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HIGH-FREQUENCY LIMIT OF THE LONGITUDINAL IMPEDANCE

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A method of iterations that takes into account finite widths of resonances is developed and used to calculate the longitudinal impedance for a cylindrical cavity with side pipes. In the high-frequency limit, the dependence of the real part of the impedance on the frequency is $\omega^{-1/2}$, in agreement with the estimate for the average impedance obtained by G. Dôme.

1. INTRODUCTION

At present there is a certain controversy in the problem of high-frequency impedance behavior. Some numerical calculations are consistent with the optical resonator model,[1] which predicts the decrease of the impedance as $\omega^{-3/2}$. Correspondingly, the total energy loss of a pointlike particle and the value of the wake potential $w(0)$ immediately behind the particle are finite. On the other hand, the analytical calculations of Dôme[2] give quite a different asymptotic dependence, as $\omega^{-1/2}$. The total energy loss and the value of the wake potential $\omega(0)$ diverge in this case.

The basic assumption used in Ref. 2 that actually made the calculation possible is that the presence of the side pipes *does not* change the resonant frequencies and the fields at the radius of the beam pipe. These were assumed to be the same as for a pillbox cavity. Under this assumption the time dependence of the wake potential and, from it, the average impedance were found. Insofar as the losses in the closed cavity are infinitely small, the real part of the impedance is represented as an infinite sum over δ-functions corresponding to different pillbox cavity modes and can become a smooth function of the frequency only after it is averaged over some frequency interval.

In an accelerator cavity with side pipes the energy stored in the field leaks out of the cavity. The resonant curves for each mode excited in the cavity became finite both in their hieghts and widths. In the high-frequency limit the width of each curve grows faster than the distance between the neighboring resonances,

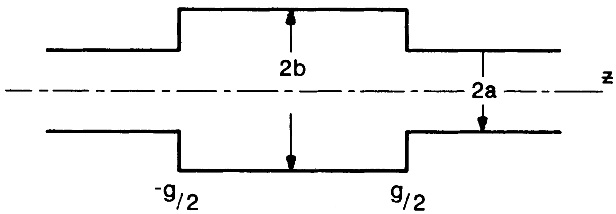

FIGURE 1 Cylindrically symmetric cavity of radius b and length g with side pipes of radius a. The origin of the cylindrical coordinate system (r, z) is placed in the middle point of the cavity axis. The system is excited by a point charge Q moving with velocity very close to c along the axis.

causing the curves to overlap. Hence, the impedance becomes finite and smooth without any averaging.

For each given wave number there are a finite number of modes that can oscillate in the cavity and an infinite number of evanescent waves that exponentially decrease in their amplitude when the distance increases from the pipe radius toward the cavity interior. One may say that a metallic boundary is effectively reinstated for the evanescent waves. This makes the widths of the corresponding resonances small. As we will show, this in turn makes the contribution from these waves to the impedance negligibly small. Hence, the effective cutoff enters the calculations of the impedance in a quite natural way.

Here we present the analytical calculation of the real part of the longitudinal impedance for a cylindrical cavity with side pipes. In Fig. 1 the geometry and coordinate system of the problem are shown. In Section 2 we derive the basic exact integral equation for the function from which the impedance can be deduced. The equation is obtained from the boundary and the field-matching conditions at the pipe radius $r = a$. In Section 3 the integral equation is rewritten to eliminate the singularities of the kernel of the integral equation so that its solution can be found by the method of iterations. The approximation we use is closely related to Feshbach's general theory of nuclear reactions,[3] where the widths of the energy levels in nuclei are generated by the coupling of the discrete spectrum with the unbounded motion in the infinite spectrum.

The estimate of the real part of the impedance in the high-frequency limit agrees with that obtained in Ref. 1. The correction coming from the next approximation is estimated and shown to be small with respect to the main term. The evaluation of certain sums can be found in the appendix.

2. THE BASIC EQUATION

If one wishes to work in the frequency domain to find the impedance in the high-frequency limit, it is natural to expand the fields in the waveguide eigenmodes. Then the exact expressions for the fields in the region inside the waveguide $(r < a)$ and in the region inside the cavity $(r > a)$ may be matched at the boundary $(r = a)$.

Inside the cavity $(r > a)$ the Fourier components at frequency ω of the electric field are superpositions

$$E_z = \sum_{n\geq 0} A_n f_n^z, \tag{1}$$

$$E_r = \sum_{n\geq 0} A_n f_n^r, \tag{2}$$

of the eigenmodes

$$f_n^r(r, z) = \lambda_n \mu_n [J_1(\mu_n r) N_0(\mu_n b) - N_1(\mu_n r) J_0(\mu_n b)] \begin{pmatrix} \sin \lambda_n z \\ -\cos \lambda_n z \end{pmatrix}, \tag{3}$$

$$f_n^z(r, z) = \mu_n^2 [J_0(\mu_n r) N_0(\mu_n b) - N_0(\mu_n r) J_0(\mu_n b)] \begin{pmatrix} \cos \lambda_n z \\ \sin \lambda_n z \end{pmatrix}, \tag{4}$$

that satisfy the corresponding boundary conditions on the cavity walls. Here the following notations are used:

$$k = \omega / c, \qquad \mu_n = \sqrt{k^2 - \lambda_n^2}, \qquad \lambda_n = \frac{2\pi}{g} \begin{pmatrix} n \\ n + 1/2 \end{pmatrix}.$$

The lower and the upper lines describe the odd and even modes, and J, N are the Bessel functions of the first and second kind, respectively.

For $r < a$ the Fourier components of the field are

$$E_z(r, z) = -iMG_0(r, a)e^{ikz} - \frac{Q}{4\pi^2} \int_{-\infty}^{\infty} dq Z(q, k) e^{iqz} J_0(r\sqrt{k^2 - q^2}), \tag{5}$$

$$E_r(r, z) = \gamma M G_1(r, a) e^{ikz} + i \frac{Q}{4\pi^2} \int_{-\infty}^{\infty} q \, dq Z(q, k) e^{iqz} \frac{J_1(R\sqrt{k^2 - q^2})}{\sqrt{k^2 - q^2}}, \tag{6}$$

where Q is the charge of the particle, $M = Qk/\pi c \gamma^2$, and

$$G_1(r, a) = K_1(kr/\gamma) + I_1(kr/\gamma) \frac{K_0(ka/\gamma)}{I_0(ka/\gamma)}, \tag{7}$$

$$G_0(r, a) = K_0(kr/\gamma) - I_0(kr/\gamma) \frac{K_0(ka/\gamma)}{I_0(ka/\gamma)}. \tag{8}$$

Notice that in the limit of $\gamma \gg 1$

$$\gamma M G_1(a, a) = \frac{Z_0 Q}{4\pi^2 a}, \qquad Z_0 = 377\ \Omega. \tag{9}$$

To get the correct behavior of the fields at $|z| \to \infty$, k^2 has to be understood as having an infinitely small positive imaginary part $k^2 \to k^2 + i\epsilon$.

The impedance is defined as the component of the radiated field synchronous with the particle [the second term in Eq. (5)]

$$Z_{\text{imp}} = -\frac{2\pi}{Q} \int_{-\infty}^{\infty} dz e^{-ikz} E_z^{\text{rad}}(r = 0, z), \tag{10}$$

and is given by the value of the (unknown) amplitude $Z(q, k)$ at k; i.e., $Z^{\text{imp}} = Z(k, k)$.

Matching the radial components of the field at $r = a$ defines the amplitudes A_n:

$$A_n = \frac{\int_{-g/2}^{g/2} dz f_n^r(a, z) E_r(a, z)}{\int_{-g/2}^{g/2} dz [f_n^r(a, z)]^2}. \tag{11}$$

The boundary condition for the longitudinal component of the field in the waveguide for $|z| > g/2$, and the matching condition for longitudinal fields for $|z| < g/2$ at $r = a$ may be written in one equation:

$$-\frac{Q}{4\pi^2} \int_{-\infty}^{\infty} dq Z(q, k) e^{iqz} J_0(a\sqrt{k^2 - q^2}) = \Theta(g/2 - |z|) \sum_n A_n f_n^z(a, z), \tag{12}$$

where the step function $\Theta(x) = 1$ for $x > 0$ and $\Theta(x) = 0$ for $x < 0$.

Now the Fourier transform of this equation gives the following integral equation for $Z(q, k)$:

$$Z(q, k) = -\frac{1}{2\pi J_0(a\sqrt{k^2 - q^2})} \sum_n j_n^z(q) C_n(k)$$
$$\times \left[\frac{Z_0}{a} j_n^r(k) + i \int_{-\infty}^{\infty} u \, du Z(u, k) j_n^r(u) \frac{J_1(a\sqrt{k^2 - u^2})}{\sqrt{k^2 - u^2}} \right]. \tag{13}$$

Here the following notations are introduced:

$$j_n^{(r)}(q) = \int_{-g/2}^{g/2} dz e^{iqz} \begin{pmatrix} \sin \lambda_n z \\ -\cos \lambda_n z \end{pmatrix}, \tag{14}$$

$$j_n^{(z)}(q) = \int_{-g/2}^{g/2} dz e^{-iqz} \begin{pmatrix} \cos \lambda_n z \\ \sin \lambda_n z \end{pmatrix}, \tag{15}$$

$$C_n(k) = \frac{2}{g} \frac{\mu_n}{\lambda_n} \frac{[J_0(a\mu_n) N_0(b\mu_n) - N_0(a\mu_n) J_0(b\mu_n)]}{[J_1(a\mu_n) N_0(b\mu_n) - N_1(a\mu_n) J_0(b\mu_n)]}. \tag{16}$$

Generally speaking, we can expect that in the high-frequency limit Eq. (13) can be solved by iterations. However, the coefficients $C_n(k)$ in it have an infinite number of singularities, and therefore the method of iterations fails for the values of k that are close to a singularity. Hence, the integral equation has to be rewritten using the exact solution in the proximity of the singularity before the method of iterations can be applied.

3. THE FINITE RESONANCE WIDTH APPROXIMATION

We search for the solution of Eq. (13) in the following form:

$$Z(q, k) = -\frac{1}{2\pi J_0(a\sqrt{k^2 - q^2})} \sum_n j_n^{(z)}(q) B_n(k). \tag{17}$$

Then the functions $B_n(k)$ satisfy an infinite set of algebraic equations

$$B_n(k)[C_n^{-1} + i\Gamma_n(k)] = \frac{Z_0}{a} j_n^{(r)}(k) - i \sum_{m \neq n} \Gamma_{nm}(k) B_m(k), \tag{18}$$

where

$$\Gamma_{nm}(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} u\, du j_n^{(r)}(u) j_m^{(z)}(u) \frac{J_1(a\sqrt{k^2 - u^2})}{\sqrt{k^2 - u^2}\, J_0(a\sqrt{k^2 - u^2})}, \tag{19}$$

and the function $\Gamma_n(k) = \Gamma_{nn}(k)$. Since functions $j_n^{(r)}$ and $j_m^{(z)}$ for $n \neq m$ oscillate with different frequencies, nondiagonal Γ_{nm} is much smaller than Γ_n for the same value k. This assumption is similar to the basic assumption of the random phase approximation, commonly used in nuclear and plasma physics.

The first two iterations of Eq. (18) give

$$\begin{aligned} Z(q, k) = &-\frac{Z_0}{2\pi a} \sum_n \frac{j_n^{(z)}(q) j_n^{(r)}(k)}{J_0(a\sqrt{k^2 - q^2})[C_n^{-1}(k) + i\Gamma_n(k)]} \\ &\times \left[1 - \frac{i}{j_n^{(r)}(k)} \sum_{m \neq n} \frac{\Gamma_{nm}(k) j_m^{(r)}(k)}{C_m^{-1}(k) + i\Gamma_m(k)}\right]. \end{aligned} \tag{20}$$

Due to the above-mentioned interference of different modes we can expect a good convergence of this expansion even in the close vicinity of a resonance.

In the zeroth approximation the solution of the integral of Eq. (13) for an arbitrary value of q is given now as the sum:

$$Z(q, k) = -\frac{Z_0}{2\pi a} \sum_n \frac{j_n^{(z)}(q) j_n^{(r)}(k)}{J_0(a\sqrt{k^2 - q^2})[C_n^{-1}(k) + i\Gamma_n(k)]}. \tag{21}$$

This solution differs from the first term of Eq. (13) only by the presence of the term $i\Gamma_n$ in the denominator. If we neglect here all the Γ_n terms, the obtained solution coincides with the δ-functional solution for the closed pillbox cavity.

Near a zero $\hat{k}$ of the function $C_n^{-1}(k)$, the function Z has the well-known Breit-Wigner form, so that $\Gamma_n(\hat{k})$ of the function $C_n^{-1}(k)$, the function Z has the well-known Breit-Wigner form, so that $\Gamma_n(\hat{k})$ defines the width of the corresponding resonance. The impedance in zeroth approximation is

$$Z_{\text{imp}}(k) = -\frac{Z_0}{2\pi a} \sum_n \frac{j_n^{(z)}(k) j_n^{(r)}(k)}{C_n^{-1}(k) + i\Gamma_n(k)}. \tag{22}$$

The numerator of each term in this expression is purely imaginary:

$$j_n^{(z)}(k) j_n^{(r)}(k) = \frac{4ik\lambda_n}{(k^2 - \lambda_n^2)^2} \sin^2 \frac{g}{2}(k - \lambda_n). \tag{23}$$

The coefficients $C_n(k)$ are real both for $k > \lambda_n$ and for $k < \lambda_n$. For $\mu_n a > 1$ they are asymptotically

$$C_n(k) \approx \frac{2\mu_n}{g\lambda_n} \tan[(b - a)\mu_n]. \tag{24}$$

The zeros of denominators in Eq. (22) define the positions of poles of the impedance considered as a function of a complex wave number k. Consequently, each function $C_n(k)$ has an infinite number of zeros that are located at the values of k given by

$$k_{nl} = \left[\lambda_n^2 + \left(\frac{\pi}{b-a}\right)^2\left(l+\frac{1}{2}\right)^2\right]^{1/2} \qquad (n,\, l \text{ positive integers}). \tag{25}$$

Notice that all $k_{nl} > \lambda_n$.

The imaginary part of $\Gamma_n(k)$ gives small shifts of the resonant frequencies. The real part of $\Gamma_n(k)$ shifts the pole from the real axis and produces the real part of the impedance.

The function $\Gamma_n(k)$ can be calculated in the following way. Substitute Eq. (14) for $j_n^{(r)}$ with the integration over z and Eq. (15) for $j_n^{(z)}$ with the integration over z' into Eq. (19) for Γ_{nn}. The integration over u can now be performed by closing the contour of the integration by adding a circle of infinitely large radius that lies in the upper half-plane when $z - z' > 0$ and in the lower half-plane when $z - z' < 0$. The integral is now given as the sum of residues at the poles defined by the equation $J_0(\nu_m) = 0$ (where $m = 1, 2, \ldots, \infty$) and positioned at

$$u_m = \sqrt{k^2 - \frac{\nu_m^2}{a^2}}. \tag{26}$$

$\Gamma_n(k)$ can now be represented as the following sum:

$$\Gamma_n(k) = \frac{i}{a}\sum_m \int_{-g/2}^{g/2} dz \int_{-g/2}^{z} dz' e^{iu_m(z-z')} \sin[\lambda_n(z-z')]. \tag{27}$$

As can be seen from this expression, only roots with indices that satisfy the condition

$$\nu_m < ka \tag{28}$$

contribute to the $\operatorname{Re}\Gamma_n$. Hence, it is represented by a finite sum

$$\operatorname{Re}\Gamma_n(k) = -\frac{1}{a}\sum_{m=1}^{m=M}\left[\frac{\sin^2 g(u_m-\lambda_n)/2}{(u_m-\lambda_n)^2} - \frac{\sin^2 g(u_m+\lambda_n)/2}{(u_m+\lambda_n)^2}\right], \tag{29}$$

with $\nu_M < ka < \nu_{M+1}$. The rest of the roots that correspond to the evanescent waves contribute only to the imaginary part of Γ_n:

$$\begin{aligned}\operatorname{Im}\Gamma_n(k) = {} & \frac{1}{2a}\sum_{m=1}^{m=M}\left[\frac{\sin g(u_m-\lambda_n)}{(u_m-\lambda_n)^2} - \frac{\sin g(u_m+\lambda_n)}{(u_m+\lambda_n)^2} - \frac{2g\lambda_n}{u_m^2-\lambda_n^2}\right] \\ & + \frac{1}{a}\sum_{m=M}^{\infty}\frac{1}{(\hat{u}_m^2+\lambda_n^2)^2}[g\lambda_n(\hat{u}_m^2+\lambda_n^2) - 2\hat{u}_m\lambda_n + 2\hat{u}_m\lambda_n e^{-g\hat{u}_m}\cos(g\lambda_n)],\end{aligned} \tag{30}$$

where

$$\hat{u}_m = \left(\frac{\nu_m^2}{a^2} - k^2\right)^{1/2} \tag{31}$$

Since $\operatorname{Re}\Gamma_n(k) = 0$ for all $k < \nu_1/a$, the impedance below the cut-off frequency for the pipe is purely imaginary.

Consider now the case when the geometry of the cavity allows existence of the range $\nu_m/a < k < \pi/g$. The impedance in this range has both real and imaginary parts, but the first resonance lies above this range of frequencies. One obtains the nonresonant behavior of the real part of the impedance as a consequence of the finite widths of the resonances.

In the high-frequency limit, when $kg \gg 1$ and $ka \gg 1$, the main contribution to the sum in the expression for the real part of $\Gamma_n(k)$ comes from the large m. In this case the summation over m can be replaced by the integration

$$\operatorname{Re}\Gamma_n(k) \approx -\frac{g}{2\pi}\int_0^1 \frac{x\,dx}{\sqrt{1-x^2}} \frac{\sin^2 \frac{kg}{2}(x-\lambda_n/k)}{\frac{kg}{2}(x-\lambda_n/k)^2}. \tag{32}$$

The term $[\sin^2 g/2(kx-\lambda_n)]/(kx-\lambda_n)^2$ oscillates rapidly, and its contribution is small. For $kg \gg 1$ the last factor in the integrand behaves like a δ-function $\pi\delta(x-\lambda_n/k)$. This gives for $\lambda_n < k$

$$\operatorname{Re}\Gamma_n(k) = -\frac{g\lambda_n}{2\sqrt{k^2-\lambda_n^2}}. \tag{33}$$

For $(k-\lambda_n) < \pi/g$

$$\operatorname{Re}\Gamma_n(k) \approx -\left(\frac{g}{2}\right)^{3/2}\sqrt{\frac{k}{\pi}}.$$

The value of $\operatorname{Re}\Gamma_n(k)$ for $\lambda_n > k$ is small. This result means that the real part of the impedance [Eq. (22)] in the high-frequency limit is produced mainly by the terms with $\lambda_n < k$.

In Fig. 2 the estimation [Eq. (33)] is compared with numerical calculations according to Eq. (29).

The estimate of the imaginary part [Eq. (30)] gives for $\lambda_n > k + \pi/g$

$$\operatorname{Im}\Gamma_n = \frac{gk}{2\sqrt{k^2-\lambda_n^2}},$$

and for $|k-\lambda_n| < \pi/g$

$$\operatorname{Im}\Gamma_n \approx \frac{g}{4}\sqrt{\frac{2gk}{\pi}}.$$

The real part of the impedance [Eq. (22)] in the high-frequency limit can be evaluated similarly to the estimation of the real part $\operatorname{Re}\Gamma_n$. The main contribution is given by the term $\lambda_n \approx k$. The shift given by $\operatorname{Im}\Gamma_n$ is unimportant. Basically, the answer in this first approximation is given simply by the average density of the resonances [zeros in the denominator of Eq. (22)] and, therefore, coincides with the Ref. 1 result. The details of calculation can be found in the

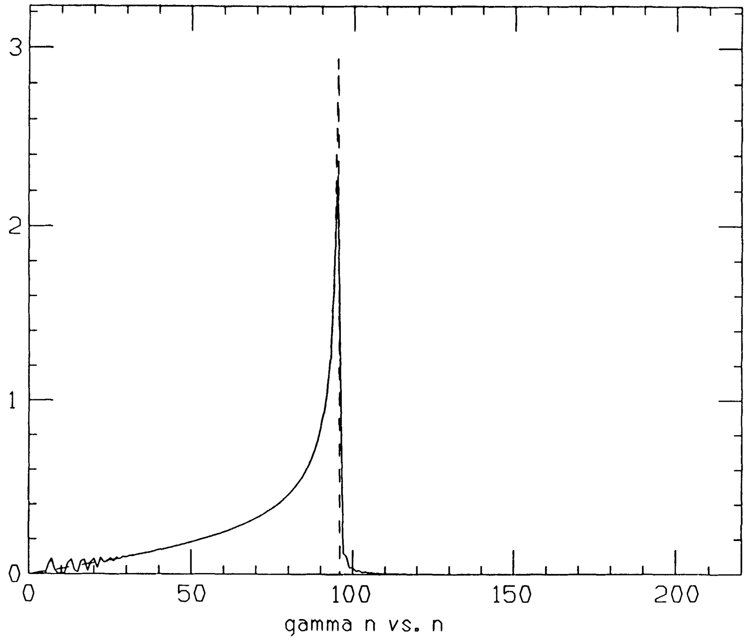

FIGURE 2 $\mathrm{Re}\,\Gamma_n(k)$ vs. n. The solid line is for the numeric result according to Eq. (29); the dashed line is plotted according to the estimate [Eq. (33)]. The parameters used were $ka = 500.0$, $g/2a = 0.25$.

appendix. The result for $kg \gg 1$, $ka \gg 1$, $k(b-a) \gg 1$ is

$$\mathrm{Re}\, Z_{\mathrm{imp}} = \frac{Z_0}{2\pi} \sqrt{\frac{g}{\pi a}} \frac{1}{\sqrt{ka}}, \tag{34}$$

which decreases as $(ka)^{-1/2}$.

The ratio of the result of numerical calculations of the $\mathrm{Re}\, Z_{\mathrm{imp}}$ according to Eq. (22) to the analytic result [Eq. (34)] is shown in Fig. 3. On average, the ratio is equal to one. This result agrees with that obtained in Ref. 1. The small rapid oscillations with typical period $ka \approx 1$, shown in Fig. 4, are due to the numerator in Eq. (22).

The impedance [Eq. (34)] does not depend on the outer radius b of the cavity. This means that the impedance is produced mainly by the diffracted waves and not by reflected ones. It should be kept in mind, though, that the condition $k(b-a) \gg 1$ has been used essentially in the derivation of Eq. (34). Hence, this feature of the impedance does not hold for extremely small b, as is clear from simple physical considerations. The more general formula [Eq. (22)] is still valid and should be used under these conditions. The same is true for the low-frequency range, where Eq. (22) can be used to evaluate the widths and eigenfrequencies of separate resonances.

The estimation of the corrections to the result [Eq. (34)] due to the next iterations in Eq. (20) is given in the appendix and proved to be of order of $(ka)^{-1/2}$, so that the corrections are negligibly small in the high-frequency limit.

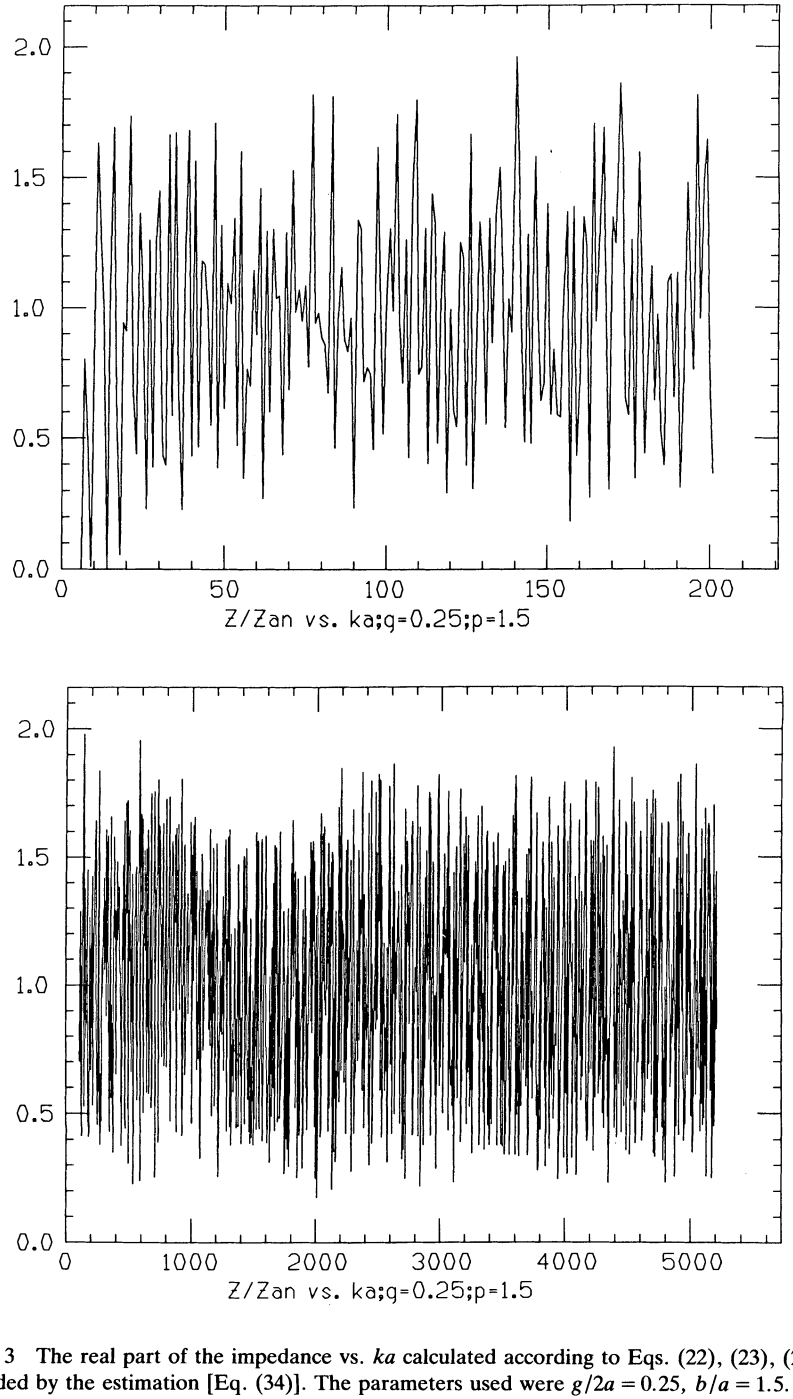

FIGURE 3 The real part of the impedance vs. *ka* calculated according to Eqs. (22), (23), (24), and (33), divided by the estimation [Eq. (34)]. The parameters used were $g/2a = 0.25$, $b/a = 1.5$.

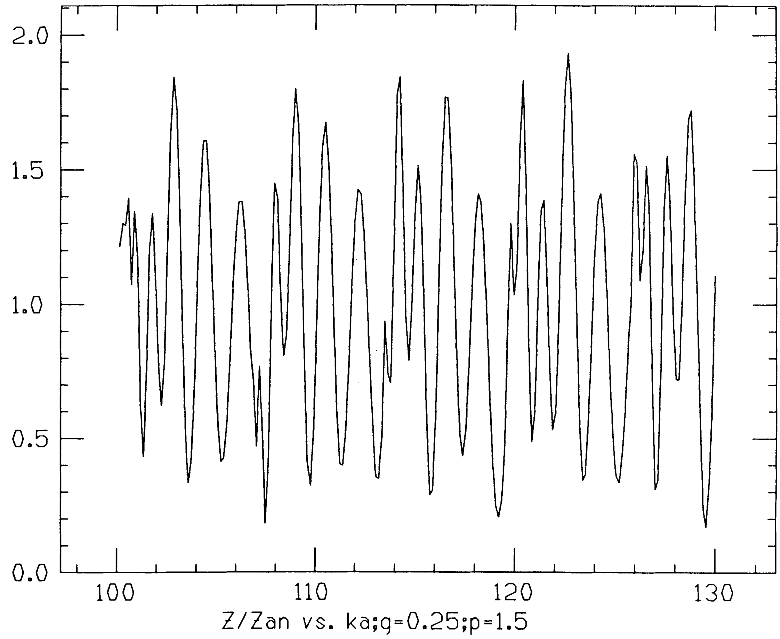

FIGURE 4 Magnification of Fig. 3 for the same parameters.

The question of the convergence of the whole series of iterations nevertheless remains and requires more thorough analysis. Here we would like to mention only that we do not expect that iterations in the problem under consideration can give asymptotically divergent series. In all known cases where asymptotic series occur (method of canonical transformations in nonlinear dynamics, quasi-energies in quantum mechanics, etc.) they are the result of a description of a system in terms that are intrinsically wrong for the system (reduction of the nonintegral system to the integrable one in the method of canonical transformations, description of a system with continuous spectrum as a system with discrete spectrum for the quasi-energy approach, etc.) and can be valid only in some restricted sense, usually for only a short time. The approach used above is consistent with the physical problem we consider, so that the higher iterations can give small corrections but cannot change the result drastically.

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APPENDIX

Here we present the evaluation of some sums encountered in the main text.

Let us first start with the expression [Eq. (22)] for the impedance. Substituting Eq. (23) for the product $j_n^{(z)}(k)j_n^{(r)}(k)$, Eq. (24) for $C_n(k)$, and Eq. (33) for the real part of the function $\Gamma_n(k)$, one gets a sum over n that can be replaced by an integral over $x = \lambda_n/k$ when kg is large:

$$\text{RE } Z_{\text{imp}} = \frac{2Z_0}{\pi^2 ka}\int_0^1 \frac{dx}{(1-x^2)^{3/2}} \sin^2\left[\frac{gk}{2}(1-x)\right] \sin^2[(b-a)k\sqrt{1-x^2}]. \quad \text{(A-1)}$$

The integrand in the integral in this expression has no singularity at the point $x=0$. It has a rather-sharp maximum near the upper end of the interval of the integration and then decreases to zero for $x=1$. The width of the maximum is of the order of π/gk. The last $\sin^2$ term oscillates rapidly everywhere except in the narrow interval with the width of the order of $[k(b-a)]^{-2}$ near the point $x=1$ and can be replaced by 1/2. The interval of integration can then be broken into two parts: $0<x<1-\pi/gk$ and $1-\pi/gk<x<1$. In the first interval both the $\sin^2$ terms again can be replaced by 1/2, after which the integration is easily done.

In the second interval $x \approx 1$. The integral can be simplified as

$$\int_{1-a}^1 \frac{dx}{(1-x)^{3/2}} \sin^2\frac{gk}{2}(1-x) = \sqrt{gk}\int_0^{\pi/2} dt \frac{\sin^2 t}{t^{3/2}} \approx \sqrt{\pi gk}\,. \quad \text{(A-2)}$$

By combining the values of the both integrals, one gets the result stated in the text.

The estimation of the second iteration in Eq. (20) is more elaborate. The second term in Eq. (20) replaces $j_n^{(r)}$ by $(j_n^{(r)})_{\text{eff}}$:

$$(j_n^{(r)})_{\text{eff}} = j_n^{(r)} - i\sum_{m\neq n}\frac{\Gamma_{nm}(k)}{C_m^{-1}+i\Gamma_m(k)} j_m^{(r)}(k). \quad \text{(A-3)}$$

This relation can be considered as rotation of a coordinate system in Hilbert space. For the calculation of a real part of the impedance, Eq. (A3) should be understood as

$$(j_n^{(r)})_{\text{eff}} = j_n^{(r)} - \sum\frac{\Gamma_m \operatorname{Re}\Gamma_{nm}}{(C_m^{-1})^2+\Gamma_m^2} j_m^{(r)}(k).$$

The second term gives a small effect if

$$\sum_{m \neq n} \left| \frac{\Gamma_m \, \mathrm{RE} \, \Gamma_{nm}}{(C_m^{-1})^2 + \Gamma_m} \right|^2 \ll 1. \tag{A-4}$$

This expression depends on the coefficients Γ_{mn}, which can be calculated as the sum over the residues in the zeroth term of the Bessel function J_0 in Eq. (19):

$$\Gamma_{nm} = \frac{i}{2a} \sum_l \int dz \int dz' e^{iu_l|z-z'|} \cos(\lambda_m z' - \psi_m) \sin(\lambda_n z - \psi_n) \epsilon(z - z'),$$

where

$$u_l = \sqrt{k^2 - \left(\frac{\nu_l}{a}\right)^2}; \qquad J_0(\nu_l) = 0,$$

$$\epsilon(z) = \begin{pmatrix} 1 & z > 0 \\ -1 & z < 0 \end{pmatrix},$$

$$\psi_n = \begin{pmatrix} 0 \\ \pi/2 \end{pmatrix}; \qquad \lambda_n = \frac{\pi}{g} \begin{pmatrix} n \\ n + \frac{1}{2} \end{pmatrix} \quad \text{for} \quad n = \begin{pmatrix} \text{integer} \\ \text{half integer} \end{pmatrix}.$$

The integration over z, z' gives $\Gamma_{nm} \neq 0$ only if both indices are integers or both are half integers. In these cases

$$\Gamma_{nm} = (-1)^{n+m} \frac{\lambda_n}{a} \sum_l \frac{u_l(e^{i(u_l - \lambda_n)} - 1}{(u_l^2 - \lambda_n^2)(u_l^2 - \lambda_m^2)}.$$

Replacing the sum over l by the integration over $u_l = \sqrt{k^2 - (\pi l/a)^2}$, $\mathrm{Re}\,\Gamma_{nm}$ can be written as

$$\mathrm{Re}\,\Gamma_{nm} = (-1)^{n+m+1} \frac{\lambda_n}{\pi} \int_0^k \frac{u^2 \, du}{\sqrt{k^2 - u^2}} \frac{\sin^2 \frac{g}{2}(u_l - \lambda_n)}{(u^2 - \lambda_n^2)(u^2 - \lambda_m^2)}.$$

If $\lambda_n \approx k$, $\lambda_m \ll k$, then

$$\mathrm{Re}\,\Gamma_{nm} \approx (-1)^{n+m} \frac{\lambda_n}{\pi k^2} \frac{1}{kg(1 - \lambda_n^2/k^2)^{3/2}}. \tag{A-5}$$

The numeric calculations confirmed this result.

The correction [Eq. (A-4)] with $\mathrm{Re}\,\Gamma_n$ and $C_n(k)$ given by Eqs. (24) and (33) builds up due to the terms $\lambda_m \ll k$ and is of the order of

$$\frac{k^2}{4\pi^2} \mathrm{Re}\,\Gamma_{nm}^2.$$

As was shown above for the calculations of $\mathrm{Re}\, z_{\mathrm{imp}}$, essential λ_n are of the order of $k - \sqrt{k/g}$. Therefore the correction to the real part of the impedance turns out to be of the order of

$$\sqrt{\frac{1}{kg}},$$

so that iterations give a small correction in the high-frequency limit $kg \gg 1$.