Particle Accelerators, 1997, Vol. 56, pp. 161–180
Reprints available directly from the publisher
Photocopying permitted by license only

NONLINEAR PHASE SPACE STUDY IN A LOW-EMITTANCE LIGHT SOURCE USING HARMONIC APPROXIMATION

E. LEVICHEV and V. SAJAEV

Budker Institute of Nuclear Physics, Novosibirsk 630090, Russia

(Received 17 January 1996; in final form 15 August 1996)

In order to have a large dynamic aperture in low-emittance light sources with strong chromatic sextupoles, the understanding of phenomena determined the phase space topology of the nonlinear system is very important. In the present paper we discuss a single particle dynamics using azimuthal harmonics expansion of the perturbed Hamiltonian. There are two aspects of our study: firstly, we show that the essential features of the nonlinear motion (invariant curves, amplitude dependent tune shift, etc.) can be described taking into account only few main (dominant) harmonics, and secondly, we establish that the magnitude of these harmonics is determined by the basic parameters of the ring — natural chromaticity and horizontal emittance of the beam. In specific case of the dedicated light source, the magnitude of dominant harmonics depends weakly on the strength and position of the chromatic sextupoles. To demonstrate the harmonic approach we solve the nonlinear Hamiltonian using Lie canonical transformations. The phase space trajectories obtained analytically and numerically show good agreement both for one and two degrees of freedom up to the limits of stable motion.

Keywords: Nonlinearities; single particle dynamics.

1 INTRODUCTION

A typical cell of the magnetic lattice of modern low-emittance light source includes an achromatic bend and dispersion-free straight section to accommodate insertion devices. Since the horizontal emittance is inversely proportional to the third power of the number of achromats, a ring, as a rule, consists of many identical cells. To achieve a high brightness of the emitted radiation, efforts are made to minimize the natural emittance by the strong horizontal focusing in dipole magnets. This procedure inevitably leads to the large chromaticity which has to be compensated by strong sextupoles placed inside the achromat. The chromatic sextupoles act on the betatron motion, distort the phase space trajectories and reduce a particle stable motion area (dynamic aperture).

The main goal of this paper is an investigation of the perturbation of the beam dynamics caused by the chromatic sextupoles. As an example we use the magnetic lattice of the SIBERIA-2 light source, which is a 2.5 GeV storage ring with 6-fold symmetry lattice, dedicated for synchrotron radiation production.[1] We use 2D Hamiltonian with the sextupole perturbation expanded into azimuthal harmonics. Assuming that the dominant harmonics make the largest contribution to the beam dynamics, we analyze the features of these harmonics whose magnitude, as will be shown, is mainly determined by such general characteristics of the ring as natural chromaticity and horizontal emittance.

It is known that, in case of sextupole perturbation, the high order terms of the solution have to be considered to reproduce analytically the phase space topology. To construct the solution systematically, order by order, we use the recurrent series of Deprit based on Lie transforms. These recurrent expressions were implemented in the algebraic manipulator program REDUCE.[2]

The solution of the nonlinear Hamiltonian is a power series with respect to the amplitude of betatron oscillations (or action variables) $A_{x,z} \propto \sqrt{J_{x,z}}$. The coefficients of this power series have rather complex form and depend on unperturbed betatron tunes as well as on amplitudes of azimuthal Fourier harmonics of the sextupole potential. When the order of approximation is increased, these coefficients became more and more complicated, and REDUCE fails fast in attempt to manipulate them. In this case, it seems very attractive to select only a few dominant harmonics and simplify the solutions. We study the solutions and find the conditions for such simplifications.

2 RING MODEL

To verify the analytical estimations and to illustrate our conclusions we use the numerical simulation for the SIBERIA-2 storage ring.[8] We have also studied the nonlinear beam behaviour for different modern light sources (ELETTRA, APS, SPring-8, PLS, etc.[9]) and obtained results similar to those presented in this paper.

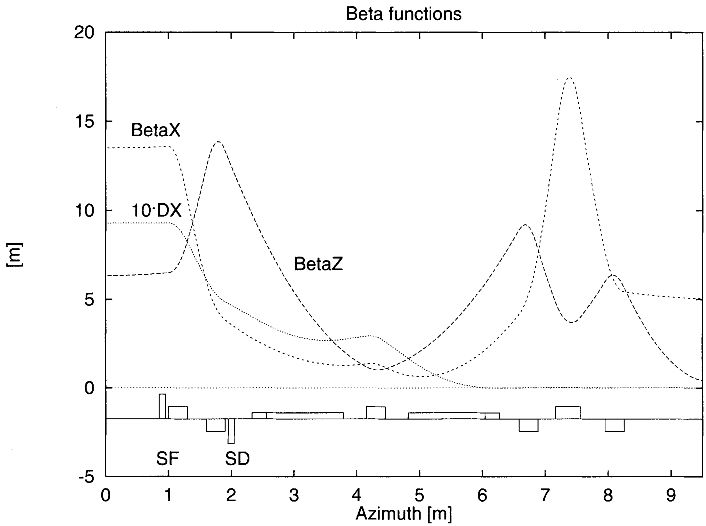

FIGURE 1 SIBERIA-2 lattice functions.

The 2.5 GeV light source SIBERIA-2 has six achromat cells with the reflection symmetry. The lattice functions for the SIBERIA-2 half-cell are demonstrated in Figure 1 and the main parameters of the ring are listed in the table below:

Energy (GeV)	2.5
Circumference (m)	124.128
Number of cells	6
ν_x, ν_z per cell	1.294, 1.116
ξ_x, ξ_z per cell	-3.99, -3.98
ϵ_x (nm)	76
β_{x0} (m)	14.0
β_{z0} (m)	7.0
η_{x0} (m)	0.9

Here, the betatron tunes $\nu_{x,z}$ and the natural chromaticity $\xi_{x,z}$ are given for one cell. Beta functions presented in the table correspond to $s = 0$ in Figure 1. In this paper, all the phase trajectories will be plotted for this azimuth.

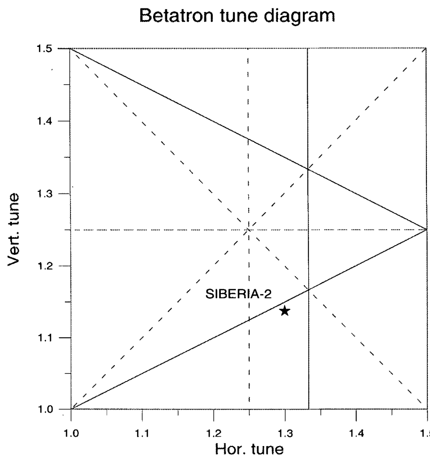

FIGURE 2 Tune diagram (solid line - first order resonances, dashed lines – second order resonances).

It is well known in perturbation theory that each *new order of perturbation* produces new *order of sextupole resonances* (where the order of the resonance $k_x \nu_x + k_z \nu_z = m$ is determined as usual $N = |k_x| + |k_z|$). For example, the main sextupole resonances (first order of perturbation) have orders $N = 1,\ 3$, the second order of perturbation brings new resonances of order $N = 2,\ 4$, and etc. Therefore, one should distinguish the order of perturbation and the order of resonance which appears for some order of perturbation.

Low-order resonances have significant influence on the storage ring nonlinear features. The SIBERIA-2 tune point for one cell is shown in Figure 2 together with the first and second order sextupole resonances.

The nearest sextupole resonances for the first order of perturbation are $\nu_x = 1,\ 3\nu_x = 4,\ \nu_x + 2\nu_z = 3,\ \nu_x - 2\nu_z = -1$. It means that the main driving terms to be considered are $A_{11},\ A_{34},\ B_{11},\ B_{+4},\ B_{--1}$. The second order of perturbation produces the following important resonances: $4\nu_{x,z} = 5$ and $2\nu_x + 2\nu_z = 5$.

Detailed nonlinear beam behaviour studies for the SIBERIA-2 light source have been carried out in [10]

3 AZIMUTHAL HARMONICS OF PERTURBATION

The Hamiltonian for an accelerator with sextupole harmonics can be expressed in terms of an action-angle (J, ϕ) as [4,6]

$$\begin{aligned} H &= \nu_x J_x + \nu_z J_z \\ &\quad + (2J_x)^{3/2} \sum_m [3A_{1m} \cos(\phi_x - m\theta) + A_{3m} \cos(3\phi_x - m\theta)] \\ &\quad - 3(2J_x)^{1/2} 2J_z \sum_m [2B_{1m} \cos(\phi_x - m\theta) \\ &\quad + B_{+m} \cos(\phi_+ - m\theta) + B_{-m} \cos(\phi_- - m\theta)]. \end{aligned} \tag{1}$$

where the amplitudes of five Fourier harmonics are

$$\begin{aligned} A_{jm} &= \frac{1}{48\pi} \sum_k [(\beta_x^{3/2} k_2 l) \cos(j(\psi_x - \nu_x\theta) + m\theta)]_k, \\ B_{1m} &= \frac{1}{48\pi} \sum_k [(\beta_x^{1/2} \beta_z k_2 l) \cos(\psi_x - \nu_x\theta + m\theta)]_k, \\ B_{\pm m} &= \frac{1}{48\pi} \sum_k [(\beta_x^{1/2} \beta_z k_2 l) \cos(\psi_\pm - \nu_\pm\theta + m\theta)]_k. \end{aligned} \tag{2}$$

Here $(k_2 l)_k$ is an effective strength of the kth sextupole (thin lens approximation is assumed), β and ψ are the amplitude and phase betatron functions, $\phi_\pm = \phi_x \pm 2\phi_z$ and similarly for $\psi_\pm$ and $\nu_\pm$. For the sake of simplicity, we assume that the observation point $s = 0$ is a reflection-symmetry point of the linear optics and the sextupoles location, and therefore $\alpha_{x,z} = 0$.

We start with the main harmonics satisfying the resonance conditions

$$\begin{aligned} j\nu_x &\simeq M, \quad (j = 1, 3), \\ \nu_\pm &\simeq M. \end{aligned} \tag{3}$$

The amplitude of these harmonics is expressed in (2) and our question is which basic accelerator parameters determine the value of this amplitude?

We consider a circular accelerator lattice with several identical cells. Each cell consists of achromatic bend and non-dispersive straight section. The natural chromaticity for one cell is $(\xi_x,\ \xi_z)$. The strength of the chromatic sextupoles, which are located in the achromat, is defined from the chromaticity compensation condition

$$4\pi\xi_{x,z} + \sum_k [(k_2 l)\beta_{x,z}\eta]_k = 0, \tag{4}$$

where $\eta(s)$ is a horizontal dispersion function.

We start with the harmonic A_{1m}, (2). For the resonant condition $m = M$ (3) gives $\nu_x\theta - M\theta \simeq 0$. Then the resonant harmonic A_{1M} can be written as (we shall omit index M)

$$A_1 \simeq \frac{1}{48\pi} \sum_k [(\beta_x^{3/2} k_2 l) \cos\psi_x]_k. \tag{5}$$

To connect the above expression with the accelerator parameters we consider the well known $\mathcal{H}$-function that defines the transverse horizontal emittance

$$\mathcal{H} = \gamma_x\eta^2 + 2\alpha_x\eta\eta' + \beta_x\eta'^2.$$

The dispersion function η satisfies the second order differential equation

$$\eta''(s) + k_x(s)\eta = \frac{1}{\rho(s)},$$

whose solution outside the bending magnets $(1/\rho(s) = 0)$ has the form similar to betatron oscillations. Hence, outside the bending magnets $\mathcal{H} =$ const, like the Courant-Snyder invariant. Following this analogy, we use the Floquet's coordinate transformation[20] for the dispersion function

$$a = \eta/\sqrt{\beta_x} \qquad b = \eta'\sqrt{\beta_x} + \eta\alpha_x/\sqrt{\beta_x},$$

and write down $\mathcal{H}_a$ for achromatic bend in the form

$$\mathcal{H}_a = a^2 + b^2,$$

$$a = \sqrt{\mathcal{H}_a}\cos\psi_x \qquad b = \sqrt{\mathcal{H}_a}\sin\psi_x$$

where ψ_x is the betatron phase advance. Introducing $\sqrt{\beta_x}/\eta = \cos\psi_x/\sqrt{\mathcal{H}_a}$ in (5) and taking into account that in achromatic bend $\mathcal{H} =$ const, we find

$$A_1 \simeq \frac{1}{48\pi\sqrt{\mathcal{H}_a}} \sum_k (k_2 l)_k \beta_{xk} \eta_k. \tag{6}$$

Using (4), one can estimate the resonant harmonic value as

$$A_1 \simeq -\frac{1}{12}\frac{\xi_x}{\sqrt{\mathcal{H}_a}}.$$

A similar formula can be found for the resonant harmonic B_{1M}, but with $-\xi_z$ instead of ξ_x (from (4), second equation). It is interesting to note that the main harmonics A_{3M} and $B_{\pm M}$ will satisfy the same expression if we assume that the value of betatron phase advance is small at the sextupole azimuths $\psi_{x,z} < 1$. The latter assumption is suited rather well for our case of light source lattice, since the chromatic sextupoles are located inside the achromat, and the major phase advance occurs in the region of small beta-functions inside bending magnets. The resulting expressions to estimate the magnitude of the resonant harmonics of A- and B-type are

$$A \simeq -\frac{1}{12}\frac{\xi_x}{\sqrt{\mathcal{H}_a}}, \quad B \simeq \frac{1}{12}\frac{\xi_z}{\sqrt{\mathcal{H}_a}}. \tag{7}$$

For the case of the light source lattice, $\mathcal{H}_a$ is proportional to the mean value in dipoles $\langle\mathcal{H}(s)\rangle$ which determines the horizontal emittance ($\varepsilon_x \propto \langle\mathcal{H}(s)\rangle$). If the latter is minimized,[21] the relation is very simple: $\mathcal{H}_a = 4\langle\mathcal{H}(s)\rangle$. Otherwise, the ratio between $\mathcal{H}_a$ and $\langle\mathcal{H}(s)\rangle$ depends on the $\beta_x(s)$ inside bending magnets. But anyway, we can say that the amplitude of dominant azimuthal harmonics of sextupole perturbation is defined by the ring fundamental parameters: natural chromaticity and horizontal emittance (for one lattice cell)

$$A \propto \xi_x/\sqrt{\varepsilon_x}, \quad B \propto \xi_z/\sqrt{\varepsilon_x}.$$

For the case of the SIBERIA-2 cell $\xi_x \simeq \xi_z = -3.98$ and $\mathcal{H}_a = 4.03$ cm the estimate in (7) yields $A = 0.165\ \text{cm}^{-1/2}$, $B = -0.165\ \text{cm}^{-1/2}$, whereas explicit calculation gives:

$$A_{11} = 0.165\ \mathrm{cm}^{-1/2}, \qquad B_{11} = -0.167\ \mathrm{cm}^{-1/2},$$
$$A_{34} = 0.177\ \mathrm{cm}^{-1/2}, \qquad B_{-1} = -0.167\ \mathrm{cm}^{-1/2},$$
$$B_{+4} = -0.103\ \mathrm{cm}^{-1/2},$$

It is seen that the correspondence is rather good except for B_{+4} which requires more rigorous phase advance consideration.

Figure 3 shows the correspondence between the explicit calculation and our estimate in (7) for different modern light sources which have different chromaticity and horizontal emittance. As can be seen from this figure, the agreement is rather good.

We see, that in the case of compact achromatic bend when the betatron phases change slowly, an amplitude of dominant driving terms depends weakly on the chromatic sextupoles location. The straightforward tracking with different positions of sextupoles, which correct the natural chromaticity shows, that the value of main harmonics and the dynamic aperture size are the same in spite of the strength of the sextupoles changes as more as twice.

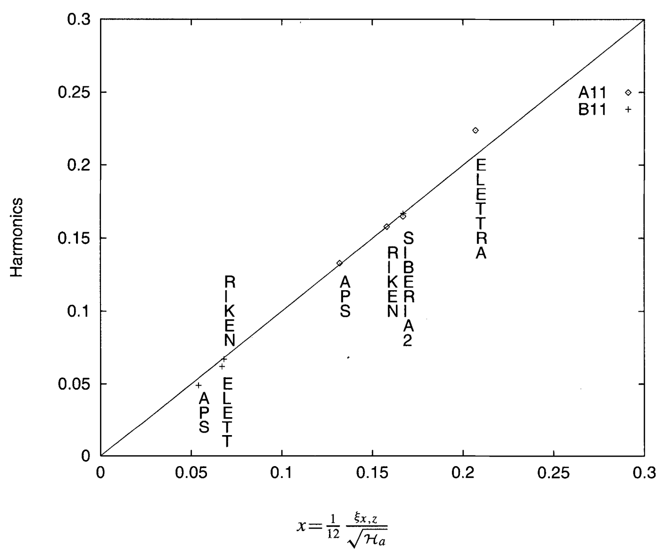

FIGURE 3 Value of main harmonics for different light sources. According to (24) the value of harmonics should correspond to the line $x = y$.

4 PHASE SPACE TOPOLOGY

4.1 Lie Perturbation Theory

The Hamiltonian (1) describes a 2D nonconservative system that cannot be solved by the straightforward integration. But we may represent this Hamiltonian as a sum of the linear term H_0 and small nonlinear term H_1, and apply some perturbation theory to find an approximate solution. We consider the par-axial case when the small parameter is equal to the amplitude of the particle deviation from the reference orbit $\epsilon \propto \sqrt{J}$ and try to find the approximate solution as a power series in ϵ. It will be shown that the coefficients of these series depend on the non-perturbed tunes in a resonant way. This is the familiar problem of small denominators. But in spite of this problem, in many cases, the results of perturbation theory are rather useful. In particular, this is true for circular accelerators, where betatron tunes are chosen far from strong resonances.

It is known that for a sextupole perturbation the first order tune shift is zero, and an analysis of high orders is required to find reasonable phase trajectories of the system. The classic Poincaré-Von Zeipel perturbation theory application tends to be rather clumsy for high orders, because in this case the canonical transformation is specified by the generating function of mixed (old and new) variables.[17] There are modern kinds of formalism for describing dynamical systems and their behaviour.[11] Among them we should note normal form analysis[12] and differential algebra methods.[13]

Here we use an efficient perturbation method utilizing the natural structure of canonical transformations is based on the Lie transforms (here we use it in the form of the Deprit perturbation theory). This theory is identical to the Poincaré-Von Zeipel approach in spirit, but the transformations themselves are much simpler. Closed form expressions, in which only the Poisson brackets appear can explicitly be written down for both the generating function and transformation equation to an arbitrary order.[18]

The detailed study of the Deprit perturbation technique in the context of accelerator theory can be found elsewhere.[14] Another approach based on the Lie transform is the Dragt and Finn perturbation theory,[16] but for our purposes both techniques are practically the same. For future reference, the recursive equations (Deprit perturbation series relations) are given below without proof. Begin by expanding the "old" Hamiltonian H, the "new" one (transformed) $\bar{H}$, and the Lie generating function w as power series in ϵ[15]:

$$w = \sum_{n=0}^{\infty} \epsilon^n w_{n+1}, \quad H = \sum_{n=0}^{\infty} \epsilon^n H_n, \quad \bar{H} = \sum_{n=0}^{\infty} \epsilon^n \bar{H}_n. \tag{8}$$

and defining an operator $\hat{D}_0$ associated with the total time derivative along the trajectory of the original system as

$$\hat{D}_0 = \partial/\partial\theta + [\ , H_0] = \frac{\partial}{\partial\theta} + \nu_x \frac{\partial}{\partial\phi_x} + \nu_z \frac{\partial}{\partial\phi_z}, \tag{9}$$

we can write the equation for w_n for the n-th order:

$$\hat{D}_0 w_n = n(\bar{H}_n - H_n) - \sum_{m=1}^{n-1} (\hat{L}_{n-m} \bar{H}_m + m \hat{T}_{n-m}^{-1} H_m), \tag{10}$$

where $\hat{L}_n = [w_n,\ ]$ is the Poisson bracket operator. The arbitrary term $\bar{H}_n$ must be chosen to make the new system more easily solved than the old one and cancel the secular terms from w_n. In the frequency region far from strong resonances for n-th step we must pick $n\bar{H}_n = \langle rhs_n \rangle$, where $\langle\rangle$ is the averaging over $\phi_{x,z}$ and θ, and rhs_n is the right hand side of (10) with $n\bar{H}_n$ excluded. So, the Lie method together with the strategy for cancel secular terms yields the transformed Hamiltonian $\bar{H}$ which is (i) integral of motion, because it is time independent, and (ii) angle independent ($\partial\bar{H}/\partial\phi_{x,z} = 0$). The latter means that for new momenta $d\bar{J}_{x,z}/d\theta = 0$ and they are invariant of motion: $\bar{J}_{x,z} = \text{const}$.

The mapping between the old canonical variables (J, ϕ) and new ones $(\bar{J}, \bar{\phi})$ is defined by the canonical transformation operator $\hat{T} = \sum_{n=0}^{\infty} \epsilon^n \hat{T}_n$:

$$\bar{J} = \hat{T} J, \qquad \bar{\phi} = \hat{T}\phi,$$

$\hat{T}$ evaluates the function at mapped point. The inverse transformation $\hat{T}^{-1}$ exists and produces the backward mapping. The phase space trajectories of the original system are obtained by applying the inverse transform. Both operators can be evaluated from the following recurrent equations:

$$\hat{T}_n = -\frac{1}{n} \sum_{m=0}^{n-1} \hat{T}_m \hat{L}_{n-m}, \quad \hat{T}_n^{-1} = \frac{1}{n} \sum_{m=0}^{n-1} \hat{L}_{n-m} \hat{T}_m^{-1}.$$

4.2 Polynomial Solution

In this section, we employ the Deprit's recursive equations for our particular case. For simplicity and to demonstrate the Lie methods, we start with a 1D horizontal motion up to the second order.

With the nonlinear perturbating term of the following form (see (1))

$$H_1 = (2J_x)^{3/2} \sum_m [3A_{1m} \cos(\phi_x - m\theta) + A_{3m} \cos(3\phi_x - m\theta)]. \quad (11)$$

the first two equations of (10) have the following solution:

$$w_1 = -(2\bar{J}_x)^{3/2} \sum_m \left[\frac{3A_{1m}}{\nu_x - m} \sin(\phi_x - m\theta) + \frac{A_{3m}}{3\nu_x - m} \sin(3\phi_x - m\theta) \right].$$

$$w_2 = 36\bar{J}_x^2 \left[\sum_{m\neq 0} \frac{\sin m\theta}{-m} \sum_l \frac{3A_{1l}A_{1l+m}}{\nu_x - l} + \frac{A_{3l}A_{3l+m}}{3\nu_x - l} \right.$$
$$+ 6 \sum_m \frac{\sin(2\phi_x - m\theta)}{2\nu_x - m} \sum_l \frac{A_{1l}A_{3m+l}(4\nu_x - 2l - m)}{(\nu_x - l)(3\nu_x - m - l)}$$
$$\left. +3 \sum_m \frac{\sin(4\phi_x - m\theta)}{4\nu_x - m} \sum_l \frac{A_{1l}A_{3m-l}(2\nu_x - 2l + m)}{(\nu_x - l)(3\nu_x - m + l)} \right].$$

Here to cancel the secular terms we have chosen new Hamiltonian $\bar{H}_n$ as

$$\bar{H}_1 = 0$$

$$\bar{H}_2 = -18\bar{J}_x^2 \sum_l \left(\frac{3A_{1l}^2}{\nu_x - l} + \frac{A_{3l}^2}{3\nu_x - l} \right).$$

Note, that if we examine new Hamiltonian $\bar{H}$, it is approximately independent of time θ and phase variable ϕ_x. Therefore, the new momentum $\bar{J}_x$ is (approximately) invariant of motion. To obtain the phase trajectory of the original system we recall that the canonical transformation $\hat{T}$ relates the old and new variables by

$$
\begin{aligned}
\bar{J}_x &= \hat{T} J_x = (\hat{T}_0 + \hat{T}_1 + \hat{T}_2) J_x \\
&= \left(1 - \hat{L}_1 - \frac{1}{2}\,\hat{L}_2 + \frac{1}{2}\,\hat{L}_1^2\right) J_x \\
&= J_x - [w_1, J_x] - \frac{1}{2}\,[w_2, J_x] + \frac{1}{2}\,[w_1\,[w_1, J_x]].
\end{aligned}
$$

The general form of the polynomial solution for $\bar{J}_x$ (or $\bar{J}_z$) is given by

$$
\bar{J}_{x,z} = \sum_{j_x, j_z, k_x, k_z} J^{\frac{j_x}{2}} J^{\frac{j_z}{2}} c^{(n)}(k_x, k_z), \tag{12}
$$

$$
c^{(n)}(k_x, k_z) = \sum_{m=-\infty}^{\infty} a^{(n)}(k_x, k_z, m) \cos(k_x \phi_x + k_z \phi_z - m\theta), \tag{13}
$$

where for n-th order $j_x + j_z = n + 2$ and the coefficients $a^{(n)}(k_x, k_z, m)$ can be found from the equation for w_n. In our case, taking into account w_1 and w_2 and applying consequently the Poisson bracket, the equations for original variables (J_x, ϕ_x) can be written as

$$
\begin{aligned}
\bar{J}_x &= J_x + J_x^{3/2}\Big[\cos\phi_x \sum_m a_x^{(1)}(1, 0, m) + \cos 3\phi_x \sum_m a_x^{(1)}(3, 0, m)\Big] \\
&\quad + J_x^2 \Bigg[\sum_m a_x^{(2)}(0, 0, m) + \cos 2\phi_x \sum_m a_x^{(2)}(2, 0, m) \\
&\quad + \cos 4\phi_x \sum_m a_x^{(2)}(4, 0, m)\Bigg],
\end{aligned} \tag{14}
$$

where we set $\theta = 0$, and non-zeroth terms of sums are defined as:

$$
\begin{aligned}
a_x^{(1)}(1, 0, m) &= 3\sqrt{8}\frac{A_{1m}}{\nu_x - m}, \\
a_x^{(1)}(3, 0, m) &= 3\sqrt{8}\frac{A_{3m}}{3\nu_x - m},
\end{aligned}
$$

$$a_x^{(2)}(0,0,m) = 108 \sum_l \frac{A_{1l}A_{1m+l}}{(\nu_x - l)(\nu_x - m - l)} + \frac{A_{3l}A_{3m+l}}{(3\nu_x - l)(3\nu_x - m - l)},$$

$$a_x^{(2)}(2,0,m) = \frac{72}{2\nu_x - m} \sum_l \frac{A_{1l}A_{3m+l}}{(\nu_x - l)(3\nu_x - l - m)}(2l - m),$$

$$a_x^{(2)}(4,0,m) = \frac{36}{4\nu_x - m} \sum_l \frac{A_{1l}A_{3m-l}}{(\nu_x - l)(3\nu_x + l - m)}(4l - m). \tag{15}$$

In the left hand side of (14) we have $\bar{J}_x =$ const , therefore, by solving this equation we can get the final solution of the original problem.

As was shown above, all the quantities obtained through the solution of the nonlinear Hamiltonian by Lie methods, such as action variables, generating functions, amplitude dependent tunes shift, etc. have the general form of power series in small parameter $\epsilon \propto \sqrt{J_{x,z}}$ (12). The coefficients of these power series are the sum of cosine terms with amplitudes defined by the original azimuthal harmonics of sextupole perturbation, (15). The number of cosine terms grows rapidly while the order of approximation is increased. A distinguished feature of these expressions is that the amplitude of azimuthal harmonics $A_{k_xk_zm}$ and $B_{k_xk_zm}$ always appear together with the related denominator $k_x\nu_x + k_z\nu_z - m$ that leads to the emphasizing of dominant harmonics.

This fact is illustrated in Figure 4 where the spectrum of $a_{x1}^{(1)}(1,0,m)$ and $a_{x1}^{(2)}(4,0,m)$ is plotted. Now we shall consider the individual contribution of harmonics to the final solutions. For the sake of brevity, we introduce the notation for m th term of the expression for $a^{(n)}(k_x, k_z, m) = a_m$ and write it in the form

$$a_m = \frac{A_m}{k_x\nu_x + k_z\nu_z - m}, \tag{16}$$

Using the notation for a distance from the nearest resonance that is classified by the integers k_x and k_z, $\delta = k_x\nu_x + k_z\nu_z - m_{k_x,k_z}$, $\delta \leq 1/2$, we rewrite (16) in the form

$$a_m = \frac{A_m}{\delta - (m - m_{k_x,k_z})},$$

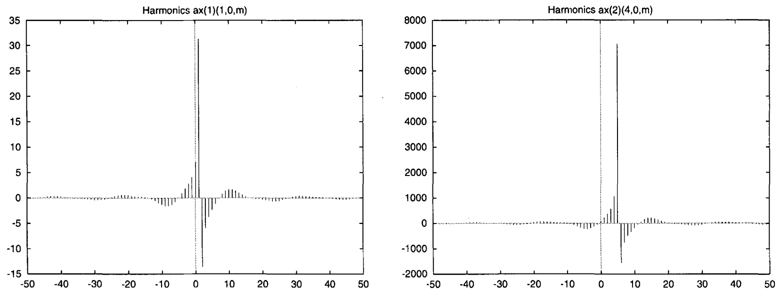

FIGURE 4 Spectrum of $a_{x1}^{(1)}(1, 0, m)$ and $a_{x1}^{(2)}(4, 0, m)$ for SIBERIA-2.

A straightforward computation shows that the amplitude absolute value for azimuthal harmonic A_m depends weaker on m compared to the resonant dependence of the denominator. So, we can take $A_m \simeq A_{m-1}$ and after the substitution $(m - m_{k_x,k_z}) \to m$ the following estimation can be made

$$R_m = \frac{|a_m|}{|a_{m-1}|} = 1 - \frac{1}{m\,(1 \mp \delta/m)},$$

where m is taken as an absolute value and sign "-" corresponds to $m \geq 0$. Recalling that $\delta \leq 1/2$, the following expansion can be written

$$R_m \simeq 1 - \frac{1}{m} \pm \frac{\delta}{m^2} + o(\delta^2). \tag{17}$$

According to (17) $R_{\pm 1} \simeq \mp\delta$, i.e. each of two harmonics, which are adjacent to the main one, contributes to the solution weaker than the main one with the factor of $1/\delta$. And it is seen that the nearer tune point to the given resonance, the more reasons to use the *dominant harmonic* approximation.

Note, that it is not the case of *a single resonance approximation* because we stay rather far from any strong resonance and we try to leave in our Hamiltonian a *few* harmonics in the frame of *non-resonance* perturbation theory.

4.3 Phase Trajectories for Horizontal Motion

Here we shall apply the results obtained in the previous sections to plot the horizontal phase trajectories using both tracking and dominant harmonic approaches for the SIBERIA-2 lattice.

When the polynomial coefficients were calculated taking into account azimuthal harmonics A_{1m} and A_{3m} ($m = -1000 \ldots + 1000$), the new action to the third order is

$$\begin{aligned}\bar{J}_x = J_x &+ J_x^{3/2}\,(2.2\ \cos\phi_x - 7.8\ \cos 3\phi_x)\\ &+ J_x^2\,(158 - 80\ \cos 2\phi_x + 144\ \cos 4\phi_x)\\ &+ J_x^{5/2}\,(-2113\ \cos\phi_x + 2357\ \cos 3\phi_x - 433\ \cos 5\phi_x).\end{aligned}$$

If we do the same but keeping only two initial harmonics A_{11} A_{34}, which are estimated according to (7), the solution for $\bar{J}_x$ will have the form

$$\begin{aligned}\bar{J}_x = J_x &+ J_x^{3/2}\,(2.9\ \cos\phi_x - 8.1\ \cos 3\phi_x)\\ &+ J_x^2\,(121 - 99\ \cos 2\phi_x + 162\ \cos 4\phi_x)\\ &+ J_x^{5/2}\,(-2218\ \cos\phi_x + 3300\ \cos 3\phi_x - 790\ \cos 5\phi_x),\end{aligned} \tag{18}$$

Due to the tune value the main contribution to the phase space topology is from $\cos 3\phi_x$ term (first and third orders) and $\cos 4\phi_x$ term (second order).

Horizontal phase curves calculated in the dominant harmonics approximation (through third order) for $\nu_x = 7.763$ and those obtained by tracking are plotted in Figure 5. The calculated points are essentially identical to the exact solution at the half-aperture level and at the border of the stable motion the agreement seems quite satisfactory.

When we fail to solve the equation $\bar{J}_x(J_x, \phi_x) =$ const for some initial conditions $J_x = J_{x0},\ \phi_x = 0$, we can consider them as a stable motion border, and in this case, the value of J_{x0} corresponds rather good to the dynamic aperture defined by tracking.

4.4 Two-dimensional Case

In the previous section we have treated one degree of freedom. The extension to a 2D motion is quite straightforward: we seek the Lie generating function to the required order and transform the Hamiltonian to the new variables where it is a function of new momenta alone. It means that $\bar{H} =$ const and $\bar{J}_{x,z} =$ const . But in the case of two degrees of freedom, all the results are rather cumbersome and it is better to treat them using REDUCE.

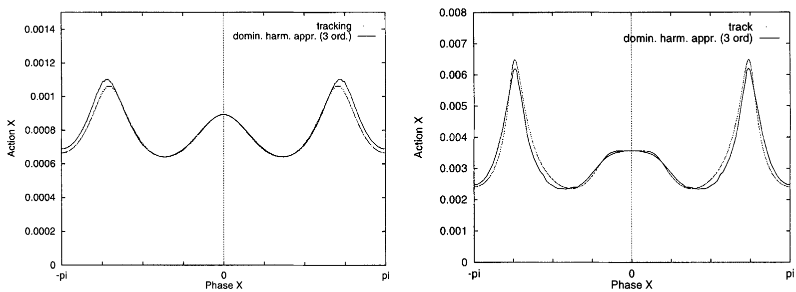

FIGURE 5 Horizontal phase trajectories: tracking (points) and approximation by dominant harmonics (lines). Left plot corresponds to half-aperture level, right – near to the limit of dynamic aperture.

The mapped Hamiltonian written up to the third order is

$$\bar{H} = \bar{H}_0 + \bar{H}_2 = \nu_x \bar{J}_x + \nu_z \bar{J}_z + h_{21} \bar{J}_x^2 + h_{22} \bar{J}_x \bar{J}_z + h_{23} \bar{J}_z^2 \tag{19}$$

from this expression we can obtain the well known amplitude dependent tune shift which appears with sextupoles as a second order effect:

$$\begin{aligned} \Delta\nu_x(\bar{J}_x, \bar{J}_z) &= 2\bar{J}_x h_{21} + \bar{J}_z h_{22} \\ \Delta\nu_z(\bar{J}_x, \bar{J}_z) &= \bar{J}_x h_{22} + 2\bar{J}_z h_{23}, \end{aligned} \tag{20}$$

here the coefficients are defined by the azimuthal harmonics (2) as:

$$\begin{aligned} h_{21} &= -18 \sum_m \left(\frac{3A_{1m}^2}{\nu_x - m} + \frac{A_{3m}^2}{3\nu_x - m} \right) \\ h_{22} &= 72 \sum_m \left(\frac{2B_{1m}A_{1m}}{\nu_x - m} - \frac{B_{+m}^2}{\nu_+ - m} + \frac{B_{-m}^2}{\nu_- - m} \right) \\ h_{23} &= -18 \sum_m \left(\frac{4B_{1m}^2}{\nu_x - m} + \frac{B_{+m}^2}{\nu_+ - m} + \frac{B_{-m}^2}{\nu_- - m} \right). \end{aligned} \tag{21}$$

In our case of SIBERIA-2 lattice $h_{21} = 449\ \text{m}^{-1}$, $h_{22} = 755\ \text{m}^{-1}$, $h_{23} = -1854\ \text{m}^{-1}$.

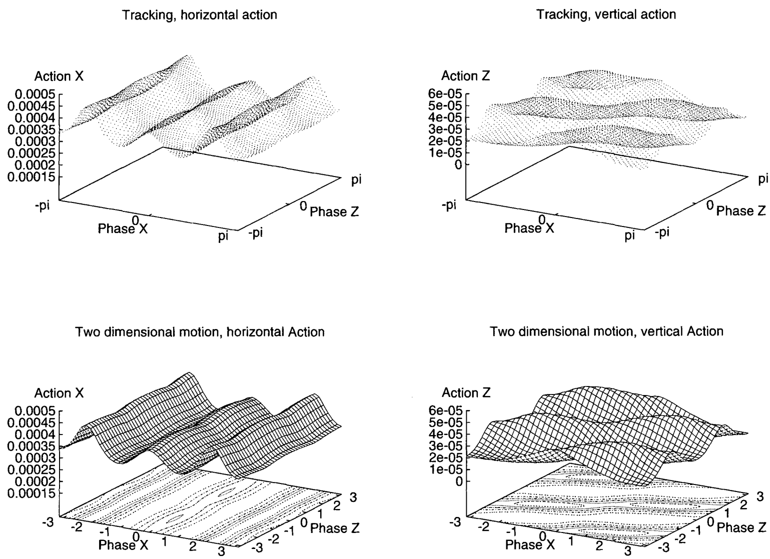

FIGURE 6 Two degrees of freedom. Upper plots show surface $J_x(\phi_x, \phi_z)$ and lower plots show $J_z(\phi_x, \phi_z)$. The theory results (line) are compared with tracking (points).

As the new Hamiltonian (19) does not depend on $\bar{\phi}_{x,z}$ and θ, it is invariant of motion together with the new action variables: $\bar{J}_{x,z} =$ const and $\bar{H} =$ const.

Applying operator $\hat{T}$, the following expressions for new momenta can be found

$$\begin{aligned} \bar{J}_x &= \bar{J}_x^{(0)} + \bar{J}_x^{(1)} + \bar{J}_x^{(2)} + \bar{J}_x^{(3)} \\ \bar{J}_z &= \bar{J}_z^{(0)} + \bar{J}_z^{(1)} + \bar{J}_z^{(2)} + \bar{J}_z^{(3)}. \end{aligned} \tag{22}$$

Due to the extreme complexity, the expressions (22) were calculated using REDUCE. These equations were solved numerically by the Newton method. The resulting phase surfaces (3 orders) are presented in Figure 6 in comparison with the tracking results. The calculations have been done for the initial conditions $J_{x0} = 4 \times 10^{-4}$ cm, $J_{z0} = 4 \times 10^{-5}$ cm while the stable motion boundary corresponds to $J_{x0} = 3 \times 10^{-3}$ cm, $J_{z0} = 2.5 \times 10^{-3}$ cm. Such relatively small starting conditions have been taken just to have

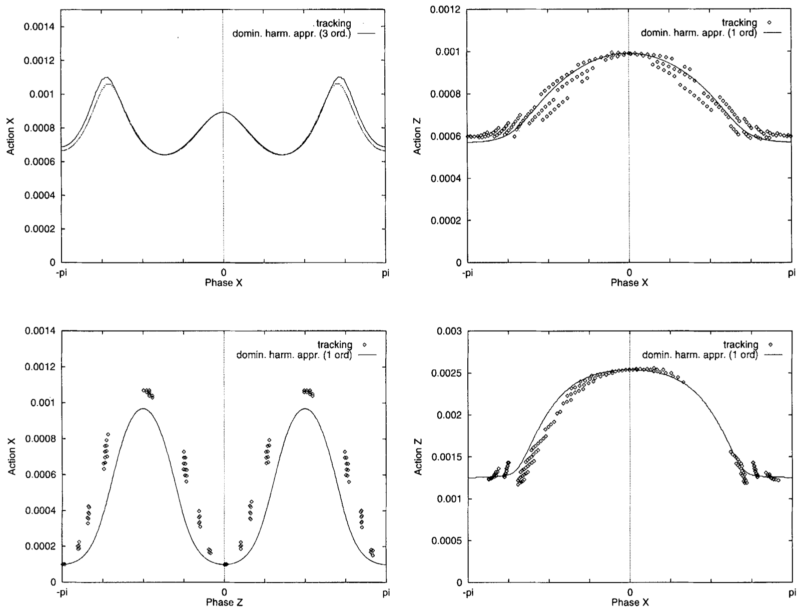

FIGURE 7 2D sections $J_x(\phi_z)$ (left plots) and $J_z(\phi_x)$ (right plots). Tracking results (points) and dominant harmonics (lines). Vertical amplitude corresponds to 1/2 of aperture limit (top) and to the border of stability (bottom), while horizontal amplitude is 1/5 of that.

readable pictures, otherwise SMEAR becomes too large. Agreement of theory and numerical results are quite impressive.

To illustrate the technique, we shall consider the Poincaré sections $J_x(\phi_x)$, $J_x(\phi_z)$, $J_z(\phi_x)$, $J_z(\phi_z)$ with different initial conditions (far from the dynamic aperture limits and directly at the border of stable motion). Figure 7 demonstrates the sections $J_x(\phi_z)$ and $J_z(\phi_x)$ under these initial conditions.

5 SUMMARY AND CONCLUSION

In the present paper, we considered nonlinear beam behaviour due to the chromatic sextupoles in the dedicated light source SIBERIA-2. The problem is studied analytically using the harmonic decomposition of the nonlinear Hamiltonian. The relevant recurrent expressions are developed by the Lie transforms theory and Deprit's perturbation methods to obtain the solution of the particular system in any arbitrary order. These expressions were programmed by the REDUCE language and provide 2D phase trajectories of a particle which agree quite good with those obtained by tracking.

It was shown that it is possible to simplify the original problem by taking into account only a few dominant harmonics that correspond to the nearest resonances. It is worth noting that this is not a single resonance approximation, because we can consider *all* of these major harmonics at one time by the non-resonant perturbation theory. The validity of this approximation is proved by tracking for a wide range of initial conditions.

We have found that in the case of the light source lattice, the magnitude of dominant harmonics which defines the phase trajectory distortion, dynamic aperture, nonlinear detuning, etc. depends on the natural chromaticity of the ring and horizontal emittance of the beam and rather weakly on the location of the chromatic sextupoles.

Acknowledgements

The authors thank Dr. V.N.Korchuganov for valuable discussions and for critically reading the manuscript. One of them (E.L.) thank Dr. V.V.Vecheslavov for some helpful explanation on Lie transforms theory.

References

[1] Anasin, V. *et al.* (1988). The dedicated synchrotron radiation source SIBERIA-2.- Proc.of EPAC'88, Rome, **1**, 380–382.
[2] Rayna, G. (1987). REDUCE: Software for Algebraic Computation. — Springer-Verlag.
[3] Ando, A. (1984). Distortion of Beam Emittance With Nonlinear Magnetic Fields. *Particle Accelerator*, **15**, 177.
[4] Ng, K.Y. (1984). Fermilab Report TM-1281.
[5] Parsa, Z. *et al.* (1987). Second-order perturbation theory for accelerators. *Particle Accelerators*, **22**, 205–230.
[6] Collins, T.L. (1984). Fermilab Internal Report 84/114.
[7] Nagaoka, R., Yoshida, K., Hara, M. (1991). Nonlinear dynamics with sextupoles in low emittance light source storage ring. *Nucl. Instrum. Methods*, **A302**, 9–26.
[8] Erg, G.I. *et al.* (1989). Magnetic lattice of SIBERIA-2 — dedicated SR source. Preprint INP 89–174.
[9] Murphy, J. (1993). Synchrotron Light Source Data Book (V.3.0), BNL 42333.
[10] Korchuganov, V.N., Levichev, E.B., Sajaev, V.V. (1993). Chromaticity compensation and dynamic aperture of the SIBERIA-2 storage ring (tracking results). BINP preprint 93–27. (In Russian).
[11] Scandale, W., Dynamic aperture, CERN SL/94-24 (AP).

[12] Berz, M. (1992). High-order computation and normal form analysis of repetitive systems, *AIP Conf. Proceed.*, **249**, 456–489.
[13] Yan, Y.T. (1992). Application of differential algebra to single-particle dynamics in storage rings, *AIP Conf. Proceed.*, **249**, 380–455.
[14] Michelotti, L. (1985). Moser-like transformations using the Lie transform. *Particle Accelerators*, **16**, 233–252.
[15] Deprit (1969). *Celestial Mechanics*, **1**, 12–30.
[16] Dragt, A.J. and Finn, J.M. (1976). *J. Math. Phys.*, **17**, 2215.
[17] Lichtenberg, A. and Liberman, M. (1983). Regular and stochastic motions. Springer-Verlag, New-York.
[18] Cary, J.R. (1981). *Phys. Reports*, **79**(2), 129–159.
[19] Dewar, R.L. (1976). *J. Phys.*, **A9**, 2043.
[20] Courant, E.D. and Snyder, H.S. (1958). *Ann. Phys.*, **3**, 1.
[21] Sommer, M. (1983). Optimization of the emittance of electrons (positrons) storage rings, Preprint LURE, LAL/RT/83–15.
[22] Dwight, H.B. (1961). Tables of integrals. The MacMillan Company, N.Y.