincide with the value of the effective BH potential of the pure $\mathcal{N} = 4$ theory at its critical points. This can be understood (see the recent discussion given in [26,38]) because this potential reads as follows ($\Lambda = 1, \ldots, 6$):

$$\begin{aligned} &V_{\mathrm{BH,pure}\mathcal{N}=4}(\phi, a, p^\Lambda, q_\Lambda) \\ &= e^{2\phi}(sp_\Lambda - q_\Lambda)(\bar{s}p^\Lambda - q^\Lambda) \\ &= (e^{2\phi}a^2 + e^{-2\phi})p^2 + e^{2\phi}q^2 - 2ae^{2\phi}p\cdot q, \end{aligned} \tag{5.26}$$

where the complex (axion-dilaton) field

$$s \equiv a + ie^{-2\phi} \tag{5.27}$$

parametrizes the coset $\frac{SU(1,1)}{U(1)}$ of $\mathcal{N} = 4$, $d = 4$ pure supergravity [39].

By computing the criticality conditions of $V_{\mathrm{BH,pure}\mathcal{N}=4}$, one obtains the following stabilization equations for the axion a and the dilaton ϕ at criticality, $(\phi, a) = (\phi_H(p,q), a_H(p,q))$ [26]:

$$\left.\frac{\partial V_{\mathrm{BH}}(\phi, a, p, q)}{\partial a}\right|_{\mathrm{crit}} = 0 \Leftrightarrow a_H(p,q) = \frac{p\cdot q}{p^2}; \tag{5.28}$$

$$\left.\frac{\partial V_{\rm BH}(\phi, a, p, q)}{\partial a}\right|_{\rm crit} = -e^{-4\phi}p^2 + q^2 - a_H(p,q)p\cdot q = -e^{-4\phi}p^2 + q^2 - \frac{(p\cdot q)^2}{p^2} = 0;$$
$$\Updownarrow$$
$$e^{-2\phi_H(p,q)} = \frac{\sqrt{p^2q^2 - (p\cdot q)^2}}{p^2}. \quad (5.29)$$

Thus, the Bekenstein-Hawking BH entropy is computed to be

$$S_{\rm BH}(p,q) = \frac{A_H(p,q)}{4} = \pi V_{\rm BH}(\phi_H(p,q), a_H(p,q), p, q) = 2\pi\sqrt{p^2q^2 - (p\cdot q)^2} = \pi\sqrt{\mathcal{I}_{4,n=0}}. \quad (5.30)$$

The third and last step, when the pure $\mathcal{N} = 4$ theory reduces to the $\mathcal{N} = 2$ quadratic theory with $n_V = 1$, is performed through the truncation $(U(1))^6 \to (U(1))^2$ of the overall Abelian gauge invariance ($\Lambda = 1, \ldots, 6 \to \Lambda = 1,, 2$). In this case, $\mathcal{I}_{4,n=0,(U(1))^6\to(U(1))^2}$ is a perfect square in both the basis of Z_{ab} and in the basis of charges (p^Λ, q_Λ), and it actually is the square of the quadratic invariant $\mathcal{I}_{2(n=1)}$ of the axion-dilaton system:

$$\mathcal{I}_{4,n=0,(U(1))^6\to(U(1))^2} = (|Z_1|^2 - |Z_2|^2)^2 = 4(p^1q_2 - p^2q_1)^2 = \mathcal{I}^2_{2(n=1)}; \quad (5.31)$$

$$\Updownarrow$$

$$\mathcal{I}_{2(n=1)} = \pm 2|p^1q_2 - p^2q_1|, \quad (5.32)$$

implying that the axion-dilaton system exhibits two types of attractors: the $\frac{1}{2}$-BPS one ($\mathcal{I}_{2(n=1)} > 0$) and the non-BPS $Z = 0$ one ($\mathcal{I}_{2(n=1)} < 0$).

By further putting

$$p^1 = 0 = q_2, \qquad p^2 \equiv p, \qquad q_1 \equiv q, \quad (5.33)$$

($\Rightarrow p\cdot q = 0$), one obtains

$$\mathcal{I}^*_{4(n=0,U(1)^6\to U(1)^2)} = \mathcal{I}^{2*}_{2(n=1)} = 4(pq)^2; \quad (5.34)$$

$$\Updownarrow$$

$$\mathcal{I}^*_{2(n=1)} = \pm 2|pq|, \quad (5.35)$$

where $\mathcal{I}^*$ means the evaluation along Eq. (5.33). For a recent treatment of the axion-dilaton–Maxwell-Einstein–(super)gravity system and of the extremal BH attractors therein, see e.g. Secs. 6 and 7 of [38].

The similarity between the right-hand sides of Eqs. (1.4) and (5.35) is only apparent. In fact, the KK extremal BH has $\sqrt{-\mathcal{I}_{4,\rm KK}}$, which necessarily implies that it is non-BPS ($Z_{AB} \neq 0$ in $\mathcal{N} = 8$ and $Z \neq 0$ in $\mathcal{N} = 2$). On the other hand, the axion-dilaton extremal BH has $\mathcal{I}^*_{2(n=1)}$ and a "$\pm$" in the right-hand side, so that it can be both $\frac{1}{2}$-BPS and non-BPS $Z = 0$ in $\mathcal{N} = 2$. Moreover, the choice (5.33) leads to vanishing axion a [see Eq. (5.28)], and this explains that Eqs. (5.35) has $SO(1,1)$ symmetry, as Eq. (1.4).

A. Truncations of the scalar sector

As reported, e.g. in Secs. 6 and 7 of [38], one can see that the attractor mechanism stabilizes the complex axion-dilaton s at the event-horizon of the axion-dilaton extremal BH itself, while, as given by Eqs. (1.12) and (1.13) within the branching (2.11), only one real scalar degree of freedom, namely, the KK radius $r_{\rm KK}$ defined by Eq. (1.11), is stabilized at the event horizon of the extremal KK BH.

The relevant branching of the scalar sector for the embedding of the axion-dilaton extremal BH into $\mathcal{N} = 8$, $d = 4$ supergravity is given by

$$SU(8) \to SU(4)\times SU(4)\times U(1),$$
$$\mathbf{70} \to (\mathbf{1},\mathbf{1},2) + (\mathbf{1},\mathbf{1},-2) + (\mathbf{6},\mathbf{6},0) + (\bar{\mathbf{4}},\mathbf{4},1) + (\mathbf{4},\bar{\mathbf{4}},-1). \quad (5.36)$$

Equation (5.36) is the analogue of Eqs. (2.10) and (2.11), holding, respectively, for the ($\mathcal{N} = 8$, $d = 4$ embedding of the) RN and KK $d = 4$ extremal (and asymptotically flat) BHs.

A remarkable feature characterizing the branchings (2.10), (2.11), and (5.36), is the possible presence of a singlet in their right-hand sides. The decomposition (5.36) contains two $SU(4)(\times SU(4))$ singlets, whereas the decomposition (2.11) contains a real singlet, and the decomposition (2.10) does not contain any singlet. The presence of the singlet may lead to an underlying maximal compact symmetry [$U(1)$ for (2.10), absent for (2.11), and $SU(4)$ for (5.36)].

(1) The first truncation ($\mathcal{N} = 8 \to \mathcal{N} = 4$, $n_V = 6$) corresponds to setting[3]

$$(\bar{\mathbf{4}},\mathbf{4},1) = 0 = (\mathbf{4},\bar{\mathbf{4}},-1). \quad (5.37)$$

Indeed, by applying the condition (5.37), one obtains the correct quantum numbers of the scalar manifold $\frac{SL(2,\mathbb{R})}{U(1)}\times\frac{SO(6,6)}{SO(6)\times SO(6)}$ of the $\mathcal{N} = 4$, $d = 4$ supergravity coupled to 6 vector multiplets.

[3]Notice the difference with respect to the analogue truncation condition (5.13) for the decomposition of the **28** and $\overline{\mathbf{28}}$ of $SU(8)$.

(2) The second truncation ($\mathcal{N} = 4$, $n_V = 6 \rightarrow$ pure $\mathcal{N} = 4$) simply consists in implementing the condition

$$(\mathbf{6}, \mathbf{6}, 0) = 0, \tag{5.38}$$

which is consistently symmetric under the exchange of the gravity sector and the matter sector. Through condition (5.38), one achieves the correct quantum numbers of the scalar manifold $\frac{SL(2,\mathbb{R})}{U(1)}$ of the pure $\mathcal{N} = 4$, $d = 4$ supergravity.

(3) The third and last step (pure $\mathcal{N} = 4 \rightarrow \mathcal{N} = 2$ quadratic, $n_V = 1$) does not change anything with respect to the previous one. Indeed, the scalar sector is unaffected by this third truncation, and the scalar manifold remains $\frac{SL(2,\mathbb{R})}{U(1)}$.

VI. CONCLUSIONS

In the present investigation, we have considered some examples of extremal BH configurations in the framework of BH attractors of $\mathcal{N} = 8$ supergravity.

The effective BH potential has been computed in different bases, namely, in the manifestly $SU(8)$-coveriant basis, as well as in the $USp(8)$-covariant one. The former is suitable to describe the (BPS) Reissner-Nördstrom extremal BH with its $U(1)$ symmetry, as a consequence of the attractor point to be the origin of the $d = 4$ scalar manifold $\frac{E_{7(7)}}{SU(8)}$. The latter has $d = 5$ origin, and it is appropriate in order to describe the non-BPS Kaluza-Klein extremal BH, with its $SO(1, 1)$ symmetry arising from the nontrivial attractor value of the KK radial mode.

We have also considered the axion-dilaton system, whose BPS or non-BPS nature depends on whether it is embedded in the $\mathcal{N} = 2$ quadratic or in $\mathcal{N} = 4$, $d = 4$ supergravity. The axion-dilaton extremal BH is obtained as a particular case of the attractor equations of the maximal $d = 4$ theory. In that case, all 70 scalars other than the $SU(4) \times SU(4)$ singlets in the decomposition (5.36) are set to vanish, and correspondingly only 12 graviphoton electric and magnetic charges are taken to be nonzero [see Eq. (5.12)]. At the level $\mathcal{N} = 2$, this attractor solution is obtained by retaining only 4 (2 electric and 2 magnetic) nonvanishing charges, according to the decomposition (5.6) of $SU(4)$.

In the appendix, we have finally considered the embedding of the *stu* model in $\mathcal{N} = 8$, $d = 4$ and $d = 5$ supergravity. In the $d = 4$ case, all nonsinglet charges in the decomposition of $E_{7(7)}$ with respect to $SO(4,4) \times (SL(2,\mathbb{R}))^3$ are set to vanish [40], whereas for $d = 5$ one obtains an axion-free framework, given by nonzero values for (p^0),q_1, q_2, q_3 or (q_0), p^1, p^2, p^3.

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APPENDIX: TRUNCATION OF $\mathcal{N} = 8$, $d = 5$ SUPERGRAVITYTO THE $d = 5$ UPLIFT OF THE *stu* MODEL

The bosonic sector of the $\mathcal{N} = 8$, $d = 5$ supergravity theory consists in the metric $g_{\mu\nu}$ (μ, $\nu = 1, \ldots, 5$), 27 vectors A_μ^Λ, and 42 scalars ϕ_{abcd} parametrizing the coset $\frac{E_{6(6)}}{USp(8)}$. The index $\Lambda = 1, \ldots 27$ is in the **27** of $E_{6(6)}$, and it can be traded for a couple of flat antisymmetric indices (ab) of $USp(8)$. Thus, the vectors A_μ^{ab} transform in the **27** of $USp(8)$, that is

$$\mathbf{27} \quad \text{of } E_{6(6)} \rightarrow \mathbf{27} \quad \text{of } USp(8). \tag{A1}$$

The 42 scalars ϕ_{abcd} are in the traceless self-real 4-fold antisymmetric representation **42** of $USp(8)$.

Upon performing the $d = 5 \rightarrow d = 4$ reduction, one gets 70 scalars, which split into the following irreps of $USp(8)$:

$$\mathbf{70} = \mathbf{42} + \mathbf{27} + \mathbf{1}. \tag{A2}$$

Here **27** accounts for the *axions* coming from the A_5^{ab} vectors of $E_{6(6)}$, **1** is the KK radius r_{KK} [see the definition (1.11)], and **42** corresponds to the scalars in $\frac{E_{6(6)}}{USp(8)}$.

In order to extract the *stu* model, we notice that its $d = 5$ uplift is the $(SO(1,1))^2$ model with cubic hypersurface [33,34] (see e.g. the treatment given in [22])

$$\hat{\lambda}^1 \hat{\lambda}^2 \hat{\lambda}^3 = 1. \tag{A3}$$

The $\mathcal{N} = 8 \rightarrow \mathcal{N} = 2$, $d = 5$ supersymmetry reduction corresponds, at the level of $E_{6(6)}$, to taking the decomposition

$$E_{6(6)} \to SO(1,1) \times SO(5,5) \to (SO(1,1))^2 \times SO(4,4), \tag{A4}$$

so that [weights with respect to $SO(1,1)$'s are disregarded, irrelevant for our purposes]

$$\mathbf{27} \to \mathbf{1} + \mathbf{16} + \mathbf{10} \to \mathbf{1} + \mathbf{8}_s + \mathbf{8}_c + \mathbf{1} + \mathbf{1} + \mathbf{8}_v. \tag{A5}$$

Thus, three $SO(4,4)$ singlets are generated; they correspond to the three Abelian vector fields of the $d = 5$ uplift of the *stu* model. By further reducing to $d = 4$, one gets a further vector from the KK vector (*alias* the $d = 4$ graviphoton). This can be easily seen by completing the decomposition (A4) starting from the U-duality group $E_{7(7)}$ of $d = 4$ maximal supergravity:

$$E_{7(7)} \to SO(1,1) \times E_{6(6)} \to (SO(1,1))^2 \times SO(5,5) \to (SO(1,1))^3 \times SO(4,4), \tag{A6}$$

so that Eq. (A5) gets completed as [as above, neglecting weights with respect to $SO(1,1)$, as they are irrelevant for our purposes]

$$\mathbf{28} \to \mathbf{27} + \mathbf{1} \to \mathbf{1} + \mathbf{16} + \mathbf{10} + \mathbf{1} \to \mathbf{1} + \mathbf{8}_s + \mathbf{8}_c + \mathbf{1} + \mathbf{1} + \mathbf{8}_v + \mathbf{1}, \tag{A7}$$

containing four $SO(4,4)$ singlets in the last term.

It is worth pointing out that at $d = 4$ the $(SO(1,1))^3$ commuting with $SO(4,4)$ gets enhanced to $(SL(2,\mathbb{R}))^3$. By further decomposing

$$SO(4,4) \to (SL(2,\mathbb{R}))^4, \tag{A8}$$

this yields the $(SL(2,\mathbb{R}))^7$ used for the seven qubit entanglement in quantum information theory [41,42].

Notice that the presence of three different $\mathbf{8}$'s of $SO(4,4)$ in the right-hand side of the decomposition (A5) [as well as of (A7)] is the origin of the *triality* symmetry [43,44] of the *stu* model [40].

The $(SO(1,1))^2$ factor in the right-hand side of the branching (A4) is nothing but the scalar manifold of the $d = 5$ counterpart of the *stu* model [spanned by $\hat{\lambda}^1$, $\hat{\lambda}^2$, and $\hat{\lambda}^3$ satisfying the cubic constraint (A3)]. On the other hand, the $(SO(1,1))^3$ factor in the right-hand side of the branching (A7) is spanned by the (unconstrained, strictly positive) $d = 4$ *dilatons* $\lambda^1 \equiv -\mathrm{Im}(s)$, $\lambda^2 \equiv -\mathrm{Im}(t)$, and $\lambda^3 \equiv -\mathrm{Im}(u)$. They are related to their hatted counterparts by $\lambda^i \equiv r_{KK}\hat{\lambda}^i$, $i = 1, 2, 3$, implying [see Eqs. (A3) and (1.11); see also e.g. [22]]

$$\lambda^1\lambda^2\lambda^3 = r_{KK}^3 \equiv \mathcal{V}. \tag{A9}$$

The decomposition of the $d = 5$ stabilizer [analogue to the decomposition (A4) of the U-duality group of the $d = 5$ maximal supergravity] reads as follows:

$$USp(8) \to USp(4) \times USp(4) = \mathrm{Spin}(5) \times \mathrm{Spin}(5) \to \mathrm{Spin}(4) \times \mathrm{Spin}(4) = (SU(2))^2 \times (SU(2))^2, \tag{A10}$$

yielding the following decomposition of the fundamental $\mathbf{8}$ of $USp(8)$:

$$\mathbf{8} \to (\mathbf{4},\mathbf{1}) + (\mathbf{1},\mathbf{4}) \to (\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1}) + (\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1}) + (\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1}) + (\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2}). \tag{A11}$$

This allows one to compute the corresponding branchings of the $\mathbf{27} = (\mathbf{8} \times \mathbf{8})_{A,0}$ and $\mathbf{42} = (\mathbf{8} \times \mathbf{8} \times \mathbf{8} \times \mathbf{8})_{A,0}$ (the subscript "A, 0" standing for "antisymmetric traceless") of $USp(8)$ [the intermediate decompositions with respect to $USp(4) \times USp(4)$ are omitted, because irrelevant for our purposes]:

$$\mathbf{27} \to (\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{1}) + (\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{1}) + (\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{2}) + (\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{1}) + (\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{2}) + (\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{2}) + 3(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1}); \tag{A12}$$

$$\mathbf{42} \to (\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{2}) + (\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{1}) + (\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{1}) + (\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{2}) + (\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{1}) + (\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{2}) + (\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{2}) + 2(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1}). \tag{A13}$$

Consistently with previous statements, the three $(SU(2))^4$ singlets in the right-hand side of the decomposition (A12) and the two $(SU(2))^4$ singlets in the right-hand side of the decomposition (A13), respectively, are the three Abelian vector fields (including the $d = 5$ graviphoton) and the two independent real scalars (say, $\hat{\lambda}^1$ and $\hat{\lambda}^2$) in the bosonic spectrum of the $(SO(1,1))^2$ model, which is the $d = 5$ uplift of the *stu* model.

Reducing to $d = 4$, the six real scalar degrees of freedom of the *stu* model are the radius r_{KK} [see Eqs. (1.11) and (A9)], the two scalars $\hat{\lambda}^1$ and $\hat{\lambda}^2$, and the three *axions* [coming from the fifth component A_5^I ($I = 1, 2, 3$) of the three $d = 5$ vectors]. As previously mentioned, the four $d = 4$ vectors come from the three $d = 5$ vectors and from the KK vector $g_{5\mu}$ ($\mu = 1, \ldots, 4$).

Finally, it should be notice that $\lambda^1\lambda^2\lambda^3$ [defining the volume of the $d = 5$ cubic hypersurface through Eqs. (1.11) and (A9)] can be obtained through a consistent truncation of the $E_{6(6)}$-invariant expression (Λ, Σ, $\Delta = 1, \ldots, 27$)

$$\frac{1}{3!}d_{\Lambda\Sigma\Delta}\lambda^\Lambda\lambda^\Sigma\lambda^\Delta \tag{A14}$$

to $(SO(1,1))^2$, by retaining only the three singlets of $SO(4,4)$ [see the decompositions (A4) and (A5) above].

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