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SORTING OF MAGNETS IN LARGE SUPERCONDUCTING SYNCHROTRONS

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The presence of unavoidable multipole errors in superconducting dipole magnets is known to be the main cause for limiting the dynamic aperture of large accelerators. Sorting of dipoles, in which dipoles are installed in the ring according to a certain sequence based on the measured multipole errors, is a way to reduce the adverse effects of the random multipole errors without an increase in the cost. In this paper, a multi-parameter sorting scheme, which can systemically handle several multipole components simultaneously, has been tried on a test lattice. It has been found that the sorting scheme is effective in enlarging the dynamic aperture and improving the linearity of the phase-space region occupied by the beam. This is true even in cases for which more than one multipoles are responsible for the aperture limitation.

1 INTRODUCTION

In large hadron accelerators, high beam energy requires very high magnetic field which can only be produced by superconducting magnets. One disadvantage of using superconducting magnets is that large multipole field errors are unavoidable. The single-particle dynamics is usually dominated by nonlinear effects arising from these multipole errors. Understanding and controlling the adverse effect of the multipole errors has been one of the primary tasks in design and construction of large hadron accelerators.

In the absence of longitudinal component, the horizontal and vertical components of magnetic field in a dipole magnet can be expressed in the multipole expansion,[1]

$$B_y + iB_x = B_0 \sum_{n=0}^{\infty} (b_n + ia_n)(x + iy)^n \tag{1}$$

where x and y are the horizontal and vertical coordinates measured from the magnet center, respectively, and B_0 is the design bend field. Field errors are specified by the normal and skew multipole coefficients b_n and a_n, respectively, and n is the order of the multipole. For a pure dipole magnet, a_n and b_n vanish except for $b_0 = 1$. In general, each a_n or b_n has two components: a systematic error which is the average over all the magnets and a random error which is the fluctuation from magnet to magnet. The systematic errors may arise, for example, from the magnet design and the persistent current magnetization in superconductors while the random errors are due to imperfections in the fabrication of magnets. The dynamics of charged particles in the transverse phase space is usually determined by these various field errors, low-order multipole errors dominating in the phase-space region near the origin while high-order multipoles contributing to the dynamic aperture limitation. The systematic errors can be compensated to a large extent by means of correction magnets.[2] It is, however, very difficult and costly to correct the random errors on individual dipoles. Sorting of dipoles, in which the magnets are installed around a ring according to measured errors so that the errors on different magnets cancel each other to some extent, has been the easiest way in many cases to reduce the detrimental effects of the random errors without introducing complications.

Sorting of dipoles was tried for the Tevatron at Fermilab with an intention to reduce a few harmonic components of sextupoles and octupoles that can drive resonances.[3] Subsequently, Gluckstern and Ohnuma proposed a simple scheme that can reduce a large number of harmonic components centered around the most troublesome ones when sextupole field is the dominant factor in the aperture reduction.[4] While significant improvements in aperture by means of the sorting of dipoles were reported,[3–12] the sorting was not effective when more than one multipole components contribute to the aperture limitation.[7,8] This is also the case when sorting of dipoles of two-in-one configuration such as LHC (Large Hadron Collider) dipoles is attempted.[7] Several modified versions of the sorting scheme have been developed with the intention of handling two multipole components.[7–9] Since all these two-parameter sorting schemes do not treat two multipoles equally, they are effective only when one multipole is dominant. Moreover, none of the existing schemes can be applied effectively when more than two multipole components are responsible for the aperture limitation.

In order to effectively suppress the adverse effects of random field errors with more than one important multipole components, a multi-parameter sorting scheme, called vector sorting, has been developed in which all multipole components involved are treated equally.[13] In this sorting scheme, the field error of each magnet is represented by a multi-dimensional vector and the sorting is based on the Euclidean distance between the vectors of different magnets. The effectiveness of the vector sorting scheme has been studied with a test lattice which is similar to the high energy booster (HEB) of Superconducting Super Collider (SSC). Tracking studies of the dynamic aperture have shown that this sorting scheme is effective in the improvement of the dynamic aperture even when large high-order multipoles are included in the field errors. Since one of the important effects of nonlinear field errors is the variation of betatron tunes with betatron-oscillation amplitudes, the effect of the sorting on this has also been investigated with the normal-form technique. The result has shown that the sorting is an effective means to reduce the amplitude dependence of tunes arising from the random field errors.

This paper is organized as follows. In Section 2, the vector sorting scheme is discussed. In Section 3, we present the test lattice. In Sections 4 and 5, the effectiveness of the vector sorting scheme on the improvement of the dynamic aperture and on the reduction of the amplitude dependence of tunes are studied, respectively. Section 6 contains further discussions on sorting.

2 VECTOR SORTING SCHEME

Since major systematic field errors can be compensated by correction magnets in most cases, only random contributions of the field error are considered here, i.e. the averages of a_n and b_n over a large number of magnets are assumed to be zero. Furthermore, random multipole components of different orders are assumed to be statistically uncorrelated.

The general principle of sorting is based on the observation that, if two magnets of approximately equal strength but opposite sign of the error are selected and placed one period of betatron oscillation apart (2π phase distance), the errors of approximately equal strength make approximately equal but opposite contribution to the particle trajectory so that the adverse effects of the errors can be substantially reduced. To understand this cancellation, let us consider two sources of the nth-order multipole separated by horizontal phase ϕ_x and vertical phase ϕ_y. For convenience, we use normalized phase-space variables $(\vec{\xi}, \vec{\eta}) = (\xi_x, \xi_y, \eta_x, \eta_y)$, where η_x and η_y are the conjugate momenta of ξ_x and ξ_y, respectively. The normalized phase-space variables are defined by $\xi_x = \beta_x^{-\frac{1}{2}} x$, $\eta_x = \alpha_x \beta_x^{-\frac{1}{2}} x + \beta_x^{\frac{1}{2}} p_x$, and similarly for ξ_y and η_y, where α and β are Courant-Snyder parameters for linear betatron oscillations.[14]

In the thin-lens approximation, the kick on the momenta due to the multipoles can be written as

$$\Delta \vec{\eta} = \epsilon_i \vec{g}(\vec{\xi}) \tag{2}$$

where ϵ_i is the integrated strength of the nth-order multipole at location $i = 1$ or 2, and $\vec{g}$ is a vectorial polynomial in $\vec{\xi}$ of degree n. Between these two kicks, the transfer map for horizontal motion is assumed to be

$$\begin{aligned} \xi'_x &= \xi_x \cos\phi_x + \eta_x \sin\phi_x + U_1 \\ \eta'_x &= -\xi_x \sin\phi_x + \eta_x \cos\phi_x + U_2 \end{aligned} \tag{3}$$

where U_1 and U_2 are nonlinear functions of phase-space variables due to field errors between two multipoles. The transfer map for vertical motion has a similar expression. Assume that a particle has the phase-space coordinate $(\vec{\xi}_0, \vec{\eta}_0)$ just before the first kick. After the second kick,

$$\begin{aligned} \xi_x &= \xi_{x0} \cos\phi_x + \eta_{x0} \sin\phi_x + \epsilon_1 g_x(\vec{\xi}_0) \sin\phi_x + U_1 + O(\vec{\xi}_0, \vec{\eta}_0, \vec{g}) \\ \eta_x &= -\xi_{x0} \sin\phi_x + \eta_{x0} \cos\phi_x + \epsilon_1 g_x(\vec{\xi}_0) \cos\phi_x + \epsilon_2 g_x(\vec{\xi}') \\ &\quad + U_2 + O(\vec{\xi}_0, \vec{\eta}_0, \vec{g}) \end{aligned} \tag{4}$$

where $O(\vec{\xi}_0, \vec{\eta}_0, \vec{g})$ represents a collection of higher-order terms of $\vec{\xi}_0$, $\vec{\eta}_0$, and $\vec{g}$, which are usually less important, and $\vec{\xi}' = (\xi_{x0} \cos\phi_x + \eta_{x0} \sin\phi_x, \xi_{y0} \cos\phi_y + \eta_{y0} \sin\phi_y)$. There is a similar expression for ξ_y and η_y. If $\phi_x \simeq \phi_y \simeq 2\pi$ and $\epsilon_1 \simeq -\epsilon_2$, the lowest-order contributions from two kicks cancel each other. In practice, however, because of an imperfect match of two magnets or phase separation different from 2π, the remainder of the lowest-order effect can still influence the beam dynamics.

If one particular multipole component is dominant, it is relatively easy to accomplish such cancellation by selecting magnets based on the value of the dominant multipole component only.[3,4] When there are n equally important multipole components, however, the sorting of magnets must be done in such a way that effects of these n multipole components are taken into account simultaneously. For this purpose, a n-dimensional vector is used to represent these n multipole components.[13] The selection of magnets for sorting is then based on the Euclidean distance between the vectors. Since the effects of different order of multipoles with different dimensionality cannot be directly compared by their multipole coefficients, the nonlinear field arising from each multipole at a certain amplitude in phase space is used for the sorting. The field associated with the nth-order normal and skew multipole at a phase-space location $x = y = x_0$ is characterized by $b_n x_0^n$ and $a_n x_0^n$ where x_0 will be used as a sorting parameter to optimize the sorting. If the primary goal of sorting is to increase the dynamic aperture, the effectiveness of sorting should be peaked at x_0 near the dynamic aperture. x_0 can therefore be chosen initially in such a way that it corresponds to the dynamic aperture of the lattice without sorting. The sorting can then be optimized by modifying x_0 to be the dynamic aperture of the sorted lattice. In fact, the sorting result is not too sensitive to the choice of x_0 as long as x_0 is close to the dynamic aperture.

Let $\vec{S}^{(i)} = (S_1^{(i)}, \ldots, S_n^{(i)}, S_{n+1}^{(i)}, \ldots, S_{2n}^{(i)})$ denote a $2n$-dimensional sorting vector where n is the highest order of multipoles considered and the superscript i specifies each magnet. Components of $\vec{S}^{(i)}$ are such that if the kth-order normal or skew multipole, $b_k^{(i)}$ or $a_k^{(i)}$, is included as a sorting parameter, $S_k^{(i)} = b_k^{(i)} x_0^k$ or $S_{n+k}^{(i)} = a_k^{(i)} x_0^k$, otherwise, $S_k^{(i)} = 0$ and $S_{n+k}^{(i)} = 0$. The strength of error field of magnet i is defined by the Euclidean normal of $\vec{S}^{(i)}$,

$$|\vec{S}^{(i)}| = \sqrt{\sum_{k=1}^{2n} \left(S_k^{(i)}\right)^2}. \tag{5}$$

Consider that there are N measured magnets available for installation at N consecutive locations with even N. These N magnets cover an even number of betatron oscillation periods. In the vector sorting scheme, the placement of magnets is done in the following manner.

A. Cancellation of the Error Fields of Each Pair of Magnets with 2π Phase Separation

The N magnets are first grouped into $N/2$ pairs, $\left(\vec{S}^{(2i-1)}, \vec{S}^{(2i)}\right)$ for $i = 1, \ldots, N/2$, such that

$$|\vec{S}^{(2i-1)}| \geq |\vec{S}^{(2i+1)}| \tag{6}$$

and

$$\vec{S}^{(2i)} = \min_{l=2i}^{N} \left(|\vec{S}^{(2i-1)} + \vec{S}^{(l)}| \right). \tag{7}$$

For example, the magnet of $\vec{S}^{(1)}$ is selected with the largest strength of error field in the group and then its pair magnet of $\vec{S}^{(2)}$ is selected by using Equation (7) with $i = 1$. This is the first pair of magnets. The next pair is selected by repeating this process to the remaining magnets. The process will be repeated until all magnets are paired. It is clear that

$$\lim_{N\to\infty} |\vec{S}^{(2i-1)} + \vec{S}^{(2i)}| = 0. \tag{8}$$

That is, magnets in each pair have almost the same strength but opposite sign of errors. They are therefore placed one period of betatron oscillation apart. Due to a finite number of magnets in a sorting group, $|\vec{S}^{(2i-1)} + \vec{S}^{(2i)}|$ is in general not exactly zero and only partial cancellation of the errors can be achieved. As the number of magnets in each sorting group is increased, the remainder of the cancellation is reduced and the sorting becomes more effective. On the other hand, for a given number of magnets, $|\vec{S}^{(2i-1)} + \vec{S}^{(2i)}|$ increases with the number of sorting parameters. Consequently, the effectiveness of sorting will diminish as more multipoles are included in the sorting.

B. Reduction of the Error Field Locally

Consider the thin-lens kick on the normalized momenta due to a nth-order multipole. The highest-order term of β_x and the highest-order term of β_y in the kick is obtained from Equation 1 as

$$\begin{cases} \Delta\eta_x = -\epsilon b_n \left[\beta_x^{\frac{n+1}{2}} \xi_x^n + (-1)^{\frac{n}{2}} \beta_x^{\frac{1}{2}} \beta_y^{\frac{n}{2}} \xi_y^n \right] + \ldots & \text{if } n \text{ is even} \\ \Delta\eta_y = \epsilon a_n \left[\beta_x^{\frac{n}{2}} \beta_y^{\frac{1}{2}} \xi_x^n + (-1)^{\frac{n}{2}} \beta_y^{\frac{n+1}{2}} \xi_y^n \right] + \ldots & \text{if } n \text{ is even} \end{cases} \tag{9}$$

$$\begin{cases} \Delta\eta_x = -\epsilon \left[b_n \beta_x^{\frac{n+1}{2}} \xi_x^n + (-1)^{\frac{n+1}{2}} a_n \beta_x^{\frac{1}{2}} \beta_y^{\frac{n}{2}} \xi_y^n \right] + \ldots & \text{if } n \text{ is odd} \\ \Delta\eta_y = \epsilon \left[a_n \beta_x^{\frac{n}{2}} \beta_y^{\frac{1}{2}} \xi_x^n - (-1)^{\frac{n+1}{2}} b_n \beta_y^{\frac{n+1}{2}} \xi_y^n \right] + \ldots & \text{if } n \text{ is odd} \end{cases} \tag{10}$$

where ϵ is a coefficient for the integrated strength of the kick. It can be seen from Equation (9) that for an even-order multipole, the normal (skew) component is more important at focusing location in the horizontal (vertical) direction where β_x (β_y) is much larger than β_y (β_x). For an odd-order multipole, Equation (10) shows that the effects of the normal and the skew components are equal at focusing and defocusing locations. To minimize the effect of error field of each magnet, magnets with larger strength of even-order normal (skew) multipole components should therefore be placed near quadrupoles defocusing in the horizontal (vertical) direction. When multipoles up to the nth order are included in the sorting, a magnet has larger even-order normal multipole components in its error field if

$$\sqrt{\sum_{k=1}^{l}\left(S_{2k}^{(i)}\right)^2} > \sqrt{\sum_{k=1}^{l}\left(S_{n+2k}^{(i)}\right)^2}, \tag{11}$$

otherwise, the magnet is considered to have larger even-order skew multipole components in its error field. Here $l = n/2$ for an even n and $l = (n-1)/2$ for an odd n.

C. Partial Cancellation of the Error Field Locally

When the phase distance between two adjacent magnets is not too large, there will be a partial cancellation of the error field locally if the sign of the errors in magnets alternates. Therefore, the adjacent magnet to a magnet represented by $\vec{S}^{(2i-1)}$ is chosen to be

$$\vec{S}^{(j)} = \min_{l \neq 2i}\left(|\vec{S}^{(2i-1)} + \vec{S}^{(l)}|\right). \tag{12}$$

For example, if the magnet of $\vec{S}^{(1)}$ is installed at a certain location according to the rule b and the magnet of $\vec{S}^{(2)}$ is installed at the location with 2π phase separation from that of $\vec{S}^{(1)}$, the next magnet to that of $\vec{S}^{(1)}$ will be selected by using Equation (12) with $i = 1$. The magnet of $\vec{S}^{(j)}$ will be placed next to that of $\vec{S}^{(1)}$ and if j is an odd (even) number the magnet of $\vec{S}^{(j+1)}$ ($\vec{S}^{(j-1)}$) will be placed next to that of $\vec{S}^{(2)}$.

To install all magnets in a ring, one repeats this procedure in each segment covering an even number of betatron oscillation periods. When one arc does not contain an even number of betatron oscillation periods, there will be leftover cells with less than two periods of betatron oscillation. For these unbalanced cells, one may set aside a number of "good" magnets from each sorting group provided that a few extra magnets are available in each group for this purpose.

It should be noted that if a single multipole component alone is included into the sorting vector $\vec{S}^{(i)}$, the vector sorting scheme will be reduced to the original Gluckstern-Ohnuma one-parameter sorting scheme.[4] When multipoles of different orders are included, the minimization of the Euclidean distance of the vectors adopted by the vector sorting scheme effectively excludes unintended cancellation of the error fields between different orders of multipoles. Any sorting scheme relied on such cancellation (e.g., cancelling sextupole field with decapole field) is harmful as the effect of sorting will then strongly depend on phase-space locations.

3 TEST LATTICE

The test lattice used in this work is similar to the high energy booster of SSC with two-fold symmetry.[1] The total number of regular cells is 116 and each cell can contain four, six, or eight dipoles. Most of the studies reported in this paper have been done with four dipoles in each regular cell which is 65 m in length. In addition, we have included two long-straight and four short-straight sections in order to make the model more realistic. The length of the long-straight section and the short-straight section is 648.35 m and 490.82 m, respectively. All insertions are assumed to be perfectly matched to the arc sections. The phase advance of a regular cell is $\pi/2$ in both transverse directions. The horizontal and vertical tunes are 39.42489 and 38.41437, respectively. The magnetic field errors in each dipole are represented by a set of thin-lens multipoles, located in the middle and at both ends. All systematic errors as well as the dipole component of random field error are taken to be zero. The other random multipole components are chosen with Gaussian distributions centered at zero and truncated at $\pm 3\sigma$ where σ is the rms value of each multipole coefficient. Table I lists values of σ used in this study, which are the specifications of SSC dipoles.[15] Random multipole components of different orders are assumed to be statistically

TABLE I The rms values of the multipole coefficients of random errors in SSC dipoles. The unit is 10^{-4} cm^{-n}.

n	1	2	3	4	5	6	7
σ_{b_n}	0.50	1.15	0.16	0.22	0.017	0.018	0.01
σ_{a_n}	1.25	0.35	0.32	0.05	0.050	0.008	0.01

uncorrelated. When the two-in-one configuration of dipoles is considered, random multipole components in two rings are also assumed to be statistically uncorrelated.

4 EFFECT OF THE SORTING ON THE DYNAMIC APERTURE

In order to reduce the sensitivity of dynamic aperture to the choice of initial launch point for tracking in phase space, we define an aperture as the shortest distance from the origin in the four-dimensional normalized phase space during the tracking. In order to find the dynamic aperture, the launch point is moved away from the origin until the particle is lost. No physical aperture limit is imposed in the ring. A particle is defined to be lost if $x^2 + y^2 \geq (10 \text{ cm})^2$ where x and y are its horizontal and vertical coordinates, respectively. The dynamic aperture defined in this manner is found to be relatively insensitive to the choice of launch point in phase space. Tracking of particle motions has been done without synchrotron oscillations, momentum deviations, or closed orbit distortions. To improve the statistical significance of the simulations, we have used a number of different samples of random multiple components, usually fifty, generated with different seed numbers in a random number generator routine. In order to facilitate the study of many cases and to increase efficiency, we developed our own tracking codes. Two codes were made independently in order to test the reliability of each.

4.1 Sextupole Components (b_2, a_2) Only

It is already known that sorting sextupole components will improve the linearity of a ring.[8] With σ_{b_2} and σ_{a_2} given in Table I, the dynamic aperture has been calculated for fifty random samples in 1000-turn tracking. Since the phase advance of a regular cell of the test lattice is $\pi/2$, the sorting has been performed for dipoles in eight cells at a time which is the minimum number of dipoles required in our sorting scheme. For the case of four dipoles in each cell, the sorting group contains 32 dipoles. Since the sorting is performed on the multipoles of the same order, the components of the sorting vector are simply taken to be the sextupole coefficients. For unsorted arrangement of dipoles, the dynamic aperture averaged over fifty samples is 3.7×10^{-3} m$^{1/2}$ With b_2 alone is used for sorting, the average dynamic aperture increases to 9.3×10^{-3} m$^{1/2}$, which is an 150% increase. With both b_2 and a_2 sorted, the average dynamic aperture is 9.8×10^{-3} m$^{1/2}$, an 165% increase over the unsorted arrangement. It is understandable that there is no substantial improvement in the dynamic aperture between two-parameter sorting and one-parameter sorting since b_2 is dominant in this case. To examine the effectiveness of two-parameter sorting with the vector sorting scheme, we also studied systems with different ratios of σ_{a_2} and σ_{b_2}. Improvements of the aperture due to the sorting are listed in Table II which shows that the effectiveness of two-parameter sorting is independent of the ratio of σ_{a_2} and σ_{b_2} while the effectiveness of the one-parameter sorting diminishes considerably as $\sigma_{a_2}/\sigma_{b_2}$ approaches 1.

In order to examine the effectiveness of the sorting in the two-in-one configuration of dipoles, two sets of fifty samples of random multipole components are generated independently with each pair of samples representing a magnet for two rings. σ_{b_2} and σ_{a_2} of both rings are taken from Table I. Without the sorting, both rings have the same average dynamic aperture of 3.7×10^{-3} m$^{1/2}$. When b_2 and a_2 in both rings are sorted simultaneously, the average dynamic aperture of each ring increases to 8.1×10^{-3} m$^{1/2}$, which is an 120% increase. As a comparison, the previous sorting scheme for the two-in-one configuration of LHC dipoles improved the dynamic aperture by 40% when only b_2 was considered.[7]

4.2 Effect of Linear Coupling

Linear coupling has been found to be an important factor to the reduction of dynamic aperture in some cases such as in LHC. Although the linear motion can be decoupled globally or even locally in most cases, a good dipole sorting scheme should be able to cope with the linear coupling. In order to examine the effectiveness of the vector sorting scheme in coupled systems, we have studied systems with a_1, b_2, and a_2 included, where σ_{a_1} is taken from Table I. Since linear coupling is not corrected, the lattice is fully coupled.

TABLE II Average dynamic aperture for different values of $\sigma_{a_2}/\sigma_{b_2}$ with $\sigma_{a_2}+\sigma_{b_2} = 2.0\,\mathrm{m}^{-2}$ AD_0, AD_1, and AD_2 are the average dynamic aperture of the unsorted arrangement, sorting b_2 only, and sorting both b_2 and a_2, respectively. The unit of the aperture is 10^{-3} $\mathrm{m}^{1/2}$

σ_{a_2}	σ_{b_2}	AD_0	AD_1	AD_2	AD_1/AD_0	AD_2/AD_0
0.5	1.5	2.8	6.9	7.7	2.5	2.8
0.7	1.3	2.9	6.0	7.8	2.1	2.7
1.0	1.0	2.9	4.6	7.7	1.6	2.7

TABLE III Average dynamic aperture for different values of $\sigma_{a_2}/\sigma_{b_2}$ with $\sigma_{a_1} = 0.0125\ \mathrm{m}^{-1}$ and $\sigma_{a_2}+\sigma_{b_2} = 2.0\,\mathrm{m}^{-2}$. AD_0, AD_2, and AD_3 are the average dynamic aperture of the unsorted arrangement, two-parameter sorting of b_2 and a_2, and three-parameter sorting of a_1, b_2, and a_2, respectively. The unit of the aperture is 10^{-3} $\mathrm{m}^{1/2}$.

σ_{a_1}	σ_{a_2}	σ_{b_2}	AD_0	AD_2	AD_3	AD_2/AD_0	AD_3/AD_0
0.0125	0.5	1.5	2.5	6.6	6.9	2.6	2.8
0.0125	0.7	1.3	2.6	6.6	6.7	2.5	2.6
0.0125	1.0	1.0	2.8	6.5	6.8	2.3	2.4

Table III lists the average dynamic aperture with or without sorting. It shows that even in a coupled system, the sorting is still very effective in the enlargement of dynamic aperture. When all a_1, b_2, and a_2 are included in the sorting, the sorting vector is $\vec{S} = (0, b_2x_0, a_1, a_2x_0)$ where x_0 is chosen initially in such a way that it approximately corresponds to the dynamic aperture. The sorting is then optimized by varying x_0. The calculation of the dynamic aperture of the system sorted with different x_0 has shown that the effectiveness of sorting is not sensitive to the choice of x_0 as long as x_0 is near the dynamic aperture.

4.3 Effect of High-order Multipoles

To understand the effect of high-order multipoles, we consider all the high-order multipoles up to the 7th order in the error field. The rms values of multipole coefficients are given in Table I. The linear lattice is assumed to be decoupled so that the skew quadrupole component a_1 is zero.

When all the multipoles up to the 7th order are included, the average dynamic aperture of the unsorted ring is 1.8×10^{-3} $\mathrm{m}^{1/2}$. With sorting the sextupole components only on each group of 32 dipoles, it increases

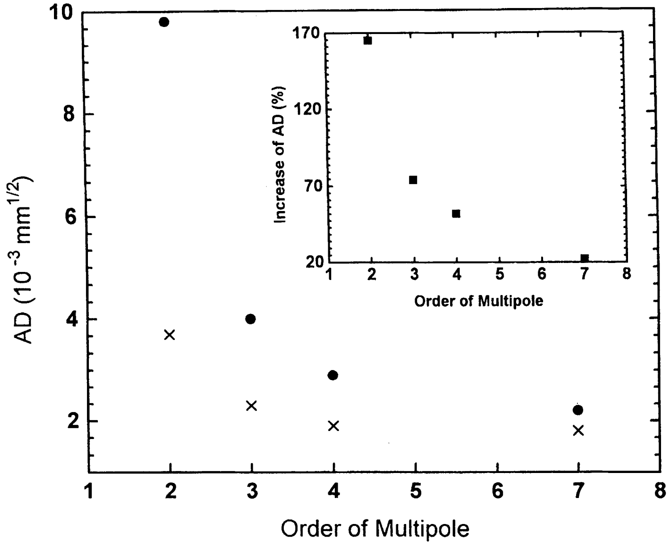


FIGURE 1 The average dynamic aperture of fifty random samples (AD) vs. the highest order of multipoles included in the tracking. The cross is the aperture of the unsorted arrangement and the dot the aperture after a multi-parameter sorting. The sorting is performed on all the multipole components included with 32 dipoles in each sorting group. The insert shows the increase of the aperture after the sorting.

to 1.9×10^{-3} m$^{1/2}$, a mere 6% improvement. It is easy to understand that sorting sextupoles alone is no longer effective. At the phase-space region near the dynamic aperture, the strength of nonlinearities arising from various multipole components is of the same order. Because of this, a multi-parameter sorting which includes all the multipoles must be used. To examine the effectiveness of multi-parameter sorting, we have studied cases in which multipoles up to the 3rd, 4th, or 7th order are included in the tracking. When multipoles up to the nth order are included, a $2n$-dimensional sorting vector is chosen to be $\vec{S} = (0, b_2, b_3x_0, ..., b_nx_0^{n-2}, 0, a_2, a_3x_0, ..., a_nx_0^{n-2})$. Figure 1 shows the average dynamic aperture before and after the multi-parameter sorting as a function of n when the sorting is performed with 32 dipoles in each sorting group. It shows that, with the field error given in Table I, the dynamic aperture of the ring is determined by many multipole components and the sorting becomes less effective as more high-order multipoles are taken into account. When all the multipoles up to the 7th order are included, the 12-parameter sorting increases the average dynamic aperture by 22%. Although this is a sizeable gain in the aperture, it is much less than that of the case with sextupoles only. This reduction of the sorting efficiency can be understood as a result of the decrease in the cancellation when more multipole components are included in sorting while the number of dipoles in each sorting group is kept unchanged. Increasing the number of dipoles in each sorting group should improve the effectiveness of the sorting. A 12-parameter sorting with 232 dipoles in each sorting group has been tested when all the multipoles up to the 7th order are included. These 232 dipoles are in 58 cells which cover half of the test lattice. Table IV lists the dynamic aperture of the unsorted arrangement, 12-parameter sorting with 32 dipoles, and 12-parameter sorting with 232 dipoles. After the sorting with 232 dipoles, the average dynamic aperture is increased to 2.9×10^{-3} $m^{1/2}$ which is a 60% improvement over the unsorted arrangement.

It should be noted that one significant result from the multi-parameter sorting is the elimination of particularly bad arrangements ("unlucky cases"). As shown in Table IV, the smallest dynamic aperture is either 33% or 87% larger than that of the worst unsorted case for the 12-parameter sorting with 32 dipoles or with 232 dipoles when all the multipoles up to the 7th order are included. Figure 2 is a plot of the distribution of the dynamic aperture of fifty random samples with and without the sorting, showing that the spread of the dynamic aperture is significantly reduced after the multi-parameter sorting.

4.4 Number of Turns Used in Tracking

Most of the tracking in this work have been limited to one thousand to five thousand turns. This is justified since our goal is to study the effectiveness of sorting scheme and not to evaluate the aperture for long-term beam survival. For a few cases, the tracking has been extended to one million turns in order to see any change in the dynamic aperture. When all multipoles except sextupoles are taken to be zero in the tracking, there is a reduction of as much as $\sim$ 30%. When high-order multipoles are included in the tracking, the change in the dynamic aperture is much less. As the number of turns in the tracking is increased from 10^3 to 10^5, as shown in Table IV, the average dynamic aperture is reduced by 11%, 10%, and 7%, respectively, for the unsorted arrangement, the 12-parameter sorting with 32 dipoles, and the 12-parameter sorting with 232 dipoles. The effect of the sorting is in fact more pronounced when the number of turns for the tracking is increased.

TABLE IV Dynamic aperture when all the multipole components up to the 7th order are included. AD and SD represent the average dynamic aperture and the smallest dynamic aperture of the fifty samples, respectively. The unit of the aperture is 10^{-3} $m^{1/2}$

	Turns of Tracking	*Unsorted*	*2-Parameter Sorting with 32 Dipoles*	*12-Parameter Sorting with 32 Dipoles*	*12-Parameter Sorting with 232 Dipoles*
AD	10^3	1.8	1.9	2.2	2.9
AD	10^5	1.6		1.9	2.7
SD	10^3	1.5	1.7	2.0	2.8
SD	10^5	1.1		1.6	2.5

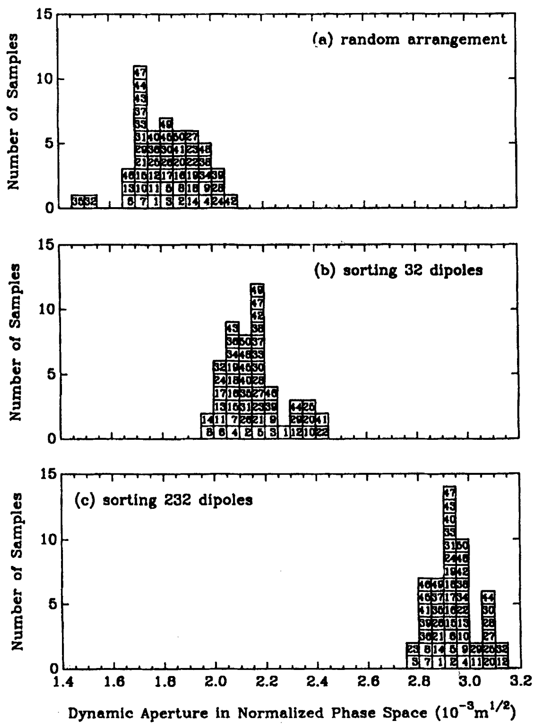


FIGURE 2 Dynamic aperture of fifty random samples when all the multipoles up to the 7th order are included. The number in each block identifies each sample. (a) dipoles are placed randomly; (b) all the multipole components are sorted with 32 dipoles in each sorting group; and (c) all the multipole components are sorted with 232 dipoles in each sorting group.

5 EFFECT OF SORTING ON THE AMPLITUDE DEPENDENCE OF BETATRON TUNES

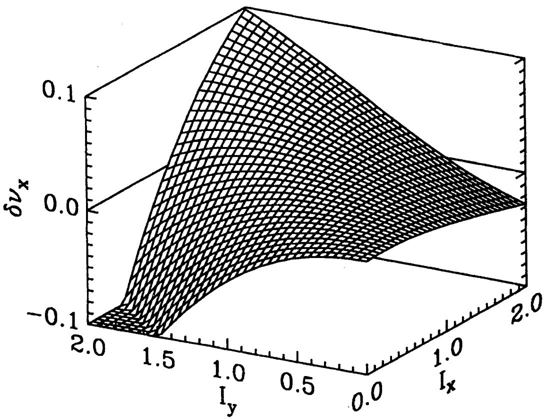
A strong nonlinearity in the lattice can lead to a substantial degree of amplitude dependence of betatron tunes even in a phase-space region near the origin, and this may result in crossings of dangerous resonances and a reduction in the dynamic aperture. Minimizing the amplitude dependence of tunes is thus desirable for a stable operation of the accelerator. For the systematic errors of dipoles, correction schemes for minimizing the amplitude dependence of tunes have been carefully studied and the effect of the correction has been demonstrated with tracking study.[16,17] We have investigated the effect of sorting on the reduction of the amplitude dependence of tunes due to the random error fields. Two methods, direct calculation of the average phase advance during the tracking and normal-form analysis of the one-turn map, have been employed for this purpose. The normal-form analysis, which has been implemented in our code in an efficient way based on the Lie algebra formalism, provides a general and easy procedure to compute the amplitude dependence. The direct calculation of the average phase advance during tracking has been used to check the result of the normal-form analysis. Good agreement between these two methods has been found in our calculations.

In normal-form space, the betatron motion is approximated by an amplitude-dependent rotation. The Hamiltonian for the rotation can be expanded in a power series of the action variables $\vec{I} = (I_x, I_y)$,

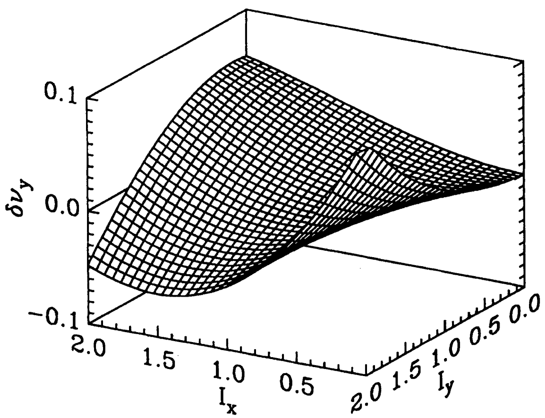
$$H = \vec{\nu}_0 \cdot \vec{I} + \sum_{k=2}^{m} \sum_{i=0}^{k} c_{k-i,i} I_x^{k-i} I_y^i + O(m+1) \tag{13}$$

where $\vec{\nu_0} = (\nu_{x0}, \nu_{y0})$ is the design betatron tune and $O(m+1)$ represents a remainder consisting of terms higher than the mth order. $\vec{I}$ can be expressed in term of the normalized variables as $I_x = \frac{1}{2}(\xi_x^2 + \eta_x^2)$ and $I_y = \frac{1}{2}(\xi_y^2 + \eta_y^2)$. By using the normal-form technique, c_{ij}s can be calculated from the one-turn map of the lattice. The shift in tune, which leads to the amplitude dependence of the tune, is calculated from

$$\delta\vec{\nu} = \frac{\partial H}{\partial \vec{I}} - \vec{\nu}_0. \tag{14}$$



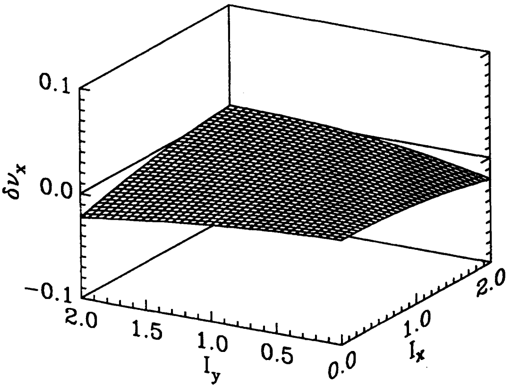
3a



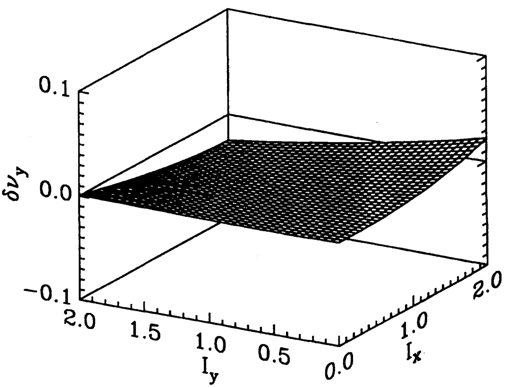
3b

FIGURE 3 (a) $\delta\nu_x$ and (b) $\delta\nu_y$, averaged over fifty random unsorted lattices, as functions of I_x and I_y. All the multipoles up to the 7th order are included. The unit of I_x and I_y is 10^{-3} mm.

Since the normal-form series in Equation (13) is an asymptotic expansion, the convergence of the series should be examined by comparing the results of different orders. In our case, the 4th-order calculation [$m = 4$ in Equation (13)] has been found to be satisfactory. In Figures 3 and 4, $\delta\nu_x$ and $\delta\nu_y$, averaged over fifty random samples, are plotted as a function of I_x and I_y when all the multipoles up to the 7th order are included. Without



4a



4b

FIGURE 4 (a) $\delta\nu_x$ and (b) $\delta\nu_y$, averaged over fifty random samples, as functions of I_x and I_y. Conditions here are the same as those in Figure 3 except that all multipole components are sorted with 232 dipoles in each sorting group. The unit of I_x and I_y is 10^{-3} mm.

sorting, both horizontal and vertical tunes strongly depend on I_x and I_y as shown in Figure 3. Since $\delta\vec{\nu}$ is calculated at one end of a long-straight section of the lattice where β_x is much larger than β_y ($\beta_x = 108.7$ m and $\beta_y = 19.1$ m), this amplitude dependence is stronger in the horizontal motion than that in the vertical motion. Figure 4 shows the tune shift of the lattice after the 12-parameter sorting of all the multipole components, where the sorting is performed with 232 dipoles in each sorting group. A comparison between the sorted and unsorted case shows that the sorting effectively suppresses the nonlinear tune shift. The linearity of the phase space is significantly improved after the sorting.

Since $\delta\vec{\nu}$ is a two-dimensional vector field in amplitude space, it is convenient to use its magnitude averaged over a phase-space region to characterize the degree of amplitude dependence in that region. The average magnitude of the tune shift can be defined as

$$\left\langle\left[(\delta\nu_x)^2+(\delta\nu_y)^2\right]^{\frac{1}{2}}\right\rangle=\frac{2}{I^2}\int_{I_x+I_y\leq I}\left[(\delta\nu_x)^2+(\delta\nu_y)^2\right]^{\frac{1}{2}}dI_x dI_y \qquad (15)$$

where the average is taken over a phase-space region bounded by a four-dimensional sphere $I_x+I_y=I$. The dependence of $\left\langle\left[(\delta\nu_x)^2+(\delta\nu_y)^2\right]^{\frac{1}{2}}\right\rangle$ on I provides an overall estimate of the amplitude dependence of the tune in a phase-space region of interest. In Figure 5, $\left\langle\left[(\delta\nu_x)^2+(\delta\nu_y)^2\right]^{\frac{1}{2}}\right\rangle$, averaged over fifty random samples, is plotted as a function of I when all the multipoles up to the 7th order are included. A comparison between the sorted and unsorted case shows that the sorting can effectively reduce the nonlinear tune shift due to the random field errors in phase space region occupied by the beam.

6 DISCUSSIONS

The multi-parameter sorting of superconducting dipoles with the vector sorting scheme has been found to be an effective means to increase the dynamic aperture and to improve the linearity of the phase-space region occupied by the beam. This is true even when more than one multipole components are responsible for the aperture limitation. When many multipoles are involved, an effective multi-parameter sorting needs to be done with a large number of dipoles in each sorting group. The number of dipoles in each sorting group is, however, limited by magnet installation schedules and magnet storage capacity. Since the installation of LHC dipoles is planned to cause as little interruption of LEP (Large Electron-Position Collider) operation as possible, it is expected that a large number of LHC dipoles will be available for sorting before installation. The multi-parameter

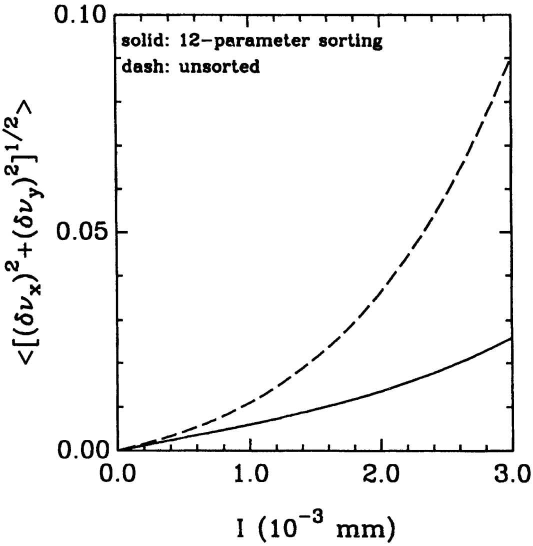


FIGURE 5 $\left\langle\left[(\delta\nu_x)^2 + (\delta\nu_y)^2\right]^{\frac{1}{2}}\right\rangle$, averaged over fifty random samples, as a function of amplitude when all the multipoles up to the 7th order are included. The dashed line is for the unsorted arrangement and the solid line is for the 12-parameter sorting of all the multipoles.

sorting with a large number of dipoles in each sorting group is thus possible for the LHC. Moreover, as the vector sorting scheme handles multiple error fields simultaneously, it is especially suitable to the dipoles of two-in-one configuration such as LHC dipoles.

Magnet sorting requires a reliable measurement, preferably cold measurement, of multipole components of all the magnets. If this is not possible for reasons of cost and schedule, one will be forced to come up with magnets satisfying the pre-determined tolerance requirements which, however, may not be easy to realize. The merit of sorting lies in the fact that it can coexist with any other correcting measures without introducing any harmful side effects and cost increase. It is, however, important to stress that magnet sorting should never be considered as "cure-all" in dealing with the nonlinear problems in superconducting rings. It is also important to realize that there is no unique way of sorting magnets. Different circumstances and different parameters certainly require different strategy optimum to the particular conditions. The vector sorting scheme, however, provides a foundation to the magnet sorting.

It should be noted that the magnet sorting is based on an assumption that random multipole components of different orders are statistically uncorrelated. The indication of such a correlation has, however, been found in a study of field measurements performed on superconducting dipoles.[18] Since no regularities of correlations can be found in existing data, it is impossible to predict these correlations for superconducting magnets of future particle accelerators. We have therefore decided not to include any correlations in the sorting scheme. As these correlations are not expected to be serious, only minor adjustments may need to be made in the sorting strategy if such correlations do appear in the future superconducting magnets.

Finally, since the combined effect of errors in arc quadrupoles is in general much less than that of errors in arc dipoles, field errors in all quadrupoles have been neglected. Problems associated with quadrupoles in the insertions are special when β-function is very large. Any compensation strategy there, including sorting of a few insertion quadruples, must be done independently from the sorting of dipoles in the arcs.

Acknowledgements

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References

[1] SSC Central Design Group (1986). *Superconducting Super Collider Conceptual Design*, SSC-SR-2020.
[2] Neuffer, D. (1990). *Particle Accelerators*, **27**, 209.
[3] Michelotti, L. and Ohnuma, S. (1983). *IEEE Trans. Nucl. Sci.*, **30**, 2472.
[4] Gluckstern, R.L. and Ohnuma, S. (1985). *IEEE Trans. Nucl. Sci.*, **NS-32**, 2314.
[5] Ohnuma, S. (1987). *IEEE Trans. Nucl. Sci.*, **NS-34**, 1176.
[6] Ohnuma, S. (1988). In *Proceedings of the Second ICFA Advanced Beam Dynamics Workshop* edited by J. Hagel and E. Keil, p. 218.
[7] Scandal, W. and Wang, L. (1989). *CERN Note 95*; *CERN Note 107*.
[8] Li, M. and Ohnuma, S., In *Proceedings of the 1989 Particle Accelerator Conference*, Chicago, March 1989, p. 897; in *Proceedings of the 2nd European Particle Accelerator Conference*, (1990), p. 1175.
[9] Schachinger, L. (1986). *SSC report SSC-N-123*.
[10] Willeke, F. (1987). *DESY report HERA 87-12*.
[11] Giovannozzi, M., Grassi, R., Scandale, W. and Todesco, E. (1996). *Phys. Rev. E*, **52**, 3093.
[12] Dinev, D. (1996). *Proceedings of the 5th European Particle Accelerator Conf.*

[13] Shi, J., Ohnuma, S. and Wang, G. (1994). In *Proceedings of the 4th European Particle Accelerator Conf.* (World Scientific, Singapore), p. 1051.
[14] Courant, E.D. and Snyder, H.S. (1958). *Ann. Phys.*, **3**, 1.
[15] Teng, L.C. (1991). *AIP Conference Proceedings No. 255*, (AIP, New York), p. 389.
[16] Scandale, W., Schmidt, F. and Todesco, T. (1991). *Particle Accelerators*, **35**, 53.
[17] Giovannozzi, M., Scandale, W. and Schmidt, F. (1993). *CERN SL/93-29 (AP)*, LHC Note 230.
[18] Galluccio, F. (1992). *CERN SL 92-04 (AP).*