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SU(3)-equivariant quiver gauge theories and nonabelian vortices

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Abstract: We consider SU(3)-equivariant dimensional reduction of Yang-Mills theory on Kähler manifolds of the form $M \times \mathrm{SU}(3)/H$, with $H = \mathrm{SU}(2) \times \mathrm{U}(1)$ or $H = \mathrm{U}(1) \times \mathrm{U}(1)$. The induced rank two quiver gauge theories on M are worked out in detail for representations of H which descend from a generic irreducible SU(3)-representation. The reduction of the Donaldson-Uhlenbeck-Yau equations on these spaces induces nonabelian quiver vortex equations on M, which we write down explicitly. When M is a noncommutative deformation of the space $\mathbb{C}^d$, we construct explicit BPS and non-BPS solutions of finite energy for all cases. We compute their topological charges in three different ways and propose a novel interpretation of the configurations as states of D-branes. Our methods and results generalize from SU(3) to any compact Lie group.

Keywords: Solitons Monopoles and Instantons, Non-Commutative Geometry, Field Theories in Higher Dimensions, Integrable Field Theories.

Contents

1. Introduction and summary

BPS type equations for gauge theories in higher dimensions were proposed long ago [1] as generalizations of the self-duality equations in four dimensions. For nonabelian gauge theory on a Kähler manifold the most natural BPS condition lies in the Donaldson-Uhlenbeck-Yau equations [2]. These equations arise in compactifications of superstring theory down to four-dimensional Minkowski spacetime as the condition for at least one unbroken supersymmetry. In this paper we will study the structure of solutions to these equations on product manifolds of the form $M_q \times G/H$, where G is the Lie group SU(3) and H is a closed subgroup of G.

Building on earlier work [3], nonabelian gauge theory on the manifold $M_q \times S^2$, the product of a real q-dimensional manifold M_q with lorentzian or riemannian nondegenerate metric and a two-sphere S^2, was considered in [4]. The SU(2)-equivariant dimensional reduction induces a Yang-Mills-Higgs theory on M_q which is a quiver gauge theory of rank one. For Kähler manifolds M_q, with $q = 2d$, the proper reduction of the Yang-Mills equations on $M_{2d} \times S^2$ induces quiver gauge theory equations on M_{2d}, and quiver vortex equations on M_{2d} in the BPS sector [5]. The Seiberg-Witten monopole equations [6] for $d = 2$ and the ordinary vortex equations [7] for $d = 1$ are particular instances of quiver vortex equations. It has also been shown that the ordinary vortex equations are integrable when M_2 is a compact Riemann surface of genus $g > 1$ [8]. In that case $M_2 \times S^2$ is a gravitational instanton, and the vortex equations are the compatibility conditions of two linear equations (Lax pair) so that the standard methods of integrable systems can be applied to the construction of their solutions. Explicit SU(2)-equivariant monopole, dyon and monopole-antimonopole pair solutions of the Yang-Mills equations on $M_2 \times S^2$ were also obtained in [9] for M_2 of lorentzian signature $(-+)$ and with the topology of $\mathbb{R}^2$ or $\mathbb{R} \times S^1$.

While the criteria for existence of solutions to all of these BPS quiver vortex equations are by now well-understood [10, 11], in practice it is usually quite difficult to write down explicit solutions of them. The construction of exact solutions is facilitated when M_{2d} is the noncommutative space $\mathbb{R}^{2d}_\theta$. In this instance both BPS and non-BPS solutions were obtained for various cases in [3, 4, 12 − 14]. In this paper we will extend these constructions by SU(3)-equivariant dimensional reduction over the coset spaces SU(3)/H, where $H = \mathrm{SU}(2) \times \mathrm{U}(1)$ or $H = \mathrm{U}(1) \times \mathrm{U}(1)$. In the former case the coset space is the complex projective plane $\mathbb{C}P^2$. We will find that many aspects of the induced rank two quiver gauge theory on M_{2d} in this case are qualitatively similar to that obtained from the symmetric space $\mathbb{C}P^1 \times \mathbb{C}P^1$ [13]. However, the technical aspects are much more involved and some new features emerge from the nonabelian SU(2) instanton degrees of freedom which now reside at the vertices of the quiver. In the latter case the coset space is the six-dimensional homogeneous manifold Q_3, and the qualitative features of the rank two quiver gauge theories are much different in this case, even though the quiver vertex degrees of freedom involve two U(1) monopole charges as in the $\mathbb{C}P^1 \times \mathbb{C}P^1$ case [13]. This space has appeared before in a variety of different physical contexts, such as in connection with the dynamics of M-theory on a manifold of G_2 holonomy which develops a conical singularity [15], and for the dimensional reduction of ten-dimensional supersymmetric gauge theories over six-dimensional coset spaces in much the same spirit as our more general reductions [16]. In these latter applications the usage of the non-symmetric space Q_3 induces a four-dimensional field theory with softly broken $\mathcal{N} = 1$ supersymmetry. We expect that our constructions will have direct applications in these classes of models, though this is left for future work.

The outline of this paper is as follows. Throughout we present detailed constructions and examples of the pertinent quiver gauge theories. Although most of our analyses are model dependent, our techniques and results apply to reductions over more general coset spaces G/H, where G is a compact Lie group and H is a closed subgroup of G. In section 2 we give explicit constructions of the quivers and their representations which will underlie the gauge theories considered in this paper. In section 3 we carry out the SU(3)-equivariant dimensional reductions and explicitly construct the fields of the quiver gauge theory. In section 4 we study the BPS equations of the quiver gauge theory which describe nonabelian quiver vortices. In section 5 we construct explicit BPS and non-BPS solutions of finite energy on the noncommutative deformation of $M_{2d} = \mathbb{C}^d$. Finally, in section 6 we compute topological charges of our noncommutative instanton solutions from various points of view, and use the constructions to propose a novel interpretation of the configurations as states of D-branes.

2. Homogeneous bundles and quiver representations

In this section we will give purely algebraic constructions of the quivers that will play a role throughout this paper, following the general formalism developed in [10]. They are based on the representation theory of the Lie group $G = \mathrm{SU}(3)$, and are naturally associated to homogeneous vector bundles whose fibres transform under irreducible representations of SU(3). We will begin with the simplest instance of the fundamental representation as illustration of the method. Then we will give the general construction, and illustrate the formalism with various other explicit examples.

2.1 Fundamental representations

We are interested in the geometry of coset spaces of the form G/H, where H is a closed subgroup of $G = \mathrm{SU}(3)$. Given a finite-dimensional representation $\underline{V}$ of H, the corresponding induced, homogeneous hermitean vector bundle over G/H is given by the fibred product

$$\mathcal{V} = G \times_H \underline{V} \ . \tag{2.1}$$

Every G-equivariant bundle of finite rank over G/H, with respect to the standard transitive action of G on the homogeneous space, is of the form (2.1). If $\underline{V}$ is irreducible, then H is the structure group of the associated principal bundle. The category of such homogeneous bundles is equivalent to the category of finite-dimensional representations of a certain quiver with relations, whose structure is determined entirely by the subgroup H. In contrast to the treatment of [10], we restrict to those representations $\underline{V}$ which descend from some irreducible representation of SU(3) by restriction to H. We will now give an elementary construction of this quiver representation in the simplest case where $\underline{V} = \underline{C}^{1,0}\big|_H$ is the restriction of the three-dimensional fundamental representation of SU(3).

The Dynkin diagram for SU(3) consists of a pair of roots α_1, α_2. The complete set Δ of non-null roots is $\pm\,\alpha_1, \pm\,\alpha_2, \pm\,(\alpha_1{+}\alpha_2)$, with the inner products $(\alpha_1,\alpha_1) = (\alpha_2,\alpha_2) = 1$ and $(\alpha_1,\alpha_2) = -\frac{1}{2}$ so that $(\alpha_1+\alpha_2, \alpha_1+\alpha_2) = 1$. For the system Δ^+ of positive roots we take $\alpha_1 = (1,0)$, $\alpha_2 = \frac{1}{2}\,(-1,\sqrt{3}\,)$ and $\alpha_1+\alpha_2 = \frac{1}{2}\,(1,\sqrt{3}\,)$. Let e_{ij} be the matrix units obeying $e_{ij}\,e_{kl} = \delta_{jk}\,e_{il}$, and abbreviate $e_i := e_{ii}$. The generators of SU(3) for the Cartan-Weyl basis in the 3×3 fundamental representation are then given by the Chevalley generators

$$E_{\alpha_1} \ = \ e_{12}\,, \qquad E_{\alpha_2} \ = \ e_{23}\,, \qquad \text{and} \qquad E_{\alpha_1+\alpha_2} \ := \ [E_{\alpha_1}, E_{\alpha_2}] \ = \ e_{13} \tag{2.2}$$

along with

$$E_{-\alpha_1} \ = \ E_{\alpha_1}^\dagger \ = \ e_{21}\,, \quad E_{-\alpha_2} \ = \ E_{\alpha_2}^\dagger \ = \ e_{32}\,, \quad \text{and} \quad E_{-\alpha_1-\alpha_2} \ = \ E_{\alpha_1+\alpha_2}^\dagger \ = \ e_{31}\,. \tag{2.3}$$

We take

$$H_{\alpha_1} \;=\; e_1 - e_2 \qquad \text{and} \qquad H_{\alpha_2} \;=\; e_1 + e_2 - 2e_3 \tag{2.4}$$

as the generators of the Cartan subalgebra $\mathrm{u}(1) \oplus \mathrm{u}(1)$. The commutation relations are

$$\begin{aligned} [H_{\alpha_1}, E_{\pm\,\alpha_1}] \;&=\; \pm\,2E_{\pm\,\alpha_1} \qquad \text{and} \qquad & [H_{\alpha_2}, E_{\pm\,\alpha_1}] \;&=\; 0\,, \\ [H_{\alpha_1}, E_{\pm\,\alpha_2}] \;&=\; \mp\,E_{\pm\,\alpha_2} \qquad \text{and} \qquad & [H_{\alpha_2}, E_{\pm\,\alpha_2}] \;&=\; \pm\,3E_{\pm\,\alpha_2} \end{aligned} \tag{2.5}$$

along with

$$\begin{aligned} [E_{\alpha_1}, E_{-\alpha_1}] \;&=\; H_{\alpha_1}\,, \qquad & [E_{\alpha_2}, E_{-\alpha_2}] \;&=\; \frac{1}{2}\,(H_{\alpha_2} - H_{\alpha_1})\,, \\ [E_{\alpha_1+\alpha_2}, E_{-\alpha_1-\alpha_2}] \;&=\; \frac{1}{2}\,(H_{\alpha_1} + H_{\alpha_2})\,, \end{aligned} \tag{2.6}$$

the Lie brackets

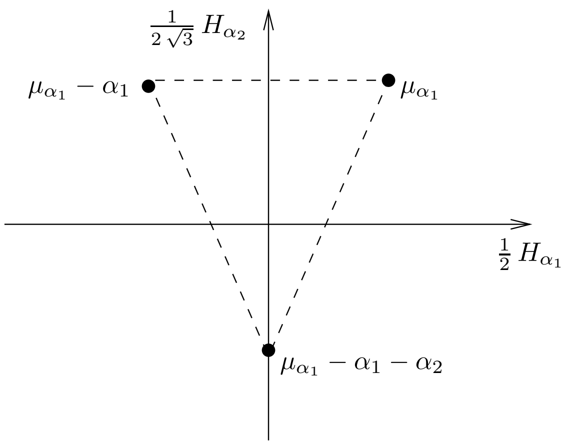
$$[H_{\alpha_1}, E_{\pm\,(\alpha_1+\alpha_2)}] \;=\; \pm\,E_{\pm\,(\alpha_1+\alpha_2)} \qquad \text{and} \qquad [H_{\alpha_2}, E_{\pm\,(\alpha_1+\alpha_2)}] \;=\; \pm\,3E_{\pm\,(\alpha_1+\alpha_2)}\,, \tag{2.7}$$

and

$$\begin{aligned} [E_{\pm\,\alpha_1}, E_{\pm\,\alpha_2}] \;&=\; E_{\pm\,(\alpha_1+\alpha_2)}, \qquad & [E_{\pm\,\alpha_1}, E_{\mp\,(\alpha_1+\alpha_2)}] \;&=\; \mp\,E_{\mp\,\alpha_2}\,, \\ [E_{\pm\,\alpha_2}, E_{\mp\,(\alpha_1+\alpha_2)}] \;&=\; \pm\,E_{\mp\,\alpha_1}\,. \end{aligned} \tag{2.8}$$

All other Lie brackets of the generators vanish.

The fundamental weights are $\mu_{\alpha_1} = \frac{1}{2}\,\big(1, \frac{1}{\sqrt{3}}\big)$ and $\mu_{\alpha_2} = \big(0, \frac{1}{\sqrt{3}}\big)$, with μ_{α_1} the highest weight of the defining representation $\underline{C}^{1,0}$. The corresponding weight diagram is



(2.9)

and the Young tableau consists of a single box. There are two homogeneous spaces of interest that we now analyse in turn.

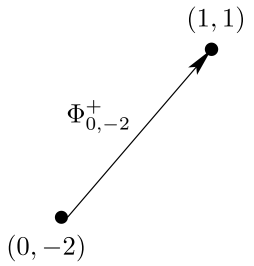
Symmetric $\underline{C}^{1,0}$ quiver. Our first example is the four-dimensional complex projective plane

$$\mathbb{C}P^2 = \mathrm{SU}(3) \,/\, \mathrm{S}\big(\mathrm{U}(2) \times \mathrm{U}(1)\big) \tag{2.10}$$

which is a symmetric space whose isometry group is isomorphic to SU(3). We can use the weight diagram (2.9) to decompose the fundamental representation of SU(3) as a representation of the subgroup $H = \mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(1)) \cong \mathrm{SU}(2) \times \mathrm{U}(1)$ (locally) to get

$$\underline{C}^{1,0}\big|_{\mathrm{SU}(2)\times\mathrm{U}(1)} = \underline{(1,1)} \ \oplus \ \underline{(0,-2)} \,, \tag{2.11}$$

since the restriction of the SU(3) operators $E_{\pm\,\alpha_1}$ to SU(2) shifts vertices along the horizontal directions of the weight diagram. The first integer $n = 2I$ in the pairs on the right-hand side labels twice the isospin I (the eigenvalue of H_{α_1}) such that $(n+1)$ is the dimension of the irreducible SU(2) representation and the second integer $m = 3Y$ labels three times the hypercharge Y which is the U(1) magnetic charge (the eigenvalue of H_{α_2}). We use the two pairs of integers in (2.11) to label vertices in a directed graph. The arrow between the two vertices comes from increasing the H_{α_1} and H_{α_2} eigenvalues using the raising operators E_{α_2} and $E_{\alpha_1+\alpha_2}$, via the commutation relations (2.5) and (2.7). This gives the elementary quiver diagram



(2.12)

associated to the symmetric space $\mathbb{C}P^2$ for the fundamental representation of SU(3), where $\Phi^+_{0,-2}$ is the H-equivariant intertwiner between the two H-modules induced by the action of the SU(3) raising operators E_{α_2} and $E_{\alpha_1+\alpha_2}$. Later on this quiver will be associated to a holomorphic triple representing the basic unstable brane-antibrane pair. A completely analogous construction applies to the conjugate representation $\underline{C}^{0,1}$, with highest weight μ_{α_2}, whose weight diagram is the counterclockwise rotation of (2.9) through angle $\frac{\pi}{3}$ about the origin. The Young diagram consists of a single column of two boxes, and the analog of the decomposition (2.11) is

$$\underline{C}^{0,1}\big|_{\mathrm{U}(1)\times\mathrm{SU}(2)} = \underline{(1,-1)} \ \oplus \ \underline{(0,2)} \ . \tag{2.13}$$

Non-symmetric $\underline{C}^{1,0}$ quiver. Our second example is the six-dimensional reducible flag manifold $\mathbb{F}(1,2;3) \cong Q_3$ given by

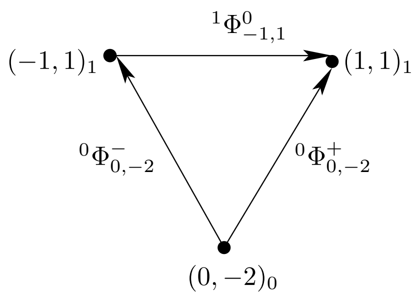
$$Q_3 = \mathrm{SU}(3) \,/\, \mathrm{U}(1) \times \mathrm{U}(1) \tag{2.14}$$

which is homogeneous but not symmetric. Now we decompose the fundamental representation of SU(3) as a representation of the maximal torus $H = T = \mathrm{U}(1) \times \mathrm{U}(1)$ of SU(3). This can be achieved starting from (2.11) by decomposing the fundamental representation of SU(2) as a U(1) representation to get $\mathbf{2}\big|_{\mathrm{U}(1)} = \underline{(1)} \oplus \underline{(-1)}$, where $\underline{(q)}$ denotes the irreducible one-dimensional representation of U(1) with magnetic charge $q \in \mathbb{Z}$ which is twice the third component of isospin of an irreducible SU(3) representation. It follows that

$$\underline{C}^{1,0}\big|_{\mathrm{U}(1)\times\mathrm{U}(1)} = \underline{(1,1)_1} \ \oplus \ \underline{(-1,1)_1} \ \oplus \ \underline{(0,-2)_0} \,, \tag{2.15}$$

JHEP08(2008)093

where now the pairs label the U(1) charges $(q, m)_n$, $m = 3Y$, of the torus T, and the subscripts label the original SU(3) isospin integer n which keeps track of multiplicities of states in the weight diagram. The three vertices are now joined by the actions of the raising operators $E_{\alpha_1+\alpha_2}$, E_{α_1} and E_{α_2} through the commutation relations (2.5) and (2.7), giving the quiver diagram



(2.16)

Compatibility with the commutation relations of the SU(3) raising operators implies the relation

$$ {}^0\Phi^+_{0,-2} - {}^1\Phi^0_{-1,1}\, {}^0\Phi^-_{0,-2} = 0 \ . \tag{2.17} $$

2.2 General constructions

We now turn to the general construction of the quiver diagram associated with the homogeneous bundle (2.1). As complex algebraic varieties, the homogeneous space G/H is diffeomorphic to the flag manifold $G^{\mathbb{C}}/P$, where P is a parabolic subgroup of $G^{\mathbb{C}} = \mathrm{SL}(3, \mathbb{C})$. Given an irreducible holomorphic representation $\underline{V}_{\,\mu}$ of P, corresponding to an integral dominant weight $\mu \in \Lambda_P^+$, the induced, irreducible, holomorphic homogeneous vector bundle over $G^{\mathbb{C}}/P$ is $\mathcal{O}_\mu := G^{\mathbb{C}} \times_P \underline{V}_{\,\mu}$. Let $L \subset P$ be the universal complexification of the subgroup H. Then there is a Levi decomposition $P = U \ltimes L$. By Engel's theorem, a holomorphic module $\underline{V}_{\,\mu}$ is an irreducible representation of P if and only if its restriction $\underline{V}_{\,\mu}\big|_L$ is an irreducible representation of the reductive Levi subgroup L and the action of U on $\underline{V}_{\,\mu}$ is trivial. Then $\mathcal{V}_\mu = G \times_H \underline{V}_{\,\mu}$ is the smooth, G-equivariant hermitean vector bundle underlying $\mathcal{O}_\mu$. The quiver is associated to an appropriate basis of weights of L which specifies the vertices. The action of U, extending the L-action to a P-action, specifies the arrows satisfying relations which invoke the commutation relations of the Lie algebra $\mathfrak{u}$ corresponding to U. Homogeneous bundles with the same slope will typically lie along diagonal lines of the quiver diagram [10, 11]. As we will see later on, the relations of the quiver correspond to integrability of the Dolbeault operator on the induced homogeneous vector bundle over $G^{\mathbb{C}}/P$, which will arise dynamically as BPS equations for a quiver gauge theory.

For each pair of non-negative integers (k, l) there is an irreducible holomorphic representation $\underline{C}^{k,l}$ of $\mathrm{SL}(3, \mathbb{C})$ of dimension

$$ d^{k,l} \ := \ \dim\big(\underline{C}^{k,l}\big) \ = \ \frac{1}{2}\,(k+l+2)\,(k+1)\,(l+1) \tag{2.18} $$

and highest weight $\mu = k\,\mu_{\alpha_1} + l\,\mu_{\alpha_2}$. The restriction $\underline{C}^{k,l}\big|_L$ then determines a finite-dimensional representation of this quiver, and conversely. Let us illustrate the construction explicitly on our two coset spaces of interest.

Symmetric $\underline{C}^{k,l}$ quiver. The complex projective plane can be modelled locally as the Grassmann manifold $\mathrm{Gr}(2,3) = \mathrm{GL}(3,\mathbb{C})/K \cong \mathbb{C}P^2$, where K is the stability subgroup of block upper triangular matrices preserving the two-dimensional lowest weight subspace of the complex vector space $\underline{C}^{1,0} \cong \mathbb{C}^3 = \mathbb{C}^2 \oplus \mathbb{C}$. Thus in this case P is the parabolic subgroup of block upper triangular matrices in $\mathrm{SL}(3,\mathbb{C})$. The Levi subgroup L is the group of unit determinant matrices in $\mathrm{GL}(2,\mathbb{C}) \times \mathbb{C}_*$, where $\mathbb{C}_* := \mathbb{C} \setminus \{0\}$. The commutative algebra $\mathfrak{u}$ corresponding to U is generated by E_{α_2} and $E_{\alpha_1+\alpha_2}$. The set of vertices Q_0 of the associated quiver consists of the weight vectors (n,m) of $\mathrm{U}(2) \cong \mathrm{SU}(2) \times \mathrm{U}(1)$ (locally) in $\mathrm{SL}(3,\mathbb{C})$ with respect to the basis $(H_{\alpha_1}, H_{\alpha_2})$. From the commutation relations (2.5) and (2.7) it follows that the arrow set Q_1 of the quiver is determined by the action of U on the weights as

$$(n,m) \longmapsto (n \pm 1, m+3)\,, \tag{2.19}$$

depending on which particular weight vectors (n,m) the raising operators $E_{\alpha_1+\alpha_2}$ and E_{α_2} act on. Since $\mathfrak{u}$ is an abelian algebra, the arrows (2.19) generate commutative quiver diagrams with quadratic holomorphic relations R around any elementary square of the quiver. These graphs are formally the same as the rectangular lattice quiver diagrams associated to the symmetric space $\mathbb{C}P^1 \times \mathbb{C}P^1$ [10, 13], which are generically a product of two chains.

For a fixed pair of non-negative integers (k,l), the irreducible SU(3) representation $\underline{C}^{k,l}$ determines a *finite* quiver vertex set $\mathsf{Q}_0(k,l)$ as follows. The quiver diagram can be obtained by collapsing the "horizontal" SU(2) representations to single nodes in the weight diagram for $\underline{C}^{k,l}$. Algebraically, the integer k is the number of fundamental representations $\underline{C}^{1,0}$ and l the number of conjugate representations $\underline{C}^{0,1}$ appearing in the tensor product construction of $\underline{C}^{k,l}$. This means that the Young diagram of $\underline{C}^{k,l}$ contains $k+l$ boxes in its first row and l boxes in its second row. To determine $\mathsf{Q}_0(k,l)$, we decompose $\underline{C}^{k,l}$ as a representation of $\mathrm{SU}(2) \times \mathrm{U}(1)$. From the weight diagram for $\underline{C}^{k,l}$ including multiplicities, one can extract the $\mathrm{SU}(2) \times \mathrm{U}(1)$ representation content for each row. For example, with $k > l$ one finds

$$\begin{aligned}
&m = k+2l\,, && n \ = \ k \\
&m = k+2l-3\,, && n \ = \ k+1\,,\ k-1 \\
&m = k+2l-6\,, && n \ = \ k+2\,,\ k\,,\ k-2 \\
& && \ \ \vdots \\
&m = k-l\,, && n \ = \ k+l\,,\ k+l-2\,,\ k+l-4\ ,\ \dots\ ,\ k-l \\
&m = k-l-3\,, && n \ = \ k+l-1\,,\ k+l-3\,,\ k+l-5\,,\ \dots\ ,\ k-l-1 \\
& && \ \ \vdots \\
&m = -2k+2l\,, && n \ = \ 2l\,,\ 2l-2\,,\ 2l-4\,,\ \dots\ ,\ 2\,,\ 0 \\
&m = -2k+2l-3\,, && n \ = \ 2l-1\,,\ 2l-3\,,\ 2l-5\,,\ \dots\ ,\ 3\,,\ 1 \\
& && \ \ \vdots \\
&m = -2k-l+3\,, && n \ = \ l+1\,,\ l-1 \\
&m = -2k-l\,, && n \ = \ l\ .
\end{aligned} \tag{2.20}$$

The set of quiver vertices (n, m) in this case generates a rectangular lattice tilted through angle $\frac{\pi}{4}$ with corners located at $(k, k+2l)$, $(k+l, k-l)$, $(0, -2k+2l)$ and $(l, -2k-l)$. The cases $k \leq l$ are treated similarly. Conversely, given any finite quiver associated to the symmetric space $\mathbb{C}P^2$, the ranges of the vertices determine a corresponding irreducible representation $\underline{C}^{k,l}$ of SU(3). The integers (n, m) have the same even/odd parity.

Non-symmetric $\underline{C}^{k,l}$ quiver. In this instance P is the Borel subgroup of (purely) upper triangular matrices in $\mathrm{SL}(3, \mathbb{C})$. The Levi subgroup is $L = (\mathbb{C}_*)^2 \subset \mathrm{SL}(3, \mathbb{C})$. The algebra $\mathfrak{u}$ is now generated by E_{α_1}, E_{α_2} and $E_{\alpha_1+\alpha_2}$, which is the non-abelian three-dimensional Heisenberg algebra with central element $E_{\alpha_1+\alpha_2}$. The set of vertices $(q, m)_n$ is now precisely the weight lattice $\Lambda \cong \mathbb{Z}^2$ of $\mathrm{SL}(3, \mathbb{C})$, and from the commutation relations (2.5) and (2.7) it follows that the action of U on the weight vectors is given by

$$\begin{aligned} E_{\alpha_1} : &\qquad (q, m)_n \longmapsto (q+2, m)_n \ , \\ E_{\alpha_2} : &\qquad (q, m)_n \longmapsto (q-1, m+3)_{n\pm 1} \ , \\ E_{\alpha_1+\alpha_2} : &\qquad (q, m)_n \longmapsto (q+1, m+3)_{n\pm 1} \ . \end{aligned} \tag{2.21}$$

The arrows of the quiver thus translate weight vectors by the set of positive roots. Moreover, the Heisenberg commutation relations induce linear terms in the relations expressing commutativity of the corresponding quiver diagrams. In contrast to the symmetric quivers above, in this case there can be multiple arrows emanating between two vertices due to degenerate weight vectors $(q, m)_n$ and $(q, m)_{n'}$ with $n \neq n'$.

Given an irreducible representation $\underline{C}^{k,l}$ of SU(3), the quiver diagram is now simply the weight diagram for $\underline{C}^{k,l}$. Thus the non-symmetric vertex set $\mathsf{Q}_0(k, l)$ is generated by lattice points $(q, m)_n$ with

$$q = -n \ , \ -n+2 \ , \ \ldots \ , \ n-2 \ , \ n \ . \tag{2.22}$$

Note that the symmetric vertex (n, m), representing a full isospin multiplet, can be obtained by collapsing the non-symmetric quiver vertices $(q, m)_n$ to (n, m). Given any finite quiver associated to the homogeneous space Q_3, the ranges of the vertices determine a corresponding representation of SU(3). Again, the integers $(q, m)_n$ have the same parity.

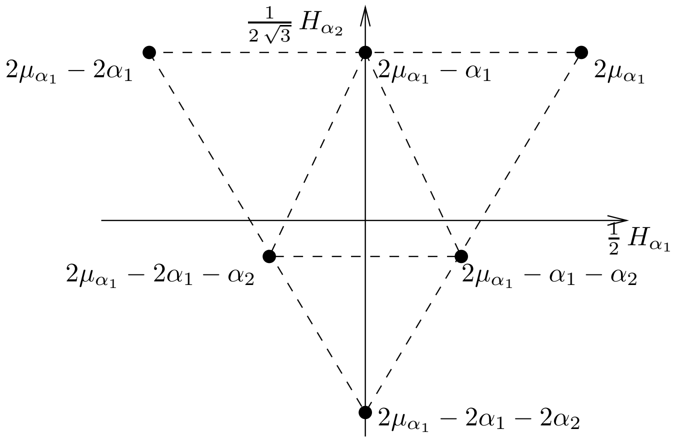
2.3 Examples

We conclude this section by giving some further explicit examples of quiver representations corresponding to irreducible SU(3) representations, as illustration of the general algorithm of section 2.2 above. From these examples the generic features of the symmetric and non-symmetric space quivers will become apparent.

Symmetric $\underline{C}^{2,0}$ quiver. Let us consider the six-dimensional representation $\underline{C}^{2,0}$ of

JHEP08(2008)093

SU(3). In this case the highest weight is $2\mu_{\alpha_1}$ and the corresponding weight diagram is

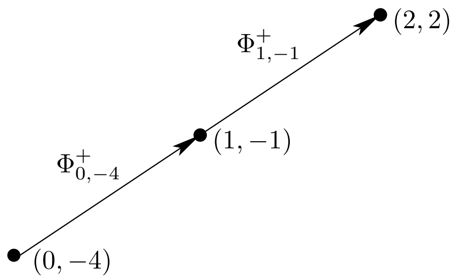


(2.23)

The Young diagram consists of a single row of two boxes, and from the weight diagram (2.23) one can work out the decomposition of $\underline{C}^{2,0}$ as a representation of $\mathrm{SU}(2)\times\mathrm{U}(1)$ to get

$$\underline{C}^{2,0}\big|_{\mathrm{SU}(2)\times\mathrm{U}(1)} = \underline{(2,2)} \;\oplus\; \underline{(1,-1)} \;\oplus\; \underline{(0,-4)}\ . \tag{2.24}$$

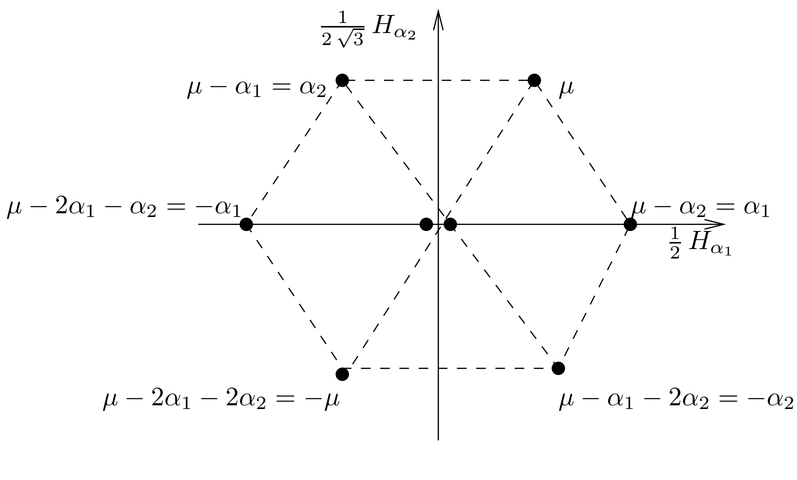
The corresponding quiver diagram is



(2.25)

which will be represented in section 3 by a holomorphic chain.

Symmetric $\underline{C}^{1,1}$ quiver. The final example of a quiver associated to the symmetric space $\mathbb{C}P^2$ corresponds to the eight-dimensional adjoint representation $\underline{C}^{1,1}$. The highest weight is given by $\mu := \mu_{\alpha_1} + \mu_{\alpha_2} = \frac{1}{2}(1, \sqrt{3}\,)$, and the weight diagram is

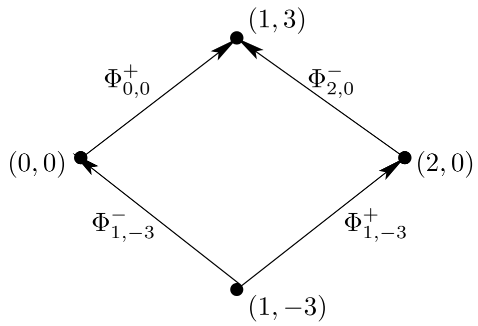


(2.26)

The corresponding decomposition of $\underline{C}^{1,1}$ is given by

$$\underline{C}^{1,1}\big|_{\mathrm{SU}(2)\times\mathrm{U}(1)} = \underline{(1,-3)} \;\oplus\; \underline{(2,0)} \;\oplus\; \underline{(0,0)} \;\oplus\; \underline{(1,3)} \ . \tag{2.27}$$

The quiver diagram is



(2.28)

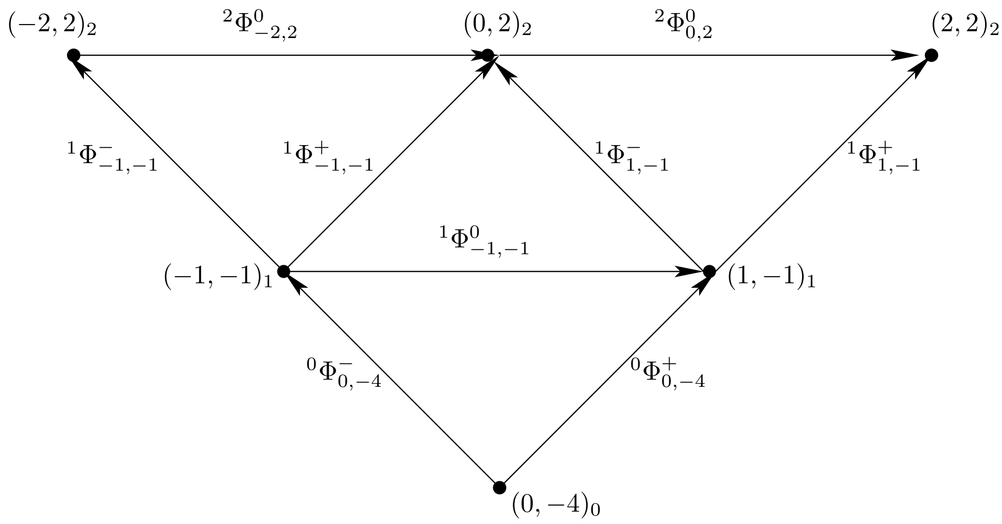
whose commutativity is expressed through the holomorphic relation

$$\Phi^-_{2,0}\,\Phi^+_{1,-3} - \Phi^+_{0,0}\,\Phi^-_{1,-3} = 0 \ . \tag{2.29}$$

Non-symmetric $\underline{C}^{2,0}$ quiver. Next we turn to the quiver associated to the non-symmetric space Q_3 which corresponds to the representation $\underline{C}^{2,0}$ of SU(3). Decomposing the SU(2)-modules in (2.24) as irreducible U(1) representations, the decomposition of $\underline{C}^{2,0}$ as a representation of the maximal torus T is given by

$$\underline{C}^{2,0}\big|_{\mathrm{U}(1)\times\mathrm{U}(1)} = \underline{(2,2)_2} \;\oplus\; \underline{(0,2)_2} \;\oplus\; \underline{(-2,2)_2} \;\oplus\; \underline{(1,-1)_1} \;\oplus\; \underline{(-1,-1)_1} \;\oplus\; \underline{(0,-4)_0} \ . \tag{2.30}$$

The corresponding quiver diagram coincides with the weight diagram (2.23) and can be presented as



(2.31)

Compatibility of the maps in (2.31) implies the set of holomorphic relations

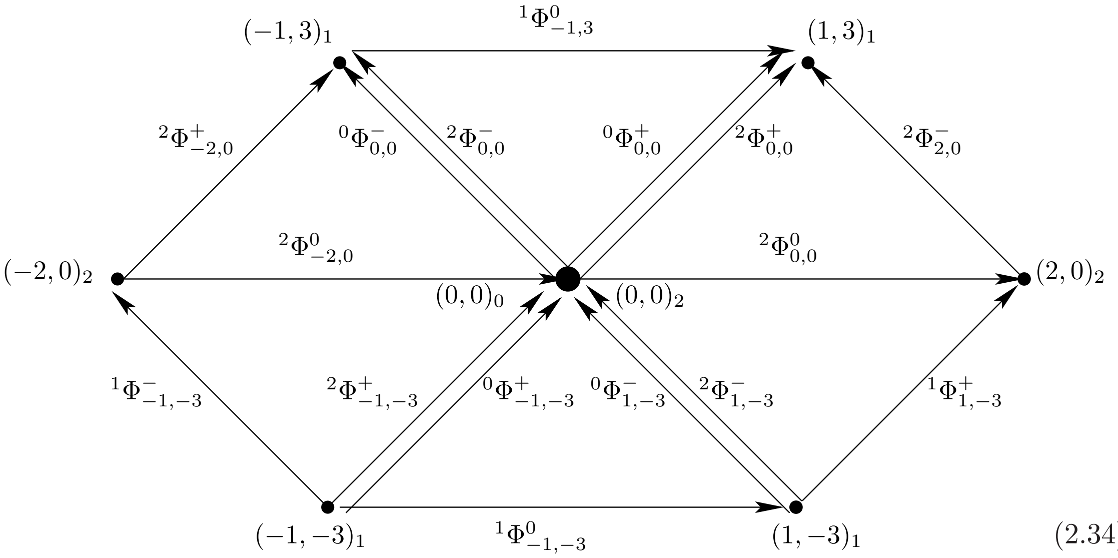
$$\begin{aligned} {}^1\Phi^+_{1,-1} - {}^2\Phi^0_{0,2}\,{}^1\Phi^-_{1,-1} &= 0\,, \\ {}^2\Phi^0_{0,2}\,{}^1\Phi^+_{-1,-1} - {}^1\Phi^+_{1,-1}\,{}^1\Phi^0_{-1,-1} &= 0\,, \end{aligned}$$

$$
\begin{aligned}
{}^1\Phi^+_{-1,-1} + {}^1\Phi^-_{1,-1}\,{}^1\Phi^0_{-1,-1} - 2\,{}^2\Phi^0_{-2,2}\,{}^1\Phi^-_{-1,-1} &= 0\,, \\
{}^1\Phi^+_{-1,-1}\,{}^0\Phi^-_{0,-4} - {}^1\Phi^-_{1,-1}\,{}^0\Phi^+_{0,-4} &= 0\,, \\
{}^0\Phi^+_{0,-4} - {}^1\Phi^0_{-1,-1}\,{}^0\Phi^-_{0,-4} &= 0\;. \qquad (2.32)
\end{aligned}
$$

Non-symmetric $\underline{C}^{1,1}$ quiver. Our final example of this section is the quiver associated to Q_3 which is built on the adjoint representation $\underline{C}^{1,1}$ of SU(3). Using (2.27) one has the decomposition

$$
\begin{aligned}
\underline{C}^{1,1}\big|_{\mathrm{U}(1)\times\mathrm{U}(1)} = \underline{(1,3)_1} \;\oplus\; \underline{(-1,3)_1} \;\oplus\; \underline{(2,0)_2} \;\oplus\; \underline{(0,0)_2} \\
\oplus\; \underline{(-2,0)_2} \;\oplus\; \underline{(0,0)_0} \;\oplus\; \underline{(1,-3)_1} \;\oplus\; \underline{(-1,-3)_1} \qquad (2.33)
\end{aligned}
$$

and the corresponding quiver diagram coincides with the weight diagram (2.26) presented as



(2.34)

The four extra arrows arise from the doubly degenerate weight vector $(0,0)_0, (0,0)_2$ occuring in (2.33). We omit the rather lengthy list of holomorphic relations expressing compatibility of the equivariant T-module morphisms in (2.34).

3. Equivariant gauge theories and quiver bundles

In this section we will consider Yang-Mills theory with G-equivariant gauge fields on manifolds of the form

$$
X \;:=\; M_D \times G/H \;=\; G \times_H M_D\,, \qquad (3.1)
$$

where M_D is a manifold of dimension D and $G = \mathrm{SU}(3)$ acts trivially on M_D. We will reduce the gauge theory on (3.1) by compensating the isometries of G/H with gauge transformations, such that the Lie derivative with respect to a Killing vector field is given by an infinitesimal gauge transformation on X. This unifies gauge and Higgs fields in the higher-dimensional gauge theory, and reduces to a quiver gauge theory on M_D. The twisted reduction will be accomplished by uniquely extending the homogeneous vector bundles (2.1) by H-equivariant bundles $E \to M_D$, providing a representation of the quivers with relations of section 2 in the category of complex equivariant vector bundles over M_D. Such a representation is called a quiver bundle. As previously, we will illustrate the idea behind the construction by considering first the quiver bundles associated to the fundamental representation of SU(3). Then we give the general construction of the G-equivariant gauge connections, and present various other explicit examples to demonstrate generic features of the quiver gauge theory. While the equivariant reduction of gauge connections has been carried out in generality in [10], the construction there relies on various formal isomorphisms. In the following we carry out the coset space dimensional reduction explicitly by exploiting the geometry of the homogeneous spaces involved and a suitable basis for the irreducible representations of SU(3).

3.1 Fundamental representations

We begin by constructing the quiver bundles associated to the fundamental representation $\underline{C}^{1,0}$ of SU(3), for the two coset spaces considered in section 2. In each case we begin by constructing the unique G-equivariant gauge connection on the homogeneous bundles (2.1), and then extend it to equivariant bundles over the product spaces (3.1).

Symmetric $\underline{C}^{1,0}$ quiver bundles. Let us describe the G-equivariant connection on $\mathbb{C}P^2$. Consider the principal $\mathrm{S}(\mathrm{U}(2)\times\mathrm{U}(1))$-bundle

$$\mathrm{SU}(3) \xrightarrow{\mathrm{S}(\mathrm{U}(2)\times\mathrm{U}(1))} \mathbb{C}P^2 \ . \tag{3.2}$$

Let $(Y_0\ Y_1\ Y_2)^\top$ denote homogeneous complex coordinates on $\mathbb{C}P^2$. The projective plane can be covered by three patches $\mathbb{C}P^2 = \mathcal{U}_0 \cup \mathcal{U}_1 \cup \mathcal{U}_2$ such that $Y_i \neq 0$ on $\mathcal{U}_i \cong \mathbb{C}^2$. Then

$$Y \ := \ \begin{pmatrix} y^1 \\ y^2 \end{pmatrix} \ \sim \ \begin{pmatrix} 1 \\ Y_1/Y_0 \\ Y_2/Y_0 \end{pmatrix} \qquad \text{and} \qquad Y^\dagger \ = \ \begin{pmatrix} \bar y^1 & \bar y^2 \end{pmatrix} \tag{3.3}$$

are coordinates on the patch $\mathcal{U}_0$ with $Y^\dagger Y = \bar y^i\, y^i$ and $i=1,2$. A local section of the fibration (3.2) is given by the 3×3 matrices

$$V \ = \ \frac{1}{\gamma}\begin{pmatrix} \Lambda & \bar Y \\ -\bar Y^\dagger & 1 \end{pmatrix} \qquad \text{and} \qquad V^\dagger \ = \ \frac{1}{\gamma}\begin{pmatrix} \Lambda & -\bar Y \\ \bar Y^\dagger & 1 \end{pmatrix} \ , \tag{3.4}$$

where

$$\Lambda \ := \ \gamma\, \mathbf{1}_2 - \frac{1}{\gamma+1}\, Y\, Y^\dagger \qquad \text{and} \qquad \gamma \ := \ \sqrt{1+Y^\dagger\, Y} \ = \ \sqrt{1+\bar y^i\, y^i} \tag{3.5}$$

obey

$$\Lambda\, Y \ = \ Y\ , \quad Y^\dagger\, \Lambda \ = \ Y^\dagger \qquad \text{and} \qquad \Lambda^2 \ = \ \gamma^2\, \mathbf{1}_2 - Y\, Y^\dagger \ . \tag{3.6}$$

Using the identities (3.6) it is easy to see that $V^\dagger\, V \ = \ V\, V^\dagger \ = \ \mathbf{1}_3$, and hence that $V \in \mathrm{SU}(3)$.

Introduce a flat connection on the trivial bundle $\mathbb{C}P^2 \times \mathbb{C}^3$ by the anti-hermitean one-form

$$A_0 \;=\; V^\dagger \, \mathrm{d}V \;=:\; \begin{pmatrix} B & \bar\beta \\ -\beta^\top & -2a \end{pmatrix} \tag{3.7}$$

with $B \in \mathrm{u}(2)$ and $a \in \mathrm{u}(1)$, where from (3.4) we obtain

$$B = \frac{1}{\gamma^2}\Big(-\frac12 \, \mathrm{d}(Y^\dagger Y)\, \mathbf{1}_2 + \bar Y \, \mathrm{d}\bar Y^\dagger + \Lambda \, \mathrm{d}\Lambda \Big) , \tag{3.8}$$

$$a = -\frac{1}{4\gamma^2}\left(\bar Y^\dagger \, \mathrm{d}\bar Y - \mathrm{d}\bar Y^\dagger \, \bar Y \right) , \tag{3.9}$$

$$\bar\beta = \frac{1}{\gamma^2}\Lambda \, \mathrm{d}\bar Y \;=\; \frac{1}{\gamma}\, \mathrm{d}\bar Y - \frac{1}{\gamma^2\,(\gamma+1)}\, \bar Y \, \bar Y^\dagger \, \mathrm{d}\bar Y \, , \tag{3.10}$$

$$\beta = \frac{1}{\gamma^2}\Lambda \, \mathrm{d}Y \;=\; \frac{1}{\gamma}\, \mathrm{d}Y - \frac{1}{\gamma^2\,(\gamma+1)}\, Y \, Y^\dagger \, \mathrm{d}Y \, . \tag{3.11}$$

Introducing components of the column one-forms in (3.10) and (3.11), one has

$$\bar\beta := \begin{pmatrix} \bar\beta^1 \\ \bar\beta^2 \end{pmatrix} \qquad \text{with} \qquad \bar\beta^i \;=\; \frac{1}{\gamma}\, \mathrm{d}\bar y^i - \frac{\bar y^i}{\gamma^2\,(\gamma+1)}\, y^j \, \mathrm{d}\bar y^j \, , \tag{3.12}$$

$$\beta \;=\; \begin{pmatrix} \beta^1 \\ \beta^2 \end{pmatrix} \qquad \text{with} \qquad \beta^i \;=\; \frac{1}{\gamma}\, \mathrm{d}y^i - \frac{y^i}{\gamma^2\,(\gamma+1)}\, \bar y^j \, \mathrm{d}y^j \, . \tag{3.13}$$

The one-forms $B - \frac12 \,\mathrm{tr}(B)\, \mathbf{1}_2$ and a on $\mathbb{C}P^2$ give the vertical components of A_0 with values in the tangent space $\mathrm{su}(2) \oplus \mathrm{u}(1)$ to the fibre of the bundle (3.2), while $\bar\beta$ and β are basis one-forms on $\mathbb{C}P^2$ taking values in the complexified cotangent bundle of $\mathbb{C}P^2$ and giving the horizontal components of A_0 tangent to the base $\mathbb{C}P^2$. The $(1,0)$-forms β^i and the $(0,1)$-forms $\bar\beta^i$ constitute a G-equivariant basis for the complex vector spaces of forms of type $(1,0)$ and $(0,1)$ on $\mathbb{C}P^2$, respectively.

Since $V \in \mathrm{SU}(3)$, the canonical one-form (3.7) on $\mathrm{SU}(3)$ satisfies the Cartan-Maurer equation

$$\mathrm{d}A_0 + A_0 \wedge A_0 = 0 \, . \tag{3.14}$$

This leads to the component equations

$$F_{\mathrm{u}(2)} := \mathrm{d}B + B \wedge B \;=\; \bar\beta \wedge \beta^\top \;=\; \begin{pmatrix} \bar\beta^1 \wedge \beta^1 & \bar\beta^1 \wedge \beta^2 \\ \bar\beta^2 \wedge \beta^1 & \bar\beta^2 \wedge \beta^2 \end{pmatrix} , \tag{3.15}$$

$$f_{\mathrm{u}(1)} := \mathrm{d}a \;=\; \frac12 \, \beta^\dagger \wedge \beta \;=\; \frac12 \left(\bar\beta^1 \wedge \beta^1 + \bar\beta^2 \wedge \beta^2 \right) \tag{3.16}$$

along with

$$\mathrm{d}\bar\beta + B \wedge \bar\beta - 2\bar\beta \wedge a \;=\; 0 \qquad \text{and} \qquad \mathrm{d}\beta^\top + \beta^\top \wedge B - 2a \wedge \beta^\top \;=\; 0 \, . \tag{3.17}$$

The abelian field strength (3.16) can be written explicitly as

$$f_{\mathrm{u}(1)} = -\frac{1}{2\gamma^4}\left(\gamma^2 \, \mathrm{d}y^i \wedge \mathrm{d}\bar y^i - \bar y^i \, \mathrm{d}y^i \wedge y^j \, \mathrm{d}\bar y^j \right) \tag{3.18}$$

while the u(2)-valued curvature (3.15) can be reduced to

$$F_{\mathrm{u}(2)} = F_{\mathrm{su}(2)} + f_{\mathrm{u}(1)}\ \mathbf{1}_2\ , \tag{3.19}$$

where

$$F_{\mathrm{su}(2)} := \begin{pmatrix} \frac12\left(\bar\beta^1\wedge\beta^1-\bar\beta^2\wedge\beta^2\right) & \bar\beta^1\wedge\beta^2 \\ \bar\beta^2\wedge\beta^1 & -\frac12\left(\bar\beta^1\wedge\beta^1-\bar\beta^2\wedge\beta^2\right)\end{pmatrix} = \mathrm{d}B_{(1)}+B_{(1)}\wedge B_{(1)} \tag{3.20}$$

is the curvature of the gauge potential $B_{(1)} = B - a\ \mathbf{1}_2 \in \mathrm{su}(2)$.

By construction, the fields a, $B_{(1)}$, $f_{\mathrm{u}(1)}$ and $F_{\mathrm{su}(2)} = F_{B_{(1)}}$ are all G-equivariant and can be understood geometrically as follows. The one-form (3.9) is the u(1)-valued monopole potential on $\mathbb{C}P^2$ which can be described as the canonical abelian connection on the Hopf bundle

$$S^5 = \mathrm{U}(3)\,/\,\mathrm{U}(2) \xrightarrow{\mathrm{U}(1)} \mathbb{C}P^2\ . \tag{3.21}$$

Choose a linearly embedded projective line $\mathbb{C}P^1 \subset \mathbb{C}P^2$ defined by the equation $y^2 = \bar y^2 = 0$. Its homology class is the generator of the group $\mathrm{H}_2(\mathbb{C}P^2;\mathbb{Z}) = \mathbb{Z}$ which is Poincaré dual to the generator $[\frac{\mathrm{i}}{2\pi}\,f_{\mathrm{u}(1)}] \in \mathrm{H}^2(\mathbb{C}P^2;\mathbb{Z}) = \mathbb{Z}$. Since the abelian group $\mathrm{H}^2(\mathbb{C}P^2;\mathbb{Z}) \cong \mathrm{H}^1(\mathbb{C}P^2;\mathrm{U}(1))$ classifies complex line bundles over $\mathbb{C}P^2$, we can pick a representative $\mathcal{L}\to\mathbb{C}P^2$ of the isomorphism class corresponding to $f_{\mathrm{u}(1)}$. This complex line bundle, associated with the principal U(1)-bundle (3.21), is the monopole bundle over $\mathbb{C}P^2$ which we take to be endowed with the same u(1)-connection a. Higher degree monopole bundles $\mathcal{L}_m := \mathcal{L}^{\otimes m}$ are endowed with the connection $m\,a$ and first Chern number $m\in\mathbb{Z}$, and are associated to higher irreducible representations $\underline{(m)}$ of the fibres of (3.21). Note that the monopole field strength $m\,f_{\mathrm{u}(1)}$ of charge m is a $(1,1)$-form proportional to the canonical Kähler two-form on $\mathbb{C}P^2$ given locally on the patch $\mathcal{U}_0$ by

$$\omega_{\mathbb{C}P^2} = -\tfrac{\mathrm{i}}{2}\,\beta^\top\wedge\bar\beta\ . \tag{3.22}$$

On the other hand, the one-form $B_{(1)} = B - a\ \mathbf{1}_2$ is the su(2)-valued one-instanton field on $\mathbb{C}P^2$ considered as the canonical connection on the rank two vector bundle $\mathcal{I}=\mathcal{I}_1$ associated with the Stiefel bundle

$$\mathrm{U}(3)\,/\,\mathrm{U}(1)\times\mathrm{U}(1) \xrightarrow{\mathrm{SU}(2)} \mathbb{C}P^2\ . \tag{3.23}$$

Its curvature $F_{\mathrm{su}(2)}$ is also a $(1,1)$-form on $\mathbb{C}P^2$. Higher rank instanton bundles $\mathcal{I}_n$ are endowed with G-equivariant one-instanton connections $B_{(n)}\in\mathrm{su}(n+1)$ and fibre spaces

$$\boldsymbol{(n+1)} \cong \mathbb{C}^{n+1} \tag{3.24}$$

in higher-dimensional representations of the SU(2) fibres of the bundle (3.23). For a given representation $\underline{(n,m)}$ of $H = \mathrm{S}(\mathrm{U}(2)\times\mathrm{U}(1))$, the corresponding homogeneous vector bundle is given by $\mathcal{H}_{n,m} = G\times_H \underline{(n,m)}$ and can be identified with $\mathcal{I}_n\otimes\mathcal{L}_m$. It follows that the flat connection (3.7) is a connection on the homogeneous bundle (2.1) induced by the decomposition of the fundamental SU(3) representation $\underline{C}^{1,0}$ in (2.11), with $\mathcal{I} \oplus \mathcal{L}_{-2} \cong \mathbb{C}P^2 \times \mathbb{C}^3$. This defines a connection on the elementary quiver bundle

$$\mathcal{L}_{-2} \xrightarrow{\wedge\bar{\beta}} \mathcal{I} \otimes \mathcal{L} \tag{3.25}$$

over $\mathbb{C}P^2$ representing (2.12). The diagonal elements $B = B_{(1)} + a\, \mathbf{1}_2$ and $-2a$ naturally define connections on $\mathcal{H}_{1,1} \cong \mathcal{I} \otimes \mathcal{L}$ and $\mathcal{H}_{0,-2} \cong \mathcal{L}_{-2}$, respectively. By (3.17), the off-diagonal elements $\bar{\beta}$ and $-\beta^\top$ implement the G-actions which connect the H-modules at the vertices of the quiver.

Let us now extend this quiver bundle with the H-equivariant vector bundle $E^{1,0} \to M_D$ given by

$$E^{1,0} = \big(E_{p_{0,-2}} \otimes \underline{(0,-2)}_{\,M_D}\big) \ \oplus\ \big(E_{p_{1,1}} \otimes \underline{(1,1)}_{\,M_D}\big)\ , \tag{3.26}$$

where $\underline{(n,m)}_{\,M_D}$ denotes the trivial H-equivariant vector bundle $M_D \times \underline{(n,m)}$, and $E_{p_{n,m}}$ are hermitean vector bundles over M_D of rank $p_{n,m}$ with the trivial H-action and the gauge connections $A^{n,m} = A^{n,m}(x) \in \mathrm{u}(p_{n,m})$ for $x \in M_D$. The action of the SU(3) operators $E_{\pm\,\alpha_2}$ and $E_{\pm\,(\alpha_1+\alpha_2)}$ is implemented by bundle morphisms given by sections $\phi^+_{0,-2} = \phi^+_{0,-2}(x) \in \mathrm{Hom}(E_{p_{0,-2}}, E_{p_{1,1}})$ and ${\phi^+_{0,-2}}^\dagger = {\phi^+_{0,-2}}^\dagger(x) \in \mathrm{Hom}(E_{p_{1,1}}, E_{p_{0,-2}})$, yielding a quiver bundle which corresponds to the holomorphic triple

$$\begin{array}{ccc} & & E_{p_{1,1}} \\ & \overset{\phi^+_{0,-2}}{\nearrow} & \\ E_{p_{0,-2}} & & \end{array} \tag{3.27}$$

over M_D representing (2.12).

The induced G-equivariant bundle over $M_D \times \mathbb{C}P^2$ is then constructed as the fibred product

$$\mathcal{E}^{1,0} \ :=\ G \times_H E^{1,0} \ =\ \big(E_{p_{0,-2}} \boxtimes \mathcal{L}_{-2}\big) \ \oplus\ \big(E_{p_{1,1}} \boxtimes (\mathcal{I} \otimes \mathcal{L})\big)\ . \tag{3.28}$$

A G-equivariant connection on this complex vector bundle is given by naturally extending the flat connection (3.7) to get

$$\mathcal{A} = \begin{pmatrix} A^{1,1} \otimes \mathbf{1}_2 + \mathbf{1}_{p_{1,1}} \otimes (B_{(1)} + a) & \phi^+_{0,-2} \otimes \bar{\beta} \\ -{\phi^+_{0,-2}}^\dagger \otimes \beta^\top & A^{0,-2} \otimes 1 + \mathbf{1}_{p_{0,-2}} \otimes (-2a) \end{pmatrix}\ . \tag{3.29}$$

The curvature of the connection (3.29) is given by

$$\mathcal{F} = \mathrm{d}\mathcal{A} + \mathcal{A} \wedge \mathcal{A}\ . \tag{3.30}$$

Using (3.15)–(3.17) and abbreviating $\phi := \phi^+_{0,-2}$, it is given explicitly by

$$\mathcal{F} = \begin{pmatrix} F^{1,1} \otimes \mathbf{1}_2 + \big(\mathbf{1}_{p_{1,1}} - \phi\,\phi^\dagger\big) \otimes \big(\bar{\beta} \wedge \beta^\top\big) & \big(\mathrm{d}\phi + A^{1,1}\,\phi - \phi\,A^{0,-2}\big) \wedge \bar{\beta} \\ -\big(\mathrm{d}\phi^\dagger + \phi^\dagger\,A^{1,1} - A^{0,-2}\,\phi^\dagger\big) \wedge \beta^\top & F^{0,-2} \otimes 1 + \big(\mathbf{1}_{p_{0,-2}} - \phi^\dagger\,\phi\big) \otimes \big(\beta^\top \wedge \bar{\beta}\big) \end{pmatrix} \tag{3.31}$$

where $F^{n,m} = \mathrm{d}A^{n,m} + A^{n,m} \wedge A^{n,m}$ is the curvature of the vector bundle $E_{p_{n,m}} \to M_D$.

Non-symmetric $\underline{C}^{1,0}$ quiver bundles. Now let us consider quiver bundles associated with the fundamental representation of SU(3) and the coset space Q_3. As a complex manifold, Q_3 is a quadric in $\mathbb{C}P^2 \times \mathbb{C}P^2_*$. In homogeneous complex coordinates p^a and q_a with $a = 1, 2, 3$ on $\mathbb{C}P^2$ and $\mathbb{C}P^2_*$, respectively, it is defined by the equation $p^a\, q_a = 0$. This embedding also identifies Q_3 as the twistor space of $\mathbb{C}P^2$ through the sphere fibration

$$\pi \,:\, Q_3 \ \xrightarrow{\mathbb{C}P^1}\ \mathbb{C}P^2 \tag{3.32}$$

defined by forgetting the dependence on the coordinates q_a. This twistor fibration is a geometric version of the algebraic relationship derived in the previous section between the quivers associated to the homogeneous spaces Q_3 and $\mathbb{C}P^2$. As we shall see, integration over the $\mathbb{C}P^1$ fibres of (3.32) is mimicked by a sum over monopole charges q for fixed isospin n which maps the $\underline{(q, m)_n}$ representation onto $\underline{(n, m)}$. This enables one to map configurations built on the two spaces using the projection π and the homomorphisms induced by it.

Consider the principal torus bundle

$$\mathrm{SU}(3) \ \xrightarrow{\mathrm{U}(1)\times\mathrm{U}(1)}\ Q_3 \ . \tag{3.33}$$

A local section of the bundle (3.33) is given by 3×3 matrices of the form

$$\Psi \ = \ \begin{pmatrix} u^1 & \epsilon^{1bc}\,\bar v_b\,\bar u_c & v^1 \\ u^2 & \epsilon^{2bc}\,\bar v_b\,\bar u_c & v^2 \\ u^3 & \epsilon^{3bc}\,\bar v_b\,\bar u_c & v^3 \end{pmatrix} \qquad \text{and} \qquad \Psi^\dagger \ = \ \begin{pmatrix} \bar u_1 & \bar u_2 & \bar u_3 \\ \epsilon_{1bc}\,v^b\,u^c & \epsilon_{2bc}\,v^b\,u^c & \epsilon_{3bc}\,v^b\,u^c \\ \bar v_1 & \bar v_2 & \bar v_3 \end{pmatrix} \ . \tag{3.34}$$

To ensure that $\Psi \in \mathrm{SU}(3)$, i.e. $\Psi^\dagger\,\Psi = \Psi\,\Psi^\dagger = \mathbf{1}_3$, the complex three-vectors $u = (u^a)$ and $v = (v^a)$ must obey the constraints

$$v^\dagger\, u \ = \ 0 \qquad \text{and} \qquad v^\dagger\, v \ = \ u^\dagger\, u \ = \ 1 \ . \tag{3.35}$$

The space Q_3 can be covered by four $\mathbb{C}^3$ patches, and on one of these patches we can solve the constraints (3.35) as

$$\begin{aligned} u \ = \ \big(u^a\big) \ &= \ \alpha\,\big(p^a\big) \ = \ \alpha \begin{pmatrix} 1 \\ y^1 \\ y^2 \end{pmatrix} \qquad \text{with} \quad \alpha \ = \ \frac{1}{\sqrt{1 + \bar y^i\, y^i}} \ , \\ v^\dagger \ = \ \big(\bar v_a\big) \ &= \ \alpha_*\,\big(q_a\big) \\ &= \ \alpha_*\,\big(z^1\,,\, z^2\,,\, 1\big) \ := \ \alpha_*\,\big(y^1\,y^3 - y^2\,,\, -y^3\,,\, 1\big) \qquad \text{with} \quad \alpha_* \ = \ \frac{1}{\sqrt{1 + \bar z^i\, z^i}} \ . \end{aligned} \tag{3.36}$$

Here y^i and z^i with $i = 1, 2$ are local complex coordinates on $\mathbb{C}P^2$ and $\mathbb{C}P^2_*$, respectively, such that z^1 is expressed in terms of the other three complex coordinates on the quadric $Q_3 \hookrightarrow \mathbb{C}P^2 \times \mathbb{C}P^2_*$.

A flat connection on the trivial bundle $Q_3 \times \mathbb{C}^3$ is then given by the anti-hermitean one-form

$$A_0 \ = \ \Psi^\dagger\,\mathrm{d}\Psi \ =: \ \begin{pmatrix} a_1 & \bar\gamma^3 & \bar\gamma^1 \\ -\gamma^3 & -a_1 - a_2 & \bar\gamma^2 \\ -\gamma^1 & -\gamma^2 & a_2 \end{pmatrix} \ , \tag{3.37}$$

where from (3.34) the connection one-forms $a_1, a_2 \in \mathrm{u}(1)$ are given by

$$a_1 \;=\; \bar u_a \, \mathrm{d}u^a \qquad \text{and} \qquad a_2 \;=\; \bar v_a \, \mathrm{d}v^a \tag{3.38}$$

while

$$\gamma^1 \;=\; u^a \, \mathrm{d}\bar v_a \, , \qquad \gamma^2 \;=\; \epsilon^{abc} \, \bar v_a \, \bar u_b \, \mathrm{d}\bar v_c \qquad \text{and} \qquad \gamma^3 \;=\; \epsilon_{abc} \, u^a \, v^b \, \mathrm{d}u^c \ . \tag{3.39}$$

The one-forms γ^a, $a = 1,2,3$, form a G-equivariant basis for the vector space of $(1,0)$-forms on Q_3. Since $\Psi \in \mathrm{SU}(3)$, the canonical connection (3.37) again obeys the Cartan-Maurer equation (3.14), from which we obtain the abelian curvature equations

$$f_1 \;:=\; \mathrm{d}a_1 \;=\; \bar\gamma^1 \wedge \gamma^1 + \bar\gamma^3 \wedge \gamma^3 \qquad \text{and} \qquad f_2 \;:=\; \mathrm{d}a_2 \;=\; -\bar\gamma^1 \wedge \gamma^1 - \bar\gamma^2 \wedge \gamma^2 \tag{3.40}$$

along with the bi-covariant constancy equations

$$\begin{aligned} \mathrm{d}\gamma^1 - (a_1 - a_2) \wedge \gamma^1 - \gamma^2 \wedge \gamma^3 &= 0 \, , \\ \mathrm{d}\gamma^2 + (a_1 + 2a_2) \wedge \gamma^2 + \gamma^1 \wedge \bar\gamma^3 &= 0 \, , \\ \mathrm{d}\gamma^3 - (2a_1 + a_2) \wedge \gamma^3 - \gamma^1 \wedge \bar\gamma^2 &= 0 \ . \end{aligned} \tag{3.41}$$

The geometrical meaning of these G-equivariant fields is as follows. For a weight $\mu = (q,m)_n$ of the maximal torus $T = \mathrm{U}(1) \times \mathrm{U}(1)$, the induced bundle (2.1) is the usual homogeneous line bundle ${}^n\mathcal{L}_{q,m} \to Q_3$ of the Borel-Weil-Bott theory for the semisimple Lie group SU(3). Since topologically $\mathrm{SU}(3) \cong S^5 \times S^3$, one has $\mathrm{H}^1(\mathrm{SU}(3);\mathbb{Z}) = \mathrm{H}^2(\mathrm{SU}(3);\mathbb{Z}) = \{0\}$ and thus the Leray-Serre spectral sequence for the fibration (3.33) gives

$$\mathrm{H}^2(Q_3;\mathbb{Z}) \;=\; \mathrm{H}^1(T;\mathbb{Z}) \;=\; \mathbb{Z} \oplus \mathbb{Z} \ . \tag{3.42}$$

It follows that there are complex line bundles $\mathcal{L}_{(i)} \to Q_3$, $i = 1,2$, whose u(1) fluxes g_i define first Chern classes which are the generators of the free abelian group (3.42). We can identify ${}^n\mathcal{L}_{q,m}$ with $(\mathcal{L}_{(1)})^{\otimes q} \otimes (\mathcal{L}_{(2)})^{\otimes m}$. By Poincaré-Hodge duality, one also has $\mathrm{H}_2(Q_3;\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$. This homology group is generated by a pair of disjoint projective lines $\mathbb{C}P^1_{(i)} \subset Q_3$ in the quadric $Q_3 \hookrightarrow \mathbb{C}P^2 \times \mathbb{C}P^2_*$. The two-forms g_i intersect dually with these projective lines as

$$\frac{\mathrm{i}}{2\pi} \int_{\mathbb{C}P^1_{(i)}} g_j = \delta_{ij} \, , \tag{3.43}$$

and thus generate the monopole charges of the line bundles ${}^n\mathcal{L}_{q,m} \to Q_3$. With the change of basis

$$f_1 \;:=\; g_1 + g_2 \qquad \text{and} \qquad f_2 \;:=\; -\,2g_2 \tag{3.44}$$

for the abelian group (3.42), the corresponding monopole gauge potentials a_i have respective $(H_{\alpha_1}, H_{\alpha_2})_n$ charges $(1,1)_1$ and $(0,-2)_0$. Thus the diagonal entries of the flat connection (3.37) are connections on the respective monopole line bundles ${}^1\mathcal{L}_{1,1}$, ${}^1\mathcal{L}_{-1,1}$ and ${}^0\mathcal{L}_{0,-2}$ over Q_3, while from (3.41) it follows that the off-diagonal elements define bundle morphisms

$${}^0\mathcal{L}_{0,-2} \xrightarrow{\wedge\bar\gamma^1} {}^1\mathcal{L}_{1,1} \, , \qquad {}^0\mathcal{L}_{0,-2} \xrightarrow{\wedge\bar\gamma^2} {}^1\mathcal{L}_{-1,1} \qquad \text{and} \qquad {}^1\mathcal{L}_{-1,1} \xrightarrow{\wedge\bar\gamma^3} {}^1\mathcal{L}_{1,1} \ . \tag{3.45}$$

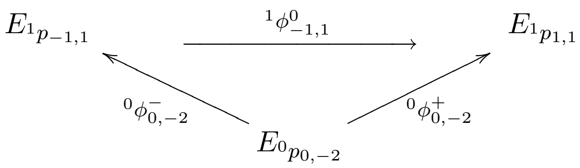
Hence (3.37) naturally defines a connection of the elementary quiver bundle over Q_3 representing (2.16). Note that as a Kähler form on Q_3 one can choose

$$\omega_{Q_3} \;=\; -\tfrac{3\,\mathrm{i}}{4}\,(f_1 - f_2) \;=\; -\tfrac{3\,\mathrm{i}}{4}\,\big(2\bar\gamma^1\wedge\gamma^1 + \bar\gamma^2\wedge\gamma^2 + \bar\gamma^3\wedge\gamma^3\big)\ . \tag{3.46}$$

Analogously to the $\mathbb{C}P^2$ case above, we consider a T-equivariant hermitean bundle $E^{1,0}\to M_D$ given by

$$E^{1,0} = \big(E_{{}^1p_{1,1}}\otimes \underline{(1,1)_1}_{\,M_D}\big)\ \oplus\ \big(E_{{}^1p_{-1,1}}\otimes \underline{(-1,1)_1}_{\,M_D}\big)\ \oplus\ \big(E_{{}^0p_{0,-2}}\otimes \underline{(0,-2)_0}_{\,M_D}\big) \tag{3.47}$$

along with appropriate bundle morphisms between the $E_{{}^np_{q,m}}$ such that



(3.48)

is a quiver bundle over M_D representing (2.16). Then a G-equivariant connection on the induced bundle

$$\mathcal{E}^{1,0} = \big(E_{{}^1p_{1,1}}\boxtimes {}^1\mathcal{L}_{1,1}\big)\ \oplus\ \big(E_{{}^1p_{-1,1}}\boxtimes {}^1\mathcal{L}_{-1,1}\big)\ \oplus\ \big(E_{{}^0p_{0,-2}}\boxtimes {}^0\mathcal{L}_{0,-2}\big) \tag{3.49}$$

over $M_D\times Q_3$ is given by

$$\mathcal{A} = \begin{pmatrix} {}_1A^{1,1}\otimes 1 + \mathbf{1}_{{}^1p_{1,1}}\otimes a_1 & {}^1\phi^0_{-1,1}\otimes\bar\gamma^3 & {}^0\phi^+_{0,-2}\otimes\bar\gamma^1 \\ -{}^1\phi^0_{-1,1}{}^\dagger\otimes\gamma^3 & {}_1A^{-1,1}\otimes 1 + \mathbf{1}_{{}^1p_{-1,1}}\otimes(-a_1-a_2) & {}^0\phi^-_{0,-2}\otimes\bar\gamma^2 \\ -{}^0\phi^+_{0,-2}{}^\dagger\otimes\gamma^1 & -{}^0\phi^-_{0,-2}{}^\dagger\otimes\gamma^2 & {}_0A^{0,-2}\otimes 1 + \mathbf{1}_{{}^0p_{0,-2}}\otimes a_2 \end{pmatrix}\ . \tag{3.50}$$

For the curvature $\mathcal{F}=(\mathcal{F}^{ab})$ given by (3.30), from the identities (3.40) and (3.41) we obtain

$$\begin{aligned}
\mathcal{F}^{11} &= {}_1F^{1,1} + \big(\mathbf{1}_{{}^1p_{1,1}} - {}^0\phi^+_{0,-2}\,{}^0\phi^+_{0,-2}{}^\dagger\big)\,\bar\gamma^1\wedge\gamma^1 + \big(\mathbf{1}_{{}^1p_{1,1}} - {}^1\phi^0_{-1,1}\,{}^1\phi^0_{-1,1}{}^\dagger\big)\,\bar\gamma^3\wedge\gamma^3\,, \\
\mathcal{F}^{22} &= {}_1F^{-1,1} - \big(\mathbf{1}_{{}^1p_{-1,1}} - {}^1\phi^0_{-1,1}{}^\dagger\,{}^1\phi^0_{-1,1}\big)\,\bar\gamma^3\wedge\gamma^3 + \big(\mathbf{1}_{{}^1p_{-1,1}} - {}^0\phi^-_{0,-2}\,{}^0\phi^-_{0,-2}{}^\dagger\big)\,\bar\gamma^2\wedge\gamma^2\,, \\
\mathcal{F}^{33} &= {}_0F^{0,-2} - \big(\mathbf{1}_{{}^0p_{0,-2}} - {}^0\phi^+_{0,-2}{}^\dagger\,{}^0\phi^+_{0,-2}\big)\,\bar\gamma^1\wedge\gamma^1 - \big(\mathbf{1}_{{}^0p_{0,-2}} - {}^0\phi^-_{0,-2}{}^\dagger\,{}^0\phi^-_{0,-2}\big)\,\bar\gamma^2\wedge\gamma^2\,, \\
\mathcal{F}^{12} &= \big(\mathrm{d}\,{}^1\phi^0_{-1,1} + {}_1A^{1,1}\,{}^1\phi^0_{-1,1} - {}^1\phi^0_{-1,1}\,{}_1A^{-1,1}\big)\wedge\bar\gamma^3 + \big({}^1\phi^0_{-1,1} - {}^0\phi^+_{0,-2}\,{}^0\phi^-_{0,-2}{}^\dagger\big)\,\bar\gamma^1\wedge\gamma^3\,, \\
\mathcal{F}^{13} &= \big(\mathrm{d}\,{}^0\phi^+_{0,-2} + {}_1A^{1,1}\,{}^0\phi^+_{0,-2} - {}^0\phi^+_{0,-2}\,{}_0A^{0,-2}\big)\wedge\bar\gamma^1 + \big({}^0\phi^+_{0,-2} - {}^1\phi^0_{-1,1}\,{}^0\phi^-_{0,-2}\big)\,\bar\gamma^2\wedge\bar\gamma^3\,, \\
\mathcal{F}^{23} &= \big(\mathrm{d}\,{}^0\phi^-_{0,-2} + {}_1A^{-1,1}\,{}^0\phi^-_{0,-2} - {}^0\phi^-_{0,-2}\,{}_0A^{0,-2}\big)\wedge\bar\gamma^2 + \big({}^0\phi^-_{0,-2} - {}^1\phi^0_{-1,1}{}^\dagger\,{}^0\phi^+_{0,-2}\big)\,\bar\gamma^3\wedge\gamma^1
\end{aligned} \tag{3.51}$$

along with the hermitean conjugates $\mathcal{F}^{ba} = -(\mathcal{F}^{ab})^\dagger$ for $a<b$. In (3.51) we have suppressed tensor products from the notation in order to simplify the expressions somewhat. Note that the off-diagonal matrix elements of the field strength carry information about both holomorphic and non-holomorphic relations for the quiver (2.16). In particular, the relation (2.17) is contained in $\mathcal{F}^{13}$.

3.2 General constructions

Let $\mathcal{E}^{k,l} \to X$ be a rank p hermitean vector bundle over the space (3.1), associated to an irreducible representation $\underline{C}^{k,l}$ of SU(3), with the structure group U(p) and gauge connection $\mathcal{A} \in \mathrm{u}(p)$. There is a one-to-one correspondence between G-equivariant hermitean vector bundles over X and H-equivariant hermitean vector bundles over M_D, with H acting trivially on M_D [10]. Given an H-equivariant bundle $E^{k,l} \to M_D$ of rank p associated to the representation $\underline{C}^{k,l}\big|_H$ of H, the corresponding G-equivariant bundle over X is defined by induction as

$$\mathcal{E}^{k,l} = G \times_H E^{k,l} \ . \tag{3.52}$$

As a holomorphic vector bundle, the action of the Levi subgroup $L = H \otimes \mathbb{C}_*$ on $E^{k,l}$ is defined by the isotopical decomposition

$$E^{k,l} \ \cong \ \bigoplus_{v\in\mathsf{Q}_0(k,l)} E_{p_v} \otimes \underline{v}_{\,M_D} \qquad \text{with} \quad E_{p_v} \ = \ \mathrm{Hom}_L\big(\underline{v}_{\,M_D}\,,\,E^{k,l}\big)\,, \tag{3.53}$$

where $\underline{v}_{\,M_D}$ denotes the trivial H-equivariant bundle $M_D \times \underline{v}$ over M_D corresponding to the irreducible H-module $\underline{v}$. The holomorphic vector bundles $E_{p_v} \to M_D$ of rank p_v, at the vertices $v \in \mathsf{Q}_0(k,l)$ of the pertinent quiver, have trivial L-actions. The rank p of $E^{k,l}$ is given by

$$p \ = \sum_{v\in\mathsf{Q}_0(k,l)} d_v\, p_v \qquad \text{with} \quad d_v \ = \ \dim\big(\underline{v}\big)\ . \tag{3.54}$$

The extension of the L-action to a P-action is defined by means of bi-fundamental Higgs fields $\phi_{v,\Phi(v)} \in \mathrm{Hom}(E_{p_v}, E_{p_{\Phi(v)}})$ which, after imposing BPS conditions, will satisfy the holomorphic relations $\mathsf{R}(k,l)$ of the quiver. These holomorphic bundle morphisms realize the G-action of the coset generators which twists the naive dimensional reduction by "off-diagonal" terms. This construction breaks the gauge group of the bundle $E^{k,l}$ as

$$\mathrm{U}(p) \ \longrightarrow \prod_{v\in\mathsf{Q}_0(k,l)} \mathrm{U}(p_v) \tag{3.55}$$

via the Higgs effect.

With $\mathcal{V}_v$ the homogeneous bundle (2.1) induced by the irreducible H-module $\underline{v}$, the structure group of the principal bundle associated to

$$\mathcal{E}^{k,l} = \bigoplus_{v\in\mathsf{Q}_0(k,l)} E_{p_v} \boxtimes \mathcal{V}_v \tag{3.56}$$

is then $H \times \prod_{v\in\mathsf{Q}_0(k,l)} \mathrm{U}(p_v)$. The space of G-equivariant sections of a homogeneous bundle (2.1) induced by a representation $\underline{V}$ of H is in a one-to-one correspondence with the set of H-invariant subspaces of $\underline{V}$. It follows that the vector space of G-equivariant $\mathcal{V}$-valued $(0,1)$-forms on the homogeneous space G/H is given by

$$\Omega^{0,1}(\mathcal{V})^G \cong \big(\mathfrak{u}^\vee \otimes \underline{V}\big)^H \ . \tag{3.57}$$

To determine the generic form of a G-equivariant, holomorphic connection one-form $\mathcal{A}$ on the bundle $\mathcal{E}^{k,l} \to X$, we decompose the space $\Omega^{0,1}(\mathrm{End}(\mathcal{E}^{k,l}))^G$ using the Whitney sum (3.56). Since by Schur's lemma $\mathrm{Hom}(\underline{v}\,,\,\underline{v'}\,)^G \cong \delta_{v,v'}\ \mathbb{C}$, corresponding to each vertex $v \in \mathsf{Q}_0(k,l)$ there is a "diagonal" subspace

$$\big(\Omega^{0,1}(\mathrm{End}(E_{p_v})) \otimes \mathbf{1}_{d_v}\big) \ \oplus \ \big(\mathbf{1}_{p_v} \otimes \Omega^{0,1}(\mathrm{End}(\mathcal{V}_v))^G\,\big) \tag{3.58}$$

in which we can choose a connection A^v on the bundle $E_{p_v} \to M_D$ twisted by a G-equivariant connection on the homogeneous vector bundle $\mathcal{V}_v \to G/H$. To each arrow $\Phi \in \mathsf{Q}_1(k,l)$ there is an "off-diagonal" subspace $\Omega^0(\mathrm{Hom}(E_{p_v}, E_{p_{\Phi(v)}}))\otimes\Omega^{0,1}(\mathrm{Hom}(\mathcal{V}_v, \mathcal{V}_{\Phi(v)}))^G$, with

$$\Omega^{0,1}\big(\mathrm{Hom}(\mathcal{V}_v, \mathcal{V}_{\Phi(v)})\big)^G \cong \Big(\mathfrak{u}^\vee \otimes \mathrm{Hom}\big(\,\underline{v}\,,\,\underline{\Phi(v)}\,\big)\Big)^H\ , \tag{3.59}$$

in which we twist the Higgs fields $\phi_{v,\Phi(v)}$ by suitable invariant $d_{\Phi(v)}\times d_v$ matrix-valued $(0,1)$-forms built from basis $(0,1)$-forms spanning $\Omega^{0,1}(G/H)^G \cong (\mathfrak{u}^\vee\,)^H$. Thus the condition of G-equivariance dictates the form of the gauge connection $\mathcal{A}$ in $p_v\, d_v \times p_{\Phi(v)}\, d_{\Phi(v)}$ blocks.

The Biedenharn basis. The key to making the above construction explicit is finding a suitable basis for the irreducible representations $\underline{C}^{k,l}$ of SU(3) that is tailored to the structure of the arrows and vertices of the pertinent quiver, which requires appropriately assembling the invariant $(0,1)$-forms into rectangular block matrices. A particularly nice basis for this is the Biedenharn representation [17, 18] which takes care of both symmetric and non-symmetric quivers simultaneously. The complete set of $d^{k,l}$ orthonormal basis vectors in this basis set are denoted $\big|\overset{n}{q}, m\big\rangle$ and are labelled by the isospin quantum numbers $n = 2I$, $q = 2I_z$ and the hypercharge $m = 3Y$. As such, they will encode the vertices and arrows of the quivers through the actions of the SU(3) operators as given in (2.19) and (2.21). These states define the spin $\frac{n}{2}$ representation of the isospin subgroup $\mathrm{SU}(2) \subset \mathrm{SU}(3)$ through

$$H_{\alpha_1}\big|\overset{n}{q}\,,\,m\big\rangle \;=\; q\,\big|\overset{n}{q}\,,\,m\big\rangle\ , \tag{3.60}$$

$$E_{\pm\,\alpha_1}\big|\overset{n}{q}\,,\,m\big\rangle \;=\; \frac{1}{2}\,\sqrt{(n\mp q)\,(n\pm q+2)}\ \big|\overset{n}{q\pm 2}\,,\,m\big\rangle\ . \tag{3.61}$$

They are also hypercharge eigenstates with

$$H_{\alpha_2}\big|\overset{n}{q}\,,\,m\big\rangle = m\,\big|\overset{n}{q}\,,\,m\big\rangle\ . \tag{3.62}$$

The remaining matrix elements can be determined by noting that the generators $E_{\alpha_1+\alpha_2}, E_{\alpha_2}$ form the $\pm\,\frac{1}{2}$ spin components of a spherical tensor operator of rank $\frac{1}{2}$ with respect to the isospin subgroup $\mathrm{SU}(2) \subset \mathrm{SU}(3)$. All of their matrix elements can thus be determined by applying the Wigner-Eckart theorem for SU(2) and relating the resulting reduced matrix elements to an irreducible SU(3) tensor [18]. The latter computation determines numerical coefficient functions

$$\lambda^{+}_{k,l}(n,m) \;=\; \sqrt{\frac{\big(\frac{k+2l}{3}+\frac{n}{2}+\frac{m}{6}+2\big)\,\big(\frac{k-l}{3}+\frac{n}{2}+\frac{m}{6}+1\big)\,\big(\frac{2k+l}{3}-\frac{n}{2}-\frac{m}{6}\big)}{n+2}}\ ,$$

$$\lambda^{-}_{k,l}(n,m) \;=\; \sqrt{\frac{\big(\frac{k+2l}{3}-\frac{n}{2}+\frac{m}{6}+1\big)\,\big(\frac{l-k}{3}+\frac{n}{2}-\frac{m}{6}\big)\,\big(\frac{2k+l}{3}+\frac{n}{2}-\frac{m}{6}+1\big)}{n}}\ . \tag{3.63}$$

The latter constants are defined for $n > 0$ and we set $\lambda^{-}_{k,l}(0,m) := 0$. For fixed k, l, many of the constants (3.63) vanish. Then with the Biedenharn phase convention one has

$$\begin{aligned} E_{\alpha_2}\big|\overset{n}{q}\,,\,m\big\rangle &= \begin{bmatrix} \frac{n}{2} & \frac{1}{2} & \frac{n+1}{2} \\ \frac{q}{2} & -\frac{1}{2} & \frac{q-1}{2} \end{bmatrix} \lambda^{+}_{k,l}(n,m)\,\big|\overset{n+1}{q-1}\,,\,m+3\big\rangle \\ &\quad + \begin{bmatrix} \frac{n}{2} & \frac{1}{2} & \frac{n-1}{2} \\ \frac{q}{2} & -\frac{1}{2} & \frac{q-1}{2} \end{bmatrix} \lambda^{-}_{k,l}(n,m)\,\big|\overset{n-1}{q-1}\,,\,m+3\big\rangle\,, \end{aligned} \tag{3.64}$$

$$\begin{aligned} E_{\alpha_1+\alpha_2}\big|\overset{n}{q}\,,\,m\big\rangle &= \begin{bmatrix} \frac{n}{2} & \frac{1}{2} & \frac{n+1}{2} \\ \frac{q}{2} & \frac{1}{2} & \frac{q+1}{2} \end{bmatrix} \lambda^{+}_{k,l}(n,m)\,\big|\overset{n+1}{q+1}\,,\,m+3\big\rangle \\ &\quad + \begin{bmatrix} \frac{n}{2} & \frac{1}{2} & \frac{n-1}{2} \\ \frac{q}{2} & \frac{1}{2} & \frac{q+1}{2} \end{bmatrix} \lambda^{-}_{k,l}(n,m)\,\big|\overset{n-1}{q+1}\,,\,m+3\big\rangle\,, \end{aligned} \tag{3.65}$$

where the square brackets denote SU(2) Clebsch-Gordan coefficients which in the present case can be computed explicitly from the known values

$$\begin{bmatrix} j & \frac{1}{2} & j+\frac{1}{2} \\ m & \alpha & m+\alpha \end{bmatrix} = \sqrt{\frac{j+2\alpha\,m+1}{2j+1}} \quad \text{and} \quad \begin{bmatrix} j & \frac{1}{2} & j-\frac{1}{2} \\ m & \alpha & m+\alpha \end{bmatrix} = 2|\alpha|\sqrt{\frac{j-2\alpha\,m}{2j+1}}\,. \tag{3.66}$$

The analogous relations for $E_{-\alpha_2}$ and $E_{-\alpha_1-\alpha_2}$ can be derived by hermitean conjugation of (3.64) and (3.65), respectively.

The Biedenharn basis is related to the more conventional basis of irreducible SU(3) tensors $\boldsymbol{T}$ for the representation $\underline{C}^{k,l}$ as follows. Introducing the SU(2) spins

$$j_{\pm} := \tfrac{n}{4} \pm \tfrac{m}{12} \pm \tfrac{1}{6}\,(k-l) \tag{3.67}$$

with $2j_+ = 0, 1, \ldots, k$ and $2j_- = 0, 1, \ldots, l$, one has the change of basis formulas

$$\begin{aligned} \big|\overset{n}{q}\,,\,m\big\rangle &= N_3(n,m) \sum_{|m_\pm| \le j_\pm} \begin{bmatrix} j_+ & j_- & \frac{n}{2} \\ m_+ & m_- & \frac{q}{2} \end{bmatrix} N_1(j_\pm, m_\pm)\; \boldsymbol{T}^{(j_+ + m_+,\, j_+ - m_+,\, k-2j_+)}_{(j_- - m_-,\, j_- + m_-,\, l-2j_-)}\,, \\ \boldsymbol{T}^{(j_+ + m_+,\, j_+ - m_+,\, k-2j_+)}_{(j_- - m_-,\, j_- + m_-,\, l-2j_-)} &= \frac{1}{N_1(j_\pm, m_\pm)} \sum_{n=0}^{k+l} \begin{bmatrix} j_+ & j_- & \frac{n}{2} \\ m_+ & m_- & \frac{q}{2} \end{bmatrix} \frac{N_2(j_\pm, n)}{N_3(n,m)}\; \big|\overset{n}{q}\,,\,m\big\rangle\,, \end{aligned} \tag{3.68}$$

where the coefficients $N_1(j_\pm, m_\pm)$, $N_2(j_\pm, n)$ and $N_3(n,m)$ can be found in [18], and the brackets (r, s, t) indicate that the tensor $\boldsymbol{T}$ has r upper or lower indices equal to 1, s equal to 2, and t equal to 3. From the decompositions (2.11), (2.15) and their conjugates it follows that the state $\boldsymbol{T}^{(j_+ + m_+,\, j_+ - m_+,\, k-2j_+)}_{(j_- - m_-,\, j_- + m_-,\, l-2j_-)}$ possesses definite values of hypercharge and isospin with the values

$$m = 6(j_+ - j_-) - 2(k-l)\,, \qquad q = 2(m_+ + m_-) \qquad \text{and} \qquad n = 2(j_+ + j_-)\,. \tag{3.69}$$

Thus the SU(2) spin j_+ (resp. j_-) is the value of the isospin I contributed by the upper (resp. lower) indices of the SU(3) tensor $\boldsymbol{T}$. With this representation it is now straightforward to write down the G-equivariant gauge connections associated to a generic irreducible representation $\underline{C}^{k,l}$.

Symmetric $\underline{C}^{k,l}$ quiver bundles. We begin by noting that the flat connection (3.7) can be written in terms of the 3×3 matrices of the defining representation of SU(3) from section 2.1 as

$$A_0 = \left[B^{11}\,H_{\alpha_1} + B^{12}\,E_{\alpha_1} - (B^{12}\,E_{\alpha_1})^\dagger\right] \\ +a\,H_{\alpha_2} + \bar\beta^1\,E_{\alpha_1+\alpha_2} + \bar\beta^2\,E_{\alpha_2} - \beta^1\,E_{-\alpha_1-\alpha_2} - \beta^2\,E_{-\alpha_2}\,, \tag{3.70}$$

where B^{ij} are the matrix elements of the su(2)-valued instanton connection $B_{(1)} = B - a\,\mathbf{1}_2$. As expected, the (0,1)-forms $\bar\beta^i$ on $\mathbb{C}P^2$ are coupled to the generators of the abelian algebra $\mathfrak{u}$. The corresponding connection of the quiver bundle over $\mathbb{C}P^2$ associated to the representation $\underline{C}^{k,l}$ is then obtained by substituting into (3.70) the $d^{k,l}\times d^{k,l}$ matrices defined by (3.60)–(3.66).

For a fixed vertex $(n,m)\in \mathsf{Q}_0(k,l)$, one has $d_{(n,m)} = n+1$ and we write

$$B_{n,m} := \sum_{q\in\{-n+2j\}_{j=0}^n} \Big(q\,B^{11}\,\big|{\scriptstyle{n\atop q}}\,,\,m\big\rangle\big\langle{\scriptstyle{n\atop q}}\,,\,m\big| + \frac12\,B^{12}\,\sqrt{(n-q)\,(n+q+2)}\,\big|{\scriptstyle{n\atop q+2}}\,,\,m\big\rangle\big\langle{\scriptstyle{n\atop q}}\,,\,m\big| \\ -\frac12\,\overline{B^{12}}\,\sqrt{(n+q)\,(n-q+2)}\,\big|{\scriptstyle{n\atop q-2}}\,,\,m\big\rangle\big\langle{\scriptstyle{n\atop q}}\,,\,m\big|\Big) \tag{3.71}$$

for the one-instanton connection $B_{(n)} = B_{n,m}$ in the $(n+1)$-dimensional irreducible representation of SU(2). Denote by

$$\Pi_{n,m} := \sum_{q\in\{-n+2j\}_{j=0}^n} \big|{\scriptstyle{n\atop q}}\,,\,m\big\rangle\big\langle{\scriptstyle{n\atop q}}\,,\,m\big| \tag{3.72}$$

the projection of $\underline{C}^{k,l}\big|_H$ onto the irreducible representation $\underline{(n,m)}$ of $H = \mathrm{SU}(2)\times\mathrm{U}(1)$. We further write

$$\bar\beta^\pm_{n,m} := \sum_{q\in\{-n+2j\}_{j=0}^n} \lambda^\pm_{k,l}(n,m)\,\bigg(\bar\beta^1\begin{bmatrix}\frac n2 & \frac12 & \frac{n\pm1}2\\ \frac q2 & -\frac12 & \frac{q-1}2\end{bmatrix}\big|{\scriptstyle{n\pm1\atop q-1}}\,,\,m+3\big\rangle\big\langle{\scriptstyle{n\atop q}}\,,\,m\big| \\ +\bar\beta^2\begin{bmatrix}\frac n2 & \frac12 & \frac{n\pm1}2\\ \frac q2 & \frac12 & \frac{q+1}2\end{bmatrix}\big|{\scriptstyle{n\pm1\atop q+1}}\,,\,m+3\big\rangle\big\langle{\scriptstyle{n\atop q}}\,,\,m\big|\bigg) \tag{3.73}$$

for the $(n\pm1+1)\times(n+1)$ matrix blocks of G-equivariant elementary bundle morphisms between the nodes of the corresponding quiver diagram, and their hermitean conjugates $\beta^\pm_{n,m} := {\bar\beta^\pm_{n,m}}{}^\dagger$.

Then the elementary flat quiver connection can be written as

$$A_0 = \sum_{(n,m)\in\mathsf{Q}_0(k,l)} \Big(B_{n,m} + m\,a\,\Pi_{n,m} + \bar\beta^+_{n,m} + \bar\beta^-_{n,m} - \beta^+_{n,m} - \beta^-_{n,m}\Big)\,. \tag{3.74}$$

At each vertex $(n,m)\in\mathsf{Q}_0(k,l)$ the Cartan-Maurer equation (3.14) gives the curvature equations

$$F_{B_{(n)}} + m\,f_{\mathrm{u}(1)}\,\mathbf{1}_{n+1} = \beta^+_{n,m}\wedge\bar\beta^+_{n,m} + \beta^-_{n,m}\wedge\bar\beta^-_{n,m} \\ +\bar\beta^+_{n-1,m-3}\wedge\beta^+_{n-1,m-3} + \bar\beta^-_{n+1,m-3}\wedge\beta^-_{n+1,m-3}\,, \tag{3.75}$$

along with the bi-covariant constancy conditions

$$\mathrm{d}\bar\beta^{\pm}_{n,m} + \bar\beta^{\pm}_{n,m} \wedge B_{(n)} + B_{(n\pm1)} \wedge \bar\beta^{\pm}_{n,m} - 3\bar\beta^{\pm}_{n,m} \wedge a = 0 \tag{3.76}$$

and the commutation relations

$$\bar\beta^{+}_{n,m} \wedge \bar\beta^{-}_{n+1,m-3} = \bar\beta^{-}_{n+2,m} \wedge \bar\beta^{+}_{n+1,m-3} \quad \text{and} \quad \bar\beta^{+}_{n,m} \wedge \beta^{-}_{n,m} = \beta^{-}_{n+1,m+3} \wedge \bar\beta^{+}_{n-1,m+3} \tag{3.77}$$

plus their hermitean conjugates. All other operator products vanish identically. In these equations $B_{(n)} := B_{n,m}$ while $B_{(n\pm1)} := B_{n\pm1,m+3}$.

We can extend (3.74) naturally to a G-equivariant gauge connection $\mathcal{A}$ on the bundle (3.56) over $M_D \times \mathbb{C}P^2$ by introducing the projection $\pi_{p_{n,m}}$ onto the sub-bundle $E_{p_{n,m}} \to M_D$ to write

$$\begin{aligned}\mathcal{A} = \sum_{(n,m)\in \mathsf{Q}_0(k,l)} \Big(& A^{n,m} \otimes \Pi_{n,m} + \pi_{p_{n,m}} \otimes \big(B_{n,m} + m\, a\, \Pi_{n,m}\big) \\ & + \phi^{+}_{n,m} \otimes \bar\beta^{+}_{n,m} + \phi^{-}_{n,m} \otimes \bar\beta^{-}_{n,m} - {\phi^{+}_{n,m}}^{\dagger} \otimes \beta^{+}_{n,m} - {\phi^{-}_{n,m}}^{\dagger} \otimes \beta^{-}_{n,m}\Big) \,.\end{aligned} \tag{3.78}$$

This operator acts on the typical fibre space $\underline{V}^{k,l} \cong \mathbb{C}^{\sum_{(n,m)\in\mathsf{Q}_0(k,l)} p_{n,m}} \otimes \mathbb{C}^{\sum_{(n,m)\in\mathsf{Q}_0(k,l)} (n+1)}$. The matrix elements of the curvature two-form (3.30) in the Biedenharn basis can be computed by using the flatness conditions on A_0 above to simplify the diagonal and off-diagonal components. At each vertex $(n,m) \in \mathsf{Q}_0(k,l)$ one finds using (3.75) the diagonal matrix elements

$$\begin{aligned}\mathcal{F}^{n,m\,;\,n,m} = {} & F^{n,m} \otimes \mathbf{1}_{n+1} + \big(\mathbf{1}_{p_{n,m}} - {\phi^{+}_{n,m}}^{\dagger}\, \phi^{+}_{n,m}\big) \otimes \big(\beta^{+}_{n,m} \wedge \bar\beta^{+}_{n,m}\big) \\ & + \big(\mathbf{1}_{p_{n,m}} - {\phi^{-}_{n,m}}^{\dagger}\, \phi^{-}_{n,m}\big) \otimes \big(\beta^{-}_{n,m} \wedge \bar\beta^{-}_{n,m}\big) \\ & + \big(\mathbf{1}_{p_{n,m}} - \phi^{+}_{n-1,m-3}\, {\phi^{+}_{n-1,m-3}}^{\dagger}\big) \otimes \big(\bar\beta^{+}_{n-1,m-3} \wedge \beta^{+}_{n-1,m-3}\big) \\ & + \big(\mathbf{1}_{p_{n,m}} - \phi^{-}_{n+1,m-3}\, {\phi^{-}_{n+1,m-3}}^{\dagger}\big) \otimes \big(\bar\beta^{-}_{n+1,m-3} \wedge \beta^{-}_{n+1,m-3}\big)\end{aligned} \tag{3.79}$$

with $F^{n,m} = \mathrm{d}A^{n,m} + A^{n,m} \wedge A^{n,m}$, while from (3.76) and (3.77) the non-vanishing off-diagonal matrix elements are given respectively by

$$\mathcal{F}^{n\pm1,m+3\,;\,n,m} = \big(\mathrm{d}\phi^{\pm}_{n,m} + A^{n\pm1,m+3}\, \phi^{\pm}_{n,m} - \phi^{\pm}_{n,m}\, A^{n,m}\big) \wedge \bar\beta^{\pm}_{n,m} \tag{3.80}$$

and

$$\mathcal{F}^{n+1,m+3\,;\,n+1,m-3} = \big(\phi^{+}_{n,m}\, \phi^{-}_{n+1,m-3} - \phi^{-}_{n+2,m}\, \phi^{+}_{n+1,m-3}\big) \otimes \big(\bar\beta^{+}_{n,m} \wedge \bar\beta^{-}_{n+1,m-3}\big) \,, \tag{3.81}$$

$$\mathcal{F}^{n+1,m+3\,;\,n-1,m+3} = \big(\phi^{+}_{n,m}\, {\phi^{-}_{n,m}}^{\dagger} - {\phi^{-}_{n+1,m+3}}^{\dagger}\, \phi^{+}_{n-1,m+3}\big) \otimes \big(\bar\beta^{+}_{n,m} \wedge \beta^{-}_{n,m}\big) \tag{3.82}$$

along with their hermitean conjugates $\mathcal{F}^{r,s\,;\,n,m} = -(\mathcal{F}^{n,m\,;\,r,s})^{\dagger}$ for $(r,s) \neq (n,m)$. Note that the matrix elements (3.80) define bi-fundamental covariant derivatives $D\phi^{\pm}_{n,m}$ of the Higgs fields, while (3.81) (resp. (3.82)) contain holomorphic (resp. non-holomorphic) Higgs field relations arising from commutativity of the quiver diagram.

JHEP08(2008)093

Non-symmetric $\underline{C}^{k,l}$ quiver bundles. The flat connection (3.37) can be written as

$$\begin{aligned} A_0 = \Big(a_1 + \frac{1}{2}\,a_2\Big)\,H_{\alpha_1} - \frac{1}{2}\,a_2\,H_{\alpha_2} + \bar\gamma^1\,E_{\alpha_1+\alpha_2} + \bar\gamma^2\,E_{\alpha_2} \\ +\bar\gamma^3\,E_{\alpha_1} - \gamma^1\,E_{-\alpha_1-\alpha_2} - \gamma^2\,E_{-\alpha_2} - \gamma^3\,E_{-\alpha_1} \end{aligned} \tag{3.83}$$

with the $(0,1)$-forms $\bar\gamma^a$ on Q_3 coupled to the generators of the Heisenberg algebra $\mathfrak{u}$. In this case we label the vertices of the quiver by $(q,m)_n \in \mathsf{Q}_0(k,l)$, with $d_{(q,m)_n} = 1$ for each one. Let

$$^n\Pi_{q,m} := \big|\overset{n}{q}\,,\,m\big\rangle\big\langle\overset{n}{q}\,,\,m\big| \tag{3.84}$$

be the projection of $\underline{C}^{k,l}\big|_T$ onto the one-dimensional representation $\underline{(q,m)_n}$ of the maximal torus T. Introduce the corresponding G-equivariant $(0,1)$-forms

$$^n\bar\gamma^{(\pm)+}_{q,m} := \bar\gamma^1\,\begin{bmatrix} \frac{n}{2} & \frac{1}{2} & \frac{n\pm1}{2} \\ \frac{q}{2} & \frac{1}{2} & \frac{q+1}{2} \end{bmatrix}\,\lambda^{\pm}_{k,l}(n,m)\,\big|\overset{n\pm1}{q+1}\,,\,m+3\big\rangle\big\langle\overset{n}{q}\,,\,m\big|\,, \tag{3.85}$$

$$^n\bar\gamma^{(\pm)-}_{q,m} := \bar\gamma^2\,\begin{bmatrix} \frac{n}{2} & \frac{1}{2} & \frac{n\pm1}{2} \\ \frac{q}{2} & -\frac{1}{2} & \frac{q-1}{2} \end{bmatrix}\,\lambda^{\pm}_{k,l}(n,m)\,\big|\overset{n\pm1}{q-1}\,,\,m+3\big\rangle\big\langle\overset{n}{q}\,,\,m\big|\,, \tag{3.86}$$

$$^n\bar\gamma^{0}_{q,m} := \frac{1}{2}\,\bar\gamma^3\,\sqrt{(n-q)\,(n+q+2)}\,\big|\overset{n}{q+2}\,,\,m\big\rangle\big\langle\overset{n}{q}\,,\,m\big|\,, \tag{3.87}$$

and the linear combinations

$$^n\bar\gamma^{\pm}_{q,m} = {}^n\bar\gamma^{(+)\pm}_{q,m} + {}^n\bar\gamma^{(-)\pm}_{q,m} \tag{3.88}$$

along with their hermitean conjugates $^n\gamma^{(\kappa)\pm}_{q,m} := {}^n\bar\gamma^{(\kappa)\pm}_{q,m}{}^\dagger$ for $\kappa=\pm$, plus $^n\gamma^{\pm}_{q,m} := {}^n\bar\gamma^{\pm}_{q,m}{}^\dagger$ and $^n\gamma^{0}_{q,m} := {}^n\bar\gamma^{0}_{q,m}{}^\dagger$. The Cartan-Maurer equation (3.14) for the connection (3.83) then yields the curvature constraints

$$\begin{aligned} \Big(q f_1 + \frac{1}{2}(q-m) f_2\Big)\,{}^n\Pi_{q,m} = {}^n\gamma^{+}_{q,m}\wedge{}^n\bar\gamma^{+}_{q,m} + {}^n\gamma^{-}_{q,m}\wedge{}^n\bar\gamma^{-}_{q,m} + {}^n\gamma^{0}_{q,m}\wedge{}^n\bar\gamma^{0}_{q,m} \\ +{}^{n-1}\bar\gamma^{(+)+}_{q-1,m-3}\wedge{}^{n-1}\gamma^{(+)+}_{q-1,m-3} + {}^{n+1}\bar\gamma^{(-)+}_{q-1,m-3}\wedge{}^{n+1}\gamma^{(-)+}_{q-1,m-3} \\ +{}^{n-1}\bar\gamma^{(+)-}_{q+1,m-3}\wedge{}^{n-1}\gamma^{(+)-}_{q+1,m-3} + {}^{n+1}\bar\gamma^{(-)-}_{q+1,m-3}\wedge{}^{n-1}\gamma^{(-)-}_{q+1,m-3} \\ +{}^n\bar\gamma^{0}_{q-2,m}\wedge{}^n\gamma^{0}_{q-2,m}\,, \end{aligned} \tag{3.89}$$

along with the bi-covariant constancy conditions

$$\begin{aligned} \mathrm{d}\,{}^n\bar\gamma^{+}_{q,m} &= (a_1-a_2)\wedge{}^n\bar\gamma^{+}_{q,m} + {}^n\bar\gamma^{-}_{q+2,m}\wedge{}^n\bar\gamma^{0}_{q,m} + \big({}^{n+1}\bar\gamma^{0}_{q-1,m+3} + {}^{n-1}\bar\gamma^{0}_{q-1,m+3}\big)\wedge{}^n\bar\gamma^{-}_{q,m}\,, \\ \mathrm{d}\,{}^n\bar\gamma^{-}_{q,m} &= -(a_1+2a_2)\wedge{}^n\bar\gamma^{-}_{q,m} - {}^n\bar\gamma^{+}_{q-2,m}\wedge{}^n\gamma^{0}_{q-2,m} - \big({}^{n+1}\gamma^{0}_{q-1,m+3} + {}^{n-1}\gamma^{0}_{q-1,m+3}\big)\wedge{}^n\bar\gamma^{+}_{q,m}\,, \\ \mathrm{d}\,{}^n\bar\gamma^{0}_{q,m} &= (2a_1+a_2)\wedge{}^n\bar\gamma^{0}_{q,m} - {}^n\gamma^{-}_{q+2,m}\wedge{}^n\bar\gamma^{+}_{q,m} \\ &\quad - {}^{n-1}\bar\gamma^{(+)+}_{q+1,m-3}\wedge{}^{n-1}\gamma^{(+)-}_{q+1,m-3} - {}^{n+1}\bar\gamma^{(-)+}_{q+1,m-3}\wedge{}^{n+1}\gamma^{(-)-}_{q+1,m-3} \end{aligned} \tag{3.90}$$

and the commutation relations

$$\begin{aligned} {}^n\bar\gamma^{+}_{q,m}\wedge\big({}^{n-1}\bar\gamma^{(+)-}_{q+1,m-3} + {}^{n+1}\bar\gamma^{(-)-}_{q+1,m-3}\big) &= {}^n\bar\gamma^{-}_{q+2,m}\wedge\big({}^{n-1}\bar\gamma^{(+)+}_{q+1,m-3} + {}^{n+1}\bar\gamma^{(-)+}_{q+1,m-3}\big)\,, \\ {}^n\bar\gamma^{+}_{q,m}\wedge{}^n\bar\gamma^{0}_{q-2,m} &= \big({}^{n-1}\bar\gamma^{0}_{q-1,m+3} + {}^{n+1}\bar\gamma^{0}_{q-1,m+3}\big)\wedge{}^n\bar\gamma^{+}_{q-2,m}\,, \end{aligned}$$

$$
\begin{aligned}
{}^n\bar\gamma^-_{q,m}\wedge{}^n\bar\gamma^0_{q-2,m} &= \big({}^{n-1}\bar\gamma^0_{q-3,m+3}+{}^{n+1}\bar\gamma^0_{q-3,m+3}\big)\wedge{}^n\bar\gamma^-_{q-2,m}\,,\\
{}^n\bar\gamma^-_{q,m}\wedge{}^n\gamma^0_{q,m} &= \big({}^{n-1}\gamma^0_{q-1,m+3}+{}^{n+1}\gamma^0_{q-1,m+3}\big)\wedge{}^n\bar\gamma^-_{q+2,m}\,,\\
{}^{n-1}\bar\gamma^{(+)+}_{q,m}\wedge{}^{n-1}\gamma^{(+)-}_{q,m}+{}^{n+1}\bar\gamma^{(-)+}_{q,m}\wedge{}^{n+1}\gamma^{(-)-}_{q,m} &= {}^n\gamma^-_{q+1,m+3}\wedge{}^n\bar\gamma^+_{q-1,m+3}\,, \qquad (3.91)\\
{}^n\bar\gamma^+_{q,m}\wedge{}^n\gamma^0_{q,m} &= \big({}^{n-1}\gamma^0_{q+1,m+3}+{}^{n+1}\gamma^0_{q+1,m+3}\big)\wedge{}^n\bar\gamma^+_{q+2,m}\,,
\end{aligned}
$$

plus their hermitean conjugates.

Then a G-equivariant gauge connection $\mathcal{A}$ on the bundle (3.56) over $M_D\times Q_3$ is given by

$$
\begin{aligned}
\mathcal{A} = \sum_{(q,m)_n\in\mathsf{Q}_0(k,l)} \Big(& {}_nA^{q,m}\otimes{}^n\Pi_{q,m}+\pi_{{}^np_{q,m}}\otimes\big(q\,a_1+\tfrac{1}{2}\,(q-m)\,a_2\big)\,{}^n\Pi_{q,m}\\
&+{}^n\phi^+_{q,m}\otimes{}^n\bar\gamma^+_{q,m}+{}^n\phi^-_{q,m}\otimes{}^n\bar\gamma^-_{q,m}+{}^n\phi^0_{q,m}\otimes{}^n\bar\gamma^0_{q,m} \qquad (3.92)\\
&-{}^n\phi^+_{q,m}{}^\dagger\otimes{}^n\gamma^+_{q,m}-{}^n\phi^-_{q,m}{}^\dagger\otimes{}^n\gamma^-_{q,m}-{}^n\phi^0_{q,m}{}^\dagger\otimes{}^n\gamma^0_{q,m}\Big)\,.
\end{aligned}
$$

The respective constraint equations (3.89)–(3.91) may then be used to compute the diagonal field strength matrix elements

$$
\begin{aligned}
{}_n\mathcal{F}^{q,m\,;\,q,m} = {}_nF^{q,m} &+ \sum_{\kappa=0,\pm}\big(\mathbf{1}_{{}^np_{q,m}}-{}^n\phi^\kappa_{q,m}{}^\dagger\,{}^n\phi^\kappa_{q,m}\big)\,{}^n\gamma^\kappa_{q,m}\wedge{}^n\bar\gamma^\kappa_{q,m}\\
&+\big(\mathbf{1}_{{}^np_{q,m}}-{}^{n-1}\phi^+_{q-1,m-3}\,{}^{n-1}\phi^+_{q-1,m-3}{}^\dagger\big)\,{}^{n-1}\bar\gamma^{(+)+}_{q-1,m-3}\wedge{}^{n-1}\gamma^{(+)+}_{q-1,m-3}\\
&+\big(\mathbf{1}_{{}^np_{q,m}}-{}^{n+1}\phi^+_{q-1,m-3}\,{}^{n+1}\phi^+_{q-1,m-3}{}^\dagger\big)\,{}^{n+1}\bar\gamma^{(-)+}_{q-1,m-3}\wedge{}^{n+1}\gamma^{(-)+}_{q-1,m-3}\\
&+\big(\mathbf{1}_{{}^np_{q,m}}-{}^{n-1}\phi^-_{q+1,m-3}\,{}^{n-1}\phi^-_{q+1,m-3}{}^\dagger\big)\,{}^{n-1}\bar\gamma^{(+)-}_{q+1,m-3}\wedge{}^{n-1}\gamma^{(+)-}_{q+1,m-3}\\
&+\big(\mathbf{1}_{{}^np_{q,m}}-{}^{n+1}\phi^-_{q+1,m-3}\,{}^{n+1}\phi^-_{q+1,m-3}{}^\dagger\big)\,{}^{n+1}\bar\gamma^{(-)-}_{q+1,m-3}\wedge{}^{n+1}\gamma^{(-)-}_{q+1,m-3}\\
&+\big(\mathbf{1}_{{}^np_{q,m}}-{}^n\phi^0_{q-2,m}\,{}^n\phi^0_{q-2,m}{}^\dagger\big)\,{}^n\bar\gamma^0_{q-2,m}\wedge{}^n\gamma^0_{q-2,m}\,, \qquad (3.93)
\end{aligned}
$$

together with the non-vanishing off-diagonal elements

$$
\begin{aligned}
{}_{n\pm1}\mathcal{F}^{q+1,m+3\,;\,q,m} &= \big(\mathrm{d}\,{}^n\phi^+_{q,m}+{}_{n\pm1}A^{q+1,m+3}\,{}^n\phi^+_{q,m}-{}^n\phi^+_{q,m}\,{}_nA^{q,m}\big)\wedge{}^n\bar\gamma^{(\pm)+}_{q,m}\\
&\quad+\big({}^n\phi^+_{q,m}-{}^n\phi^-_{q+2,m}\,{}^n\phi^0_{q,m}\big)\,{}^n\bar\gamma^{(\pm)-}_{q+2,m}\wedge{}^n\bar\gamma^0_{q,m}\\
&\quad+\big({}^n\phi^+_{q,m}-{}^{n\pm1}\phi^0_{q-1,m+3}\,{}^n\phi^-_{q,m}\big)\,{}^{n\pm1}\bar\gamma^0_{q-1,m+3}\wedge{}^n\bar\gamma^-_{q,m}\,, \qquad (3.94)\\
{}_{n\pm1}\mathcal{F}^{q-1,m+3\,;\,q,m} &= \big(\mathrm{d}\,{}^n\phi^-_{q,m}+{}_{n\pm1}A^{q-1,m+3}\,{}^n\phi^-_{q,m}-{}^n\phi^-_{q,m}\,{}_nA^{q,m}\big)\wedge{}^n\bar\gamma^{(\pm)-}_{q,m}\\
&\quad-\big({}^n\phi^-_{q,m}-{}^n\phi^+_{q-2,m}\,{}^n\phi^0_{q-2,m}{}^\dagger\big)\,{}^n\bar\gamma^{(\pm)+}_{q-2,m}\wedge{}^n\gamma^0_{q-2,m}\\
&\quad-\big({}^n\phi^-_{q,m}-{}^{n\pm1}\phi^0_{q-1,m+3}{}^\dagger\,{}^n\phi^+_{q,m}\big)\,{}^{n\pm1}\gamma^0_{q-1,m+3}\wedge{}^n\bar\gamma^+_{q,m}\,, \qquad (3.95)\\
{}_n\mathcal{F}^{q+2,m\,;\,q,m} &= \big(\mathrm{d}\,{}^n\phi^0_{q,m}+{}_nA^{q+2,m}\,{}^n\phi^0_{q,m}-{}^n\phi^0_{q,m}\,{}_nA^{q,m}\big)\wedge{}^n\bar\gamma^0_{q,m}\\
&\quad-\big({}^n\phi^0_{q,m}-{}^n\phi^-_{q+2,m}{}^\dagger\,{}^n\phi^+_{q,m}\big)\,{}^n\gamma^-_{q+2,m}\wedge{}^n\bar\gamma^+_{q,m} \qquad (3.96)\\
&\quad-\big({}^n\phi^0_{q,m}-{}^{n-1}\phi^+_{q+1,m-3}\,{}^{n-1}\phi^-_{q+1,m-3}{}^\dagger\big)\,{}^{n-1}\bar\gamma^{(+)+}_{q+1,m-3}\wedge{}^{n-1}\gamma^{(+)-}_{q+1,m-3}\\
&\quad-\big({}^n\phi^0_{q,m}-{}^{n+1}\phi^+_{q+1,m-3}\,{}^{n+1}\phi^-_{q+1,m-3}{}^\dagger\big)\,{}^{n+1}\bar\gamma^{(-)+}_{q+1,m-3}\wedge{}^{n+1}\gamma^{(-)-}_{q+1,m-3}
\end{aligned}
$$

JHEP08(2008)093

and

$$
{}_{n\pm1}\mathcal{F}^{q+1,m+3\,;\,q+1,m-3} = \big({}^{n}\phi^{+}_{q,m}\;{}^{n\mp1}\phi^{-}_{q+1,m-3} - {}^{n}\phi^{-}_{q+2,m}\;{}^{n\mp1}\phi^{+}_{q+1,m-3}\big)\;{}^{n}\bar{\gamma}^{+}_{q,m}\wedge{}^{n\mp1}\bar{\gamma}^{(\pm)-}_{q+1,m-3}\,, \tag{3.97}
$$

$$
{}_{n\pm1}\mathcal{F}^{q+1,m+3\,;\,q-2,m} = \big({}^{n}\phi^{+}_{q,m}\;{}^{n}\phi^{0}_{q-2,m} - {}^{n\pm1}\phi^{0}_{q-1,m+3}\;{}^{n}\phi^{+}_{q-2,m}\big)\;{}^{n}\bar{\gamma}^{(\pm)+}_{q,m}\wedge{}^{n}\bar{\gamma}^{0}_{q-2,m}\,, \tag{3.98}
$$

$$
{}_{n\pm1}\mathcal{F}^{q-1,m+3\,;\,q-2,m} = \big({}^{n}\phi^{-}_{q,m}\;{}^{n}\phi^{0}_{q-2,m} - {}^{n\pm1}\phi^{0}_{q-3,m+3}\;{}^{n}\phi^{-}_{q-2,m}\big)\;{}^{n}\bar{\gamma}^{(\pm)-}_{q,m}\wedge{}^{n}\bar{\gamma}^{0}_{q-2,m}\,, \tag{3.99}
$$

$$
{}_{n\pm1}\mathcal{F}^{q-1,m+3\,;\,q+2,m} = \big({}^{n}\phi^{-}_{q,m}\;{}^{n}\phi^{0}_{q,m}{}^{\dagger} - {}^{n\pm1}\phi^{0}_{q-1,m+3}{}^{\dagger}\;{}^{n}\phi^{-}_{q+2,m}\big)\;{}^{n}\bar{\gamma}^{(\pm)-}_{q,m}\wedge{}^{n}\gamma^{0}_{q,m}\,, \tag{3.100}
$$

$$
{}_{n\pm1}\mathcal{F}^{q+1,m+3\,;\,q-1,m+3} = \big({}^{n\mp1}\phi^{+}_{q,m}\;{}^{n\mp1}\phi^{-}_{q,m}{}^{\dagger} - {}^{n}\phi^{-}_{q+1,m+3}{}^{\dagger}\;{}^{n}\phi^{+}_{q-1,m+3}\big)\;{}^{n\mp1}\bar{\gamma}^{(\pm)+}_{q,m}\wedge{}^{n\mp1}\gamma^{(\pm)-}_{q,m}\,, \tag{3.101}
$$

$$
{}_{n\pm1}\mathcal{F}^{q+1,m+3\,;\,q+2,m} = \big({}^{n}\phi^{+}_{q,m}\;{}^{n}\phi^{0}_{q,m}{}^{\dagger} - {}^{n\pm1}\phi^{0}_{q+1,m+3}{}^{\dagger}\;{}^{n}\phi^{+}_{q+2,m}\big)\;{}^{n}\bar{\gamma}^{(\pm)+}_{q,m}\wedge{}^{n}\gamma^{0}_{q,m} \tag{3.102}
$$

along with their hermitean conjugates. Note that the matrix elements (3.94)–(3.96) additionally contain linear Higgs field relations appropriate to the non-symmetric quiver diagram.

3.3 Examples

In order to illustrate how to unravel the construction of section 3.2 above, we now turn to some more explicit constructions of quiver bundles and the associated equivariant gauge connections.

Symmetric $\underline{C}^{\mathbf{2,0}}$ quiver bundles. The generators of $\mathrm{sl}(3,\mathbb{C})$ in the six-dimensional representation $\underline{C}^{2,0}$ are given by

$$
\begin{aligned}
E_{\alpha_1} &= \sqrt{2}\,\big(e_{12}+e_{23}\big)+e_{45}\,, \qquad & E_{\alpha_2} &= e_{24}+\sqrt{2}\,\big(e_{35}+e_{56}\big)\,,\\
E_{\alpha_1+\alpha_2} &= \sqrt{2}\,\big(e_{14}+e_{46}\big)+e_{25}
\end{aligned} \tag{3.103}
$$

and

$$
H_{\alpha_1} = 2(e_1-e_3)+e_4-e_5 \qquad \text{and} \qquad H_{\alpha_2} = 2(e_1+e_2+e_3)-e_4-e_5-4e_6 \tag{3.104}
$$

along with $E_{-\alpha_i}=E_{\alpha_i}^\top$ and $E_{-\alpha_1-\alpha_2}=E_{\alpha_1+\alpha_2}^\top$. They satisfy the same commutation relations as in section 2.1. Substituting these 6×6 matrices into (3.70) we find the corresponding flat connection

$$
A_0 = \begin{pmatrix} B_{(2)}+2a\;\mathbf{1}_3 & \bar{\beta}^{+}_{1,-1} & 0 \\ -\bar{\beta}^{+}_{1,-1}{}^{\dagger} & B_{(1)}-a\;\mathbf{1}_2 & \bar{\beta}^{+}_{0,-4} \\ 0 & -\bar{\beta}^{+}_{0,-4}{}^{\dagger} & -4a \end{pmatrix}\,, \tag{3.105}
$$

where

$$
B_{(2)} = \begin{pmatrix} 2B^{11} & \sqrt{2}\,B^{12} & 0 \\ -\sqrt{2}\;\overline{B^{12}} & 0 & \sqrt{2}\,B^{12} \\ 0 & -\sqrt{2}\;\overline{B^{12}} & -2B^{11} \end{pmatrix} \tag{3.106}
$$

is the one-instanton field on $\mathbb{C}P^2$ in the adjoint representation of su(2) while

$$\bar{\beta}^+_{1,-1} \;=\; \begin{pmatrix} \sqrt{2}\,\bar{\beta}^1 & 0 \\ \bar{\beta}^2 & \bar{\beta}^1 \\ 0 & \sqrt{2}\,\bar{\beta}^1 \end{pmatrix} \qquad \text{and} \qquad \bar{\beta}^+_{0,-4} \;=\; \sqrt{2}\,\begin{pmatrix} \bar{\beta}^1 \\ \bar{\beta}^2 \end{pmatrix} \ . \tag{3.107}$$

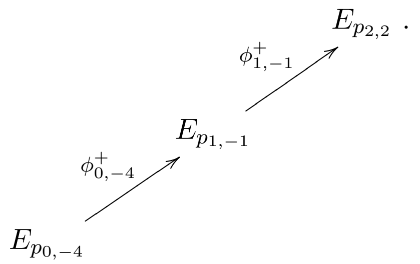
The corresponding G-equivariant connection (3.78) is given by

$$\mathcal{A} = \begin{pmatrix} A^{2,2}\otimes\mathbf{1}_3+\mathbf{1}_{p_{2,2}}\otimes(B_{(2)}+2a\ \mathbf{1}_3) & \phi^+_{1,-1}\otimes\bar{\beta}^+_{1,-1} & 0 \\ -\phi^{+}_{1,-1}{}^\dagger\otimes\bar{\beta}^+_{1,-1}{}^\dagger & A^{1,-1}\otimes\mathbf{1}_2+\mathbf{1}_{p_{1,-1}}\otimes(B_{(1)}-a\ \mathbf{1}_2) & \phi^+_{0,-4}\otimes\bar{\beta}^+_{0,-4} \\ 0 & -\phi^{+}_{0,-4}{}^\dagger\otimes\bar{\beta}^+_{0,-4}{}^\dagger & A^{0,-4}\otimes 1+\mathbf{1}_{p_{0,-4}}\otimes(-4a) \end{pmatrix} . \tag{3.108}$$

After using the flatness condition on (3.105) we find the non-vanishing curvature components

$$\begin{aligned} \mathcal{F}^{11} &= F^{2,2}\otimes\mathbf{1}_3+\big(\mathbf{1}_{p_{2,2}}-\phi^+_{1,-1}\,\phi^{+}_{1,-1}{}^\dagger\big)\otimes\big(\bar{\beta}^+_{1,-1}\wedge\bar{\beta}^{+}_{1,-1}{}^\dagger\,\big)\,, && (3.109)\\ \mathcal{F}^{22} &= F^{1,-1}\otimes\mathbf{1}_2+\big(\mathbf{1}_{p_{1,-1}}-\phi^{+}_{1,-1}{}^\dagger\,\phi^+_{1,-1}\big)\otimes\big(\bar{\beta}^{+}_{1,-1}{}^\dagger\wedge\bar{\beta}^+_{1,-1}\big) \\ &\quad+\big(\mathbf{1}_{p_{1,-1}}-\phi^+_{0,-4}\,\phi^{+}_{0,-4}{}^\dagger\big)\otimes\big(\bar{\beta}^+_{0,-4}\wedge\bar{\beta}^{+}_{0,-4}{}^\dagger\,\big)\,, && (3.110)\\ \mathcal{F}^{33} &= F^{0,-4}\otimes 1+\big(\mathbf{1}_{p_{0,-4}}-\phi^{+}_{0,-4}{}^\dagger\,\phi^+_{0,-4}\big)\otimes\big(\bar{\beta}^{+}_{0,-4}{}^\dagger\wedge\bar{\beta}^+_{0,-4}\big)\,, && (3.111)\\ \mathcal{F}^{12} &= \big(\mathrm{d}\phi^+_{1,-1}+A^{2,2}\,\phi^+_{1,-1}-\phi^+_{1,-1}\,A^{1,-1}\big)\wedge\bar{\beta}^+_{1,-1}\,, && (3.112)\\ \mathcal{F}^{23} &= \big(\mathrm{d}\phi^+_{0,-4}+A^{1,-1}\,\phi^+_{0,-4}-\phi^+_{0,-4}\,A^{0,-4}\big)\wedge\bar{\beta}^+_{0,-4} && (3.113) \end{aligned}$$

along with $\mathcal{F}^{ba} = -(\mathcal{F}^{ab})^\dagger$ for $a < b$, where here the matrix elements refer to the 3×3 block decomposition in (3.108) acting on $\underline{V}^{2,0}\cong(\mathbb{C}^{p_{2,2}}\otimes\mathbb{C}^3)\oplus(\mathbb{C}^{p_{1,-1}}\otimes\mathbb{C}^2)\oplus(\mathbb{C}^{p_{0,-4}}\otimes\mathbb{C})$, the typical fibre spaces of the vector bundle over $M_D\times\mathbb{C}P^2$ induced by the quiver bundle



(3.114)

From (3.107) one has the explicit matrix one-form products

$$\bar{\beta}^+_{1,-1}\wedge\bar{\beta}^{+}_{1,-1}{}^\dagger = \begin{pmatrix} 2\,\bar{\beta}^1\wedge\beta^1 & \sqrt{2}\ \bar{\beta}^1\wedge\beta^2 & 0 \\ \sqrt{2}\ \bar{\beta}^2\wedge\beta^1 & \bar{\beta}^1\wedge\beta^1+\bar{\beta}^2\wedge\beta^2 & \sqrt{2}\ \bar{\beta}^1\wedge\beta^2 \\ 0 & \sqrt{2}\ \bar{\beta}^2\wedge\beta^1 & 2\,\bar{\beta}^2\wedge\beta^2 \end{pmatrix} , \tag{3.115}$$

$$\bar{\beta}^{+}_{1,-1}{}^\dagger\wedge\bar{\beta}^+_{1,-1} = -\begin{pmatrix} 2\,\bar{\beta}^1\wedge\beta^1+\bar{\beta}^2\wedge\beta^2 & \bar{\beta}^1\wedge\beta^2 \\ \bar{\beta}^2\wedge\beta^1 & \bar{\beta}^1\wedge\beta^1+2\,\bar{\beta}^2\wedge\beta^2 \end{pmatrix} , \tag{3.116}$$

$$\bar{\beta}^+_{0,-4}\wedge\bar{\beta}^{+}_{0,-4}{}^\dagger = 2\begin{pmatrix} \bar{\beta}^1\wedge\beta^1 & \bar{\beta}^1\wedge\beta^2 \\ \bar{\beta}^2\wedge\beta^1 & \bar{\beta}^2\wedge\beta^2 \end{pmatrix} , \tag{3.117}$$

$$\bar{\beta}^{+}_{0,-4}{}^\dagger\wedge\bar{\beta}^+_{0,-4} = -2\,\big(\bar{\beta}^1\wedge\beta^1+\bar{\beta}^2\wedge\beta^2\big)\ . \tag{3.118}$$

Symmetric $\underline{C}^{1,1}$ quiver bundles. The 8×8 generators of $\mathrm{sl}(3,\mathbb{C})$ in the adjoint representation $\underline{C}^{1,1}$ are given by

$$\begin{aligned} E_{\alpha_1} &= e_{12} + \sqrt{2}\,(e_{45} + e_{56}) + e_{78}\,, \\ E_{\alpha_2} &= e_{14} + \sqrt{\tfrac{3}{2}}\,(e_{23} + e_{37}) + \sqrt{\tfrac{1}{2}}\,(e_{25} + e_{57}) + e_{68}\,, \\ E_{\alpha_1+\alpha_2} &= \sqrt{\tfrac{3}{2}}\,(e_{13} - e_{38}) - \sqrt{\tfrac{1}{2}}\,(e_{15} - e_{58}) + (e_{47} - e_{26}) \end{aligned} \tag{3.119}$$

along with

$$H_{\alpha_1} \;=\; (e_1 - e_2) + 2(e_4 - e_6) + (e_7 - e_8) \qquad \text{and} \qquad H_{\alpha_2} \;=\; 3(e_1 + e_2 - e_7 - e_8)\,. \tag{3.120}$$

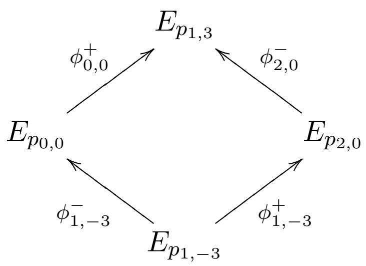
The flat connection (3.70) in this basis is given by

$$A_0 = \begin{pmatrix} B_{(1)} + 3a\ \mathbf{1}_2 & \bar\beta^+_{0,0} & \bar\beta^-_{2,0} & 0 \\ -\bar\beta^+_{0,0}{}^\dagger & 0 & 0 & \bar\beta^-_{1,-3} \\ -\bar\beta^-_{2,0}{}^\dagger & 0 & B_{(2)} & \bar\beta^+_{1,-3} \\ 0 & -\bar\beta^-_{1,-3}{}^\dagger & -\bar\beta^+_{1,-3}{}^\dagger & B_{(1)} - 3a\ \mathbf{1}_2 \end{pmatrix}\,. \tag{3.121}$$

It acts on the typical fibre space $\underline{C}^{1,1}\big|_{\mathrm{SU}(2)\times\mathrm{U}(1)} \cong \mathbb{C}^8 = \mathbb{C}^2 \oplus \mathbb{C} \oplus \mathbb{C}^3 \oplus \mathbb{C}^2$ with

$$\begin{aligned} \bar\beta^+_{0,0} &= \sqrt{\tfrac{3}{2}}\begin{pmatrix} \bar\beta^1 \\ \bar\beta^2 \end{pmatrix} \qquad \text{and} \qquad \bar\beta^-_{2,0} = \begin{pmatrix} \bar\beta^2 & -\sqrt{\tfrac{1}{2}}\,\bar\beta^1 & 0 \\ 0 & \sqrt{\tfrac{1}{2}}\,\bar\beta^2 & -\bar\beta^1 \end{pmatrix}\,, \\ \bar\beta^-_{1,-3} &= \sqrt{\tfrac{3}{2}}\,\big(\bar\beta^2\,,\,-\bar\beta^1\big) \qquad \text{and} \qquad \bar\beta^+_{1,-3} = \begin{pmatrix} \bar\beta^1 & 0 \\ \sqrt{\tfrac{1}{2}}\,\bar\beta^2 & \sqrt{\tfrac{1}{2}}\,\bar\beta^1 \\ 0 & \bar\beta^2 \end{pmatrix}\,. \end{aligned} \tag{3.122}$$

The corresponding quiver bundle over M_D is given by



(3.123)

which induces a vector bundle over $M_D \times \mathbb{C}P^2$ with typical fibre space

$$\underline{V}^{1,1} \cong \big(\mathbb{C}^{p_{1,3}} \otimes \mathbb{C}^2\big) \;\oplus\; \big(\mathbb{C}^{p_{0,0}} \otimes \mathbb{C}\big) \;\oplus\; \big(\mathbb{C}^{p_{2,0}} \otimes \mathbb{C}^3\big) \;\oplus\; \big(\mathbb{C}^{p_{1,-3}} \otimes \mathbb{C}^2\big)\,. \tag{3.124}$$

The corresponding G-equivariant connection is given by

$$\mathcal{A} = \begin{pmatrix} A^{1,3} \otimes \mathbf{1}_2 + \mathbf{1}_{p_{1,3}} \otimes (B_{(1)} + 3a\ \mathbf{1}_2) & \phi^+_{0,0} \otimes \bar\beta^+_{0,0} & \phi^-_{2,0} \otimes \bar\beta^-_{2,0} & 0 \\ -\phi^+_{0,0}{}^\dagger \otimes \bar\beta^+_{0,0}{}^\dagger & A^{0,0} \otimes 1 & 0 & \phi^-_{1,-3} \otimes \bar\beta^-_{1,-3} \\ -\phi^-_{2,0}{}^\dagger \otimes \bar\beta^-_{2,0}{}^\dagger & 0 & A^{2,0} \otimes \mathbf{1}_3 + \mathbf{1}_{p_{2,0}} \otimes B_{(2)} & \phi^+_{1,-3} \otimes \bar\beta^+_{1,-3} \\ 0 & -\phi^-_{1,-3}{}^\dagger \otimes \bar\beta^-_{1,-3}{}^\dagger & -\phi^+_{1,-3}{}^\dagger \otimes \bar\beta^+_{1,-3}{}^\dagger & A^{1,-3} \otimes \mathbf{1}_2 + \mathbf{1}_{p_{1,-3}} \otimes (B_{(1)} - 3a\ \mathbf{1}_2) \end{pmatrix}\,. \tag{3.125}$$

By using the flatness condition on (3.121) and (3.122), the matrix elements of the curvature two-form (3.30) with respect to the 4×4 block decomposition in (3.125) are found to be

$$\begin{aligned}
\mathcal{F}^{11} &= F^{1,3}\otimes \mathbf{1}_2 + \big(\mathbf{1}_{p_{1,3}} - \phi^+_{0,0}\,{\phi^+_{0,0}}^\dagger\big)\otimes\big(\bar\beta^+_{0,0}\wedge{\bar\beta^+_{0,0}}^\dagger\big)\\
&\quad + \big(\mathbf{1}_{p_{1,3}} - \phi^-_{2,0}\,{\phi^-_{2,0}}^\dagger\big)\otimes\big(\bar\beta^-_{2,0}\wedge{\bar\beta^-_{2,0}}^\dagger\big)\,, && (3.126)\\
\mathcal{F}^{22} &= F^{0,0}\otimes 1 + \big({\phi^+_{0,0}}^\dagger\,\phi^+_{0,0} - \phi^-_{1,-3}\,{\phi^-_{1,-3}}^\dagger\big)\otimes\big(\bar\beta^-_{1,-3}\wedge{\bar\beta^-_{1,-3}}^\dagger\big)\,, && (3.127)\\
\mathcal{F}^{33} &= F^{2,0}\otimes \mathbf{1}_3 + \big(\mathbf{1}_{p_{2,0}} - {\phi^-_{2,0}}^\dagger\,\phi^-_{2,0}\big)\otimes\big({\bar\beta^-_{2,0}}^\dagger\wedge\bar\beta^-_{2,0}\big)\\
&\quad + \big(\mathbf{1}_{p_{2,0}} - \phi^+_{1,-3}\,{\phi^+_{1,-3}}^\dagger\big)\otimes\big(\bar\beta^+_{1,-3}\wedge{\bar\beta^+_{1,-3}}^\dagger\big)\,, && (3.128)\\
\mathcal{F}^{44} &= F^{1,-3}\otimes \mathbf{1}_2 + \big(\mathbf{1}_{p_{1,-3}} - {\phi^-_{1,-3}}^\dagger\,\phi^-_{1,-3}\big)\otimes\big({\bar\beta^-_{1,-3}}^\dagger\wedge\bar\beta^-_{1,-3}\big)\\
&\quad + \big(\mathbf{1}_{p_{1,-3}} - {\phi^+_{1,-3}}^\dagger\,\phi^+_{1,-3}\big)\otimes\big({\bar\beta^+_{1,-3}}^\dagger\wedge\bar\beta^+_{1,-3}\big)\,, && (3.129)\\
\mathcal{F}^{12} &= \big(\mathrm{d}\phi^+_{0,0} + A^{1,3}\,\phi^+_{0,0} - \phi^+_{0,0}\,A^{0,0}\big)\wedge\bar\beta^+_{0,0}\,, && (3.130)\\
\mathcal{F}^{13} &= \big(\mathrm{d}\phi^-_{2,0} + A^{1,3}\,\phi^-_{2,0} - \phi^-_{2,0}\,A^{2,0}\big)\wedge\bar\beta^-_{2,0}\,, && (3.131)\\
\mathcal{F}^{14} &= \big(\phi^+_{0,0}\,\phi^-_{1,-3} - \phi^-_{2,0}\,\phi^+_{1,-3}\big)\otimes\big(\bar\beta^+_{0,0}\wedge\bar\beta^-_{1,-3}\big)\,, && (3.132)\\
\mathcal{F}^{23} &= \big({\phi^+_{0,0}}^\dagger\,\phi^-_{2,0} - \phi^-_{1,-3}\,{\phi^+_{1,-3}}^\dagger\big)\otimes\big(\bar\beta^-_{1,-3}\wedge{\bar\beta^+_{1,-3}}^\dagger\big)\,, && (3.133)\\
\mathcal{F}^{24} &= \big(\mathrm{d}\phi^-_{1,-3} + A^{0,0}\,\phi^-_{1,-3} - \phi^-_{1,-3}\,A^{1,-3}\big)\wedge\bar\beta^-_{1,-3}\,, && (3.134)\\
\mathcal{F}^{34} &= \big(\mathrm{d}\phi^+_{1,-3} + A^{2,0}\,\phi^+_{1,-3} - \phi^+_{1,-3}\,A^{1,-3}\big)\wedge\bar\beta^+_{1,-3} && (3.135)
\end{aligned}$$

plus their hermitean conjugates $\mathcal{F}^{ba} = -(\mathcal{F}^{ab})^\dagger$ for $a<b$. The holomorphic relation equation (2.29) is contained in (3.132). The explicit matrix one-form products can be computed from (3.122) to get

$$\bar\beta^+_{0,0}\wedge{\bar\beta^+_{0,0}}^\dagger = \frac{3}{2}\begin{pmatrix}\bar\beta^1\wedge\beta^1 & \bar\beta^1\wedge\beta^2\\ \bar\beta^2\wedge\beta^1 & \bar\beta^2\wedge\beta^2\end{pmatrix}\,, \tag{3.136}$$

$$\bar\beta^-_{2,0}\wedge{\bar\beta^-_{2,0}}^\dagger = \begin{pmatrix}\frac12\,\bar\beta^1\wedge\beta^1+\bar\beta^2\wedge\beta^2 & -\frac12\,\bar\beta^1\wedge\beta^2\\ -\frac12\,\bar\beta^2\wedge\beta^1 & \bar\beta^1\wedge\beta^1+\frac12\,\bar\beta^2\wedge\beta^2\end{pmatrix}\,, \tag{3.137}$$

$${\bar\beta^-_{2,0}}^\dagger\wedge\bar\beta^-_{2,0} = \begin{pmatrix}\beta^2\wedge\bar\beta^2 & -\sqrt{\frac12}\,\beta^2\wedge\bar\beta^1 & 0\\ -\sqrt{\frac12}\,\beta^1\wedge\bar\beta^2 & \frac12\big(\beta^1\wedge\bar\beta^1+\beta^2\wedge\bar\beta^2\big) & -\sqrt{\frac12}\,\beta^2\wedge\bar\beta^1\\ 0 & -\sqrt{\frac12}\,\beta^1\wedge\bar\beta^2 & \beta^1\wedge\bar\beta^1\end{pmatrix}\,, \tag{3.138}$$

$$\bar\beta^-_{1,-3}\wedge{\bar\beta^-_{1,-3}}^\dagger = \frac32\big(\bar\beta^1\wedge\beta^1+\bar\beta^2\wedge\beta^2\big)\,, \tag{3.139}$$

$${\bar\beta^-_{1,-3}}^\dagger\wedge\bar\beta^-_{1,-3} = \frac32\begin{pmatrix}\beta^2\wedge\bar\beta^2 & -\beta^2\wedge\bar\beta^1\\ -\beta^1\wedge\bar\beta^2 & \beta^1\wedge\bar\beta^1\end{pmatrix}\,, \tag{3.140}$$

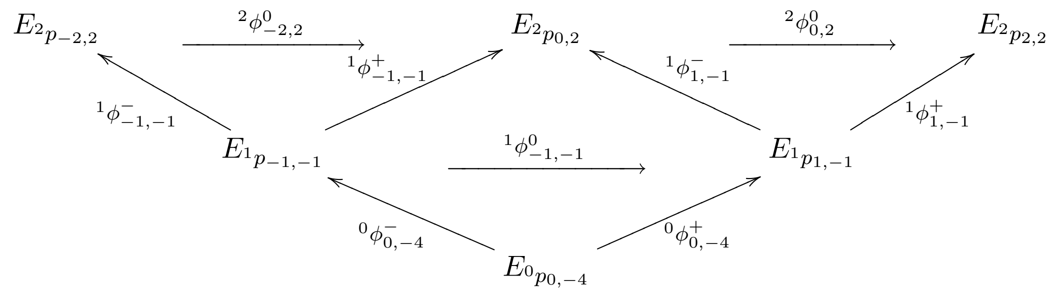
$$\bar\beta^+_{1,-3}\wedge{\bar\beta^+_{1,-3}}^\dagger = \begin{pmatrix}\bar\beta^1\wedge\beta^1 & \sqrt{\frac12}\,\bar\beta^1\wedge\beta^2 & 0\\ \sqrt{\frac12}\,\bar\beta^2\wedge\beta^1 & \frac12\big(\bar\beta^1\wedge\beta^1+\bar\beta^2\wedge\beta^2\big) & \sqrt{\frac12}\,\bar\beta^1\wedge\beta^2\\ 0 & \sqrt{\frac12}\,\bar\beta^2\wedge\beta^1 & \bar\beta^2\wedge\beta^2\end{pmatrix}\,, \tag{3.141}$$

$${\bar\beta^+_{1,-3}}^\dagger\wedge\bar\beta^+_{1,-3} = \begin{pmatrix}\beta^1\wedge\bar\beta^1+\frac12\,\beta^2\wedge\bar\beta^2 & \frac12\,\beta^2\wedge\bar\beta^1\\ \frac12\,\beta^1\wedge\bar\beta^2 & \frac12\,\beta^1\wedge\bar\beta^1+\beta^2\wedge\bar\beta^2\end{pmatrix}\,, \tag{3.142}$$

$$\bar\beta^+_{0,0}\wedge\bar\beta^-_{1,-3} = \frac{3}{2}\,\bar\beta^1\wedge\bar\beta^2\ \mathbf{1}_2\,, \tag{3.143}$$

$$\bar\beta^-_{1,-3}\wedge\bar\beta^+_{1,-3}{}^\dagger = \sqrt{\frac{3}{2}}\ \Big(\bar\beta^2\wedge\beta^1\,,\ -\sqrt{\frac{1}{2}}\ \big(\bar\beta^1\wedge\beta^1+\bar\beta^2\wedge\beta^2\big)\,,\ -\bar\beta^1\wedge\beta^2\Big)\ . \tag{3.144}$$

Non-symmetric $\underline{C}^{2,0}$ quiver bundles. For the space Q_3 and the $\underline{C}^{2,0}$ representation of SU(3), the flat connection (3.83) is obtained by substituting in (3.103) and (3.104). The quiver bundle over M_D in this case is given by



(3.145)

and the corresponding G-equivariant connection one-form on $M_D\times Q_3$ is given by

$$\mathcal{A} = \begin{pmatrix} {}_2A^{2,2}+2a_1 & \sqrt2\ {}^2\phi^0_{0,2}\,\bar\gamma^3 & 0 & \sqrt2\ {}^1\phi^+_{1,-1}\,\bar\gamma^1 & 0 & 0 \\ -\sqrt2\ {}^2\phi^0_{0,2}{}^\dagger\,\gamma^3 & {}_2A^{0,2}-a_2 & \sqrt2\ {}^2\phi^0_{-2,2}\,\bar\gamma^3 & {}^1\phi^-_{1,-1}\,\bar\gamma^2 & {}^1\phi^+_{-1,-1}\,\bar\gamma^1 & 0 \\ 0 & -\sqrt2\ {}^2\phi^0_{-2,2}{}^\dagger\,\gamma^3 & {}_2A^{-2,2}-2(a_1+a_2) & 0 & \sqrt2\ {}^1\phi^-_{-1,-1}\,\bar\gamma^2 & 0 \\ -\sqrt2\ {}^1\phi^+_{1,-1}{}^\dagger\,\gamma^1 & -{}^1\phi^-_{1,-1}{}^\dagger\,\gamma^2 & 0 & {}_1A^{1,-1}+(a_1+a_2) & {}^1\phi^0_{-1,-1}\,\bar\gamma^3 & \sqrt2\ {}^0\phi^+_{0,-4}\,\bar\gamma^1 \\ 0 & -{}^1\phi^+_{-1,-1}{}^\dagger\,\gamma^1 & -\sqrt2\ {}^1\phi^-_{-1,-1}{}^\dagger\,\gamma^2 & -{}^1\phi^0_{-1,-1}{}^\dagger\,\gamma^3 & {}_1A^{-1,-1}-a_1 & \sqrt2\ {}^0\phi^-_{0,-4}\,\bar\gamma^2 \\ 0 & 0 & 0 & -\sqrt2\ {}^0\phi^+_{0,-4}{}^\dagger\,\gamma^1 & -\sqrt2\ {}^0\phi^-_{0,-4}{}^\dagger\,\gamma^2 & {}_0A^{0,-4}+2a_2 \end{pmatrix}\ . \tag{3.146}$$

Its curvature (3.30) with respect to this 6×6 block decomposition has the non-vanishing matrix elements

$$\mathcal{F}^{11} = {}_2F^{2,2}+2\big(\mathbf{1}_{2p_{2,2}}-{}^1\phi^+_{1,-1}\,{}^1\phi^+_{1,-1}{}^\dagger\big)\,\bar\gamma^1\wedge\gamma^1+2\big(\mathbf{1}_{2p_{2,2}}-{}^2\phi^0_{0,2}\,{}^2\phi^0_{0,2}{}^\dagger\big)\,\bar\gamma^3\wedge\gamma^3\,, \tag{3.147}$$

$$\begin{aligned}\mathcal{F}^{12} &= \sqrt2\,\big(\mathrm{d}\,{}^2\phi^0_{0,2}+{}_2A^{2,2}\ {}^2\phi^0_{0,2}-{}^2\phi^0_{0,2}\ {}_2A^{0,2}\big)\wedge\bar\gamma^3 \\ &\quad +\sqrt2\,\big({}^2\phi^0_{0,2}-{}^1\phi^+_{1,-1}\,{}^1\phi^-_{1,-1}{}^\dagger\big)\,\bar\gamma^1\wedge\gamma^2\,,\end{aligned} \tag{3.148}$$

$$\begin{aligned}\mathcal{F}^{14} &= \sqrt2\,\big(\mathrm{d}\,{}^1\phi^+_{1,-1}+{}_2A^{2,2}\ {}^1\phi^+_{1,-1}-{}^1\phi^+_{1,-1}\ {}_1A^{1,-1}\big)\wedge\bar\gamma^1 \\ &\quad +\sqrt2\,\big({}^1\phi^+_{1,-1}-{}^2\phi^0_{0,2}\,{}^1\phi^-_{1,-1}\big)\,\bar\gamma^2\wedge\bar\gamma^3\,,\end{aligned} \tag{3.149}$$

$$\mathcal{F}^{15} = \sqrt2\,\big({}^2\phi^0_{0,2}\,{}^1\phi^+_{-1,-1}-{}^1\phi^+_{1,-1}\,{}^1\phi^0_{-1,-1}\big)\,\bar\gamma^3\wedge\bar\gamma^1\,, \tag{3.150}$$

$$\begin{aligned}\mathcal{F}^{22} &= {}_2F^{0,2}+\big(\mathbf{1}_{2p_{0,2}}-{}^1\phi^+_{-1,-1}\,{}^1\phi^+_{-1,-1}{}^\dagger\big)\,\bar\gamma^1\wedge\gamma^1+\big(\mathbf{1}_{2p_{0,2}}-{}^1\phi^-_{1,-1}\,{}^1\phi^-_{1,-1}{}^\dagger\big)\,\bar\gamma^2\wedge\gamma^2 \\ &\quad +2\,\big({}^2\phi^0_{0,2}{}^\dagger\,{}^2\phi^0_{0,2}-{}^2\phi^0_{-2,2}\,{}^2\phi^0_{-2,2}{}^\dagger\big)\,\bar\gamma^3\wedge\gamma^3\,,\end{aligned} \tag{3.151}$$

$$\begin{aligned}\mathcal{F}^{23} &= \sqrt2\,\big(\mathrm{d}\,{}^2\phi^0_{-2,2}+{}_2A^{0,2}\ {}^2\phi^0_{-2,2}-{}^2\phi^0_{-2,2}\ {}_2A^{-2,2}\big)\wedge\bar\gamma^3 \\ &\quad +\sqrt2\,\big({}^2\phi^0_{-2,2}-{}^1\phi^+_{-1,-1}\,{}^1\phi^-_{-1,-1}{}^\dagger\big)\,\bar\gamma^1\wedge\gamma^2\,,\end{aligned} \tag{3.152}$$

$$\begin{aligned}\mathcal{F}^{24} &= \big(\mathrm{d}\,{}^1\phi^-_{1,-1}+{}_2A^{0,2}\ {}^1\phi^-_{1,-1}-{}^1\phi^-_{1,-1}\ {}_1A^{1,-1}\big)\wedge\bar\gamma^2 \\ &\quad -\big({}^1\phi^-_{1,-1}-2\ {}^2\phi^0_{0,2}{}^\dagger\,{}^1\phi^+_{1,-1}+{}^1\phi^+_{-1,-1}\,{}^1\phi^0_{-1,-1}{}^\dagger\big)\bar\gamma^1\wedge\gamma^3\,,\end{aligned} \tag{3.153}$$

$$\begin{aligned}\mathcal{F}^{25} &= \big(\mathrm{d}\,{}^1\phi^+_{-1,-1}+{}_2A^{0,2}\ {}^1\phi^+_{-1,-1}-{}^1\phi^+_{-1,-1}\ {}_1A^{-1,-1}\big)\wedge\bar\gamma^1 \\ &\quad +\big({}^1\phi^+_{-1,-1}+{}^1\phi^-_{1,-1}\,{}^1\phi^0_{-1,-1}-2\ {}^2\phi^0_{-2,2}\,{}^1\phi^-_{-1,-1}\big)\,\bar\gamma^2\wedge\bar\gamma^3\,,\end{aligned} \tag{3.154}$$

$$\mathcal{F}^{26} = \sqrt{2}\left({}^1\phi^{+}_{-1,-1}\,{}^0\phi^{-}_{0,-4} - {}^1\phi^{-}_{1,-1}\,{}^0\phi^{+}_{0,-4}\right)\bar\gamma^1\wedge\bar\gamma^2\,, \tag{3.155}$$

$$\begin{aligned}\mathcal{F}^{33} &= {}_2F^{-2,2} + 2\big(\mathbf{1}_{{}^2p_{-2,2}} - {}^1\phi^{-}_{-1,-1}\,{}^1\phi^{-}_{-1,-1}{}^\dagger\big)\,\bar\gamma^2\wedge\gamma^2 \\ &\quad - 2\big(\mathbf{1}_{{}^2p_{-2,2}} - {}^2\phi^{0}_{-2,2}{}^\dagger\,{}^2\phi^{0}_{-2,2}\big)\,\bar\gamma^3\wedge\gamma^3\,,\end{aligned} \tag{3.156}$$

$$\mathcal{F}^{34} = \sqrt{2}\left({}^2\phi^{0}_{-2,2}{}^\dagger\,{}^1\phi^{-}_{1,-1} - {}^1\phi^{-}_{-1,-1}\,{}^1\phi^{0}_{-1,-1}{}^\dagger\right)\bar\gamma^2\wedge\gamma^3\,, \tag{3.157}$$

$$\begin{aligned}\mathcal{F}^{35} &= \sqrt{2}\,\big(\mathrm{d}\,{}^1\phi^{-}_{-1,-1} + {}_2A^{-2,2}\;{}^1\phi^{-}_{-1,-1} - {}^1\phi^{-}_{-1,-1}\;{}_1A^{-1,-1}\big)\wedge\bar\gamma^2 \\ &\quad - \sqrt{2}\left({}^1\phi^{-}_{-1,-1} - {}^2\phi^{-}_{-2,2}{}^\dagger\,{}^1\phi^{+}_{-1,-1}\right)\bar\gamma^1\wedge\gamma^3\,,\end{aligned} \tag{3.158}$$

$$\begin{aligned}\mathcal{F}^{44} &= {}_1F^{1,-1} - \big(\mathbf{1}_{{}^1p_{1,-1}} - {}^1\phi^{-}_{1,-1}{}^\dagger\,{}^1\phi^{-}_{1,-1}\big)\,\bar\gamma^2\wedge\gamma^2 + \big(\mathbf{1}_{{}^1p_{1,-1}} - {}^1\phi^{0}_{-1,-1}\,{}^1\phi^{0}_{-1,-1}{}^\dagger\big)\,\bar\gamma^3\wedge\gamma^3 \\ &\quad + 2\big({}^1\phi^{+}_{1,-1}{}^\dagger\,{}^1\phi^{+}_{1,-1} - {}^0\phi^{+}_{0,-4}\,{}^0\phi^{+}_{0,-4}{}^\dagger\big)\,\bar\gamma^1\wedge\gamma^1\,,\end{aligned} \tag{3.159}$$

$$\begin{aligned}\mathcal{F}^{45} &= \big(\mathrm{d}\,{}^1\phi^{0}_{-1,-1} + {}_1A^{1,-1}\;{}^1\phi^{0}_{-1,-1} - {}^1\phi^{0}_{-1,-1}\;{}_1A^{-1,-1}\big)\wedge\bar\gamma^3 \\ &\quad + \left({}^1\phi^{0}_{-1,-1} + {}^1\phi^{-}_{1,-1}{}^\dagger\,{}^1\phi^{+}_{-1,-1} - 2\;{}^0\phi^{+}_{0,-4}\,{}^0\phi^{-}_{0,-4}{}^\dagger\right)\bar\gamma^1\wedge\gamma^2\,,\end{aligned} \tag{3.160}$$

$$\begin{aligned}\mathcal{F}^{46} &= \sqrt{2}\,\big(\mathrm{d}\,{}^0\phi^{+}_{0,-4} + {}_1A^{1,-1}\;{}^0\phi^{+}_{0,-4} - {}^0\phi^{+}_{0,-4}\;{}_0A^{0,-4}\big)\wedge\bar\gamma^1 \\ &\quad + \sqrt{2}\left({}^0\phi^{+}_{0,-4} - {}^1\phi^{0}_{-1,-1}\,{}^0\phi^{-}_{0,-4}\right)\bar\gamma^2\wedge\bar\gamma^3\,,\end{aligned} \tag{3.161}$$

$$\begin{aligned}\mathcal{F}^{55} &= {}_1F^{-1,-1} - \big(\mathbf{1}_{{}^1p_{-1,-1}} - {}^1\phi^{+}_{-1,-1}{}^{\dagger 1}\phi^{+}_{-1,-1}\big)\bar\gamma^1\wedge\gamma^1 - \big(\mathbf{1}_{{}^1p_{-1,-1}} - {}^1\phi^{0}_{-1,-1}{}^{\dagger 1}\phi^{0}_{-1,-1}\big)\bar\gamma^3\wedge\gamma^3 \\ &\quad + 2\big({}^1\phi^{-}_{-1,-1}{}^\dagger\,{}^1\phi^{-}_{-1,-1} - {}^0\phi^{-}_{0,-4}\,{}^0\phi^{-}_{0,-4}{}^\dagger\big)\,\bar\gamma^2\wedge\gamma^2\,,\end{aligned} \tag{3.162}$$

$$\begin{aligned}\mathcal{F}^{56} &= \sqrt{2}\,\big(\mathrm{d}\,{}^0\phi^{-}_{0,-4} + {}_1A^{-1,-1}\;{}^0\phi^{-}_{0,-4} - {}^0\phi^{-}_{0,-4}\;{}_0A^{0,-4}\big)\wedge\bar\gamma^2 \\ &\quad - \sqrt{2}\,\big({}^0\phi^{-}_{0,-4} - {}^1\phi^{0}_{-1,-1}{}^\dagger\,{}^0\phi^{+}_{0,-4}\big)\,\bar\gamma^1\wedge\gamma^3\,,\end{aligned} \tag{3.163}$$

$$\begin{aligned}\mathcal{F}^{66} &= {}_0F^{0,-4} - 2\big(\mathbf{1}_{{}^0p_{0,-4}} - {}^0\phi^{+}_{0,-4}{}^\dagger\,{}^0\phi^{+}_{0,-4}\big)\,\bar\gamma^1\wedge\gamma^1 \\ &\quad -2\big(\mathbf{1}_{{}^0p_{0,-4}} - {}^0\phi^{-}_{0,-4}{}^\dagger\,{}^0\phi^{-}_{0,-4}\big)\,\bar\gamma^2\wedge\gamma^2\end{aligned} \tag{3.164}$$

plus their hermitean conjugates $\mathcal{F}^{ba} = -(\mathcal{F}^{ab})^\dagger$ for $a < b$.

Non-symmetric $\underline{C}^{1,1}$ quiver bundles. Our final example of this section is the G-equivariant connection on a G-bundle over $M_D \times Q_3$ related to the adjoint representation $\underline{C}^{1,1}$. It has the form

$$\begin{aligned}\mathcal{A} &= \begin{pmatrix} 0 & {}^1\phi^{0}_{-1,3}\,\bar\gamma^3 & \sqrt{\tfrac32}\;{}^0\phi^{+}_{0,0}\,\bar\gamma^1 & {}^2\phi^{-}_{2,0}\,\bar\gamma^2 & -\sqrt{\tfrac12}\;{}^2\phi^{+}_{0,0}\,\bar\gamma^1 & 0 & 0 & 0 \\ & 0 & \sqrt{\tfrac32}\;{}^0\phi^{-}_{0,0}\,\bar\gamma^2 & 0 & \sqrt{\tfrac12}\;{}^2\phi^{-}_{0,0}\,\bar\gamma^2 & -{}^2\phi^{+}_{-2,0}\,\bar\gamma^1 & 0 & 0 \\ & & 0 & 0 & 0 & 0 & \sqrt{\tfrac32}\;{}^0\phi^{-}_{1,-3}\,\bar\gamma^2 & -\sqrt{\tfrac32}\;{}^0\phi^{+}_{-1,-3}\,\bar\gamma^1 \\ & & & 0 & \sqrt{2}\;{}^2\phi^{0}_{0,0}\,\bar\gamma^3 & 0 & {}^1\phi^{+}_{1,-3}\,\bar\gamma^1 & 0 \\ & & & & 0 & \sqrt{2}\;{}^2\phi^{0}_{-2,0}\,\bar\gamma^3 & \sqrt{\tfrac12}\;{}^1\phi^{-}_{1,-3}\,\bar\gamma^2 & \sqrt{\tfrac12}\;{}^1\phi^{+}_{-1,-3}\,\bar\gamma^1 \\ & -\,\mathrm{h.c.} & & & & 0 & 0 & {}^1\phi^{-}_{-1,-3}\,\bar\gamma^2 \\ & & & & & & 0 & {}^1\phi^{0}_{-1,-3}\,\bar\gamma^3 \\ & & & & & & & 0 \end{pmatrix} \\ &\quad + \mathrm{diag}\big({}_1A^{1,3} + a_1 - a_2,\, {}_1A^{-1,3} - a_1 - 2a_2,\, {}_0A^{0,0},\, {}_2A^{2,0} + 2a_1 + a_2,\, {}_2A^{0,0}, \\ &\qquad\quad {}_2A^{-2,0} - 2a_1 - a_2,\, {}_1A^{1,-3} + a_1 + 2a_2,\, {}_1A^{-1,-3} - a_1 + a_2\big)\end{aligned} \tag{3.165}$$

where h.c. indicates the hermitean conjugate of the upper triangular matrix. We omit the rather complicated list of curvature matrix elements.

4. Nonabelian quiver vortex equations

In this section we will describe explicitly the dimensional reduction of gauge theory equations on manifolds of the form (3.1), in the case that M_D is a Kähler manifold of (real) dimension $D = 2d$. Using the quiver bundle constructions of the previous section, we will find that the Yang-Mills equations on X dimensionally reduce to equations on M_{2d} due to the G/H dependence of fields prescribed by G-equivariance. In particular, we will see that the BPS equations which give the natural analog of instantons on X reduce to nonabelian coupled vortex equations on M_{2d}.

4.1 BPS equations

Let us pick a complex structure on the Kähler manifold M_{2d} and local complex coordinates $(z^1, \ldots, z^d) \in \mathbb{C}^d$. In these coordinates the riemannian metric on X takes the form

$$\mathrm{d}s^2 = 2\, g_{a\bar{b}}\ \mathrm{d}z^a\ \mathrm{d}\bar{z}^{\bar{b}} + 2\, G_{i\bar{\jmath}}\ \mathrm{d}y^i\ \mathrm{d}\bar{y}^{\bar{\jmath}}\ , \tag{4.1}$$

where $g_{a\bar{b}}$, $1 \le a, b \le d$, (resp. $G_{i\bar{\jmath}}$, $1 \le i, j \le d_H/2$, $d_H := \dim(G/H)$) is the Kähler metric on M_{2d} (resp. G/H) and y^i denote local complex coordinates on the homogeneous space G/H. The Hodge duality operator on X constructed from (4.1) is denoted $*$. The Kähler two-form Ω on X is given by

$$\Omega = -2\,\mathrm{i}\, g_{a\bar{b}}\ \mathrm{d}z^a \wedge \mathrm{d}\bar{z}^{\bar{b}} - 2\,\mathrm{i}\, G_{i\bar{\jmath}}\ \mathrm{d}y^i \wedge \mathrm{d}\bar{y}^{\bar{\jmath}}\ . \tag{4.2}$$

Let $\mathcal{E} \to X$ be a hermitean vector bundle of rank p with the structure group $\mathrm{U}(p)$ and gauge connection $\mathcal{A}$. The corresponding curvature (3.30) obeys the Bianchi identity

$$D_{\mathcal{A}}\mathcal{F} = 0\,, \tag{4.3}$$

where $D_{\mathcal{A}}$ is the gauge covariant derivative constructed from $\mathcal{A}$. The vacuum Yang-Mills equations are

$$D_{\mathcal{A}}(*\mathcal{F}) = 0\ . \tag{4.4}$$

One can write the curvature two-form via its Kähler decomposition as $\mathcal{F} = \mathcal{F}^{2,0} + \mathcal{F}^{1,1} + \mathcal{F}^{0,2}$ with $\mathcal{F}^{r,s} \in \Omega^{r,s}(X, \mathrm{u}(p))$. Stable gauge bundles $\mathcal{E}$ are then those which solve the Donaldson-Uhlenbeck-Yau (DUY) equations [2]

$$*\Omega \wedge \mathcal{F} \;=\; 0 \qquad \text{and} \qquad \mathcal{F}^{2,0} \;=\; 0 \;=\; \mathcal{F}^{0,2}\ . \tag{4.5}$$

These equations generalize the usual self-duality equations in four dimensions, and they imply the Yang-Mills equations given by (4.4).

In the local complex coordinates (z^a, y^i) the DUY equations take the form

$$g^{a\bar{b}}\, \mathcal{F}_{z^a\bar{z}^{\bar{b}}} + G^{i\bar{\jmath}}\, \mathcal{F}_{y^i\bar{y}^{\bar{\jmath}}} \;=\; 0\,, \tag{4.6}$$

$$\mathcal{F}_{\bar{z}^{\bar{a}}\bar{z}^{\bar{b}}} \;=\; 0 \;=\; \mathcal{F}_{z^a z^b}\ , \tag{4.7}$$

$$\mathcal{F}_{\bar{z}^{\bar{a}}\bar{y}^{\bar{\imath}}} \;=\; 0 \;=\; \mathcal{F}_{z^a y^i}\ , \tag{4.8}$$

$$\mathcal{F}_{\bar{y}^{\bar{\imath}}\bar{y}^{\bar{\jmath}}} \;=\; 0 \;=\; \mathcal{F}_{y^i y^j} \tag{4.9}$$

for $a, b = 1, \dots, d$ and $i, j = 1, \dots, d_H/2$. For a given quiver bundle $\mathcal{E}^{k,l} \to X$, both (4.4) and (4.5) dimensionally reduce to equations on M_{2d} alone for the anti-hermitean $\mathrm{u}(p_v)$ gauge connections

$$A^v = A^v_a \, \mathrm{d}z^a + A^v_{\bar{a}} \, \mathrm{d}\bar{z}^{\bar{a}} \tag{4.10}$$

with curvatures

$$F^v = F^v_{ab} \, \mathrm{d}z^a \wedge \mathrm{d}z^b + 2F^v_{a\bar{b}} \, \mathrm{d}z^a \wedge \mathrm{d}\bar{z}^{\bar{b}} + F^v_{\bar{a}\bar{b}} \, \mathrm{d}\bar{z}^{\bar{a}} \wedge \mathrm{d}\bar{z}^{\bar{b}} \tag{4.11}$$

coupled to the Higgs fields $\phi_{v,\Phi(v)}$, for each vertex $v \in \mathsf{Q}_0(k,l)$ and arrow $\Phi \in \mathsf{Q}_1(k,l)$. With the form of the gauge potentials in the previous section, the equation (4.7) implies

$$F^v_{\bar{a}\bar{b}} \;=\; 0 \;=\; F^v_{ab} \tag{4.12}$$

which expresses holomorphicity of the vector bundle $E_{p_v} \to M_{2d}$. The remaining equations (4.6), (4.8) and (4.9) lead to nonabelian quiver vortex equations on M_{2d}.

Symmetric $\underline{C}^{k,l}$ quiver vortices. Let us fix a vertex $v = (n,m) \in \mathsf{Q}_0(k,l)$ of the symmetric quiver associated to the SU(3) representation $\underline{C}^{k,l}$ and substitute the field strength matrix elements (3.79)–(3.82) in the Biedenharn basis into the DUY equations (4.6)–(4.9). The only contribution to (4.8) comes from (3.80) which yields the bi-covariant Higgs field derivatives

$$\partial_{\bar{a}} \phi^{\pm}_{n,m} + A^{n\pm1,m+3}_{\bar{a}} \, \phi^{\pm}_{n,m} - \phi^{\pm}_{n,m} \, A^{n,m}_{\bar{a}} = 0 \tag{4.13}$$

plus their hermitean conjugates, where $\partial_{\bar{a}} := \frac{\partial}{\partial \bar{z}^{\bar{a}}}$. This expresses the fact that the BPS bundle morphisms $\phi^{\pm}_{n,m}$ are holomorphic maps. Likewise, only the matrix element (3.81) contributes to (4.9) which implies the holomorphic relations

$$\phi^{+}_{n,m} \, \phi^{-}_{n+1,m-3} - \phi^{-}_{n+2,m} \, \phi^{+}_{n+1,m-3} = 0 \ . \tag{4.14}$$

Note that non-holomorphic relations (such as those implied by the vanishing of (3.82)) do not arise as BPS conditions.

The remaining instanton equation (4.6) must be satisfied by the matrix elements (3.79). A Kähler two-form on $\mathbb{C}P^2$ can be identified locally as in (3.22). Using (3.73) the pertinent matrix one-form products are given by

$$\begin{aligned} \bar{\beta}^{\pm}_{n,m}{}^{\dagger} \wedge \bar{\beta}^{\pm}_{n,m} = & \sum_{q \in \{-n+2j\}_{j=0}^{n}} \lambda^{\pm}_{k,l}(n,m)^2 \\ & \times \Bigg\{ \Bigg(\begin{bmatrix} \frac{n}{2} & \frac{1}{2} & \frac{n\pm1}{2} \\ \frac{q}{2} & -\frac{1}{2} & \frac{q-1}{2} \end{bmatrix}^2 \beta^1 \wedge \bar{\beta}^1 + \begin{bmatrix} \frac{n}{2} & \frac{1}{2} & \frac{n\pm1}{2} \\ \frac{q}{2} & \frac{1}{2} & \frac{q+1}{2} \end{bmatrix}^2 \beta^2 \wedge \bar{\beta}^2 \Bigg) \Big|{n \atop q}, m\Big\rangle \Big\langle {n \atop q}, m\Big| \\ & \quad + \begin{bmatrix} \frac{n}{2} & \frac{1}{2} & \frac{n\pm1}{2} \\ \frac{q}{2} & -\frac{1}{2} & \frac{q-1}{2} \end{bmatrix} \begin{bmatrix} \frac{n}{2} & \frac{1}{2} & \frac{n\pm1}{2} \\ \frac{q-2}{2} & \frac{1}{2} & \frac{q-1}{2} \end{bmatrix} \beta^1 \wedge \bar{\beta}^2 \, \Big|{n \atop q}, m\Big\rangle \Big\langle {n \atop q-2}, m\Big| \\ & \quad + \begin{bmatrix} \frac{n}{2} & \frac{1}{2} & \frac{n\pm1}{2} \\ \frac{q}{2} & \frac{1}{2} & \frac{q+1}{2} \end{bmatrix} \begin{bmatrix} \frac{n}{2} & \frac{1}{2} & \frac{n\pm1}{2} \\ \frac{q+2}{2} & -\frac{1}{2} & \frac{q+1}{2} \end{bmatrix} \beta^2 \wedge \bar{\beta}^1 \Big|{n \atop q}, m\Big\rangle \Big\langle {n \atop q+2}, m\Big| \Bigg\} , \end{aligned} \tag{4.15}$$

$$\bar{\beta}^{\pm}_{n\mp1,m-3} \wedge \bar{\beta}^{\pm}_{n\mp1,m-3}{}^{\dagger} = \sum_{q\in\{-n+2j\}_{j=0}^{n}} \lambda^{\pm}_{k,l}(n\mp1,m-3)^2$$
$$\times\Bigg\{\Bigg(\begin{bmatrix} \frac{n\mp1}{2} & \frac12 & \frac n2 \\ \frac{q+1}{2} & -\frac12 & \frac q2 \end{bmatrix}^2 \bar{\beta}^1\wedge\beta^1 + \begin{bmatrix} \frac{n\mp1}{2} & \frac12 & \frac n2 \\ \frac{q-1}{2} & \frac12 & \frac q2 \end{bmatrix}^2 \bar{\beta}^2\wedge\beta^2\Bigg) \big|\overset{n}{q}\,,\,m\big\rangle\big\langle\overset{n}{q}\,,\,m\big|$$
$$+\begin{bmatrix} \frac{n\mp1}{2} & \frac12 & \frac n2 \\ \frac q2 & -\frac12 & \frac{q-1}{2} \end{bmatrix}\begin{bmatrix} \frac{n\mp1}{2} & \frac12 & \frac n2 \\ \frac q2 & \frac12 & \frac{q+1}{2} \end{bmatrix}$$
$$\times\Big(\bar{\beta}^1\wedge\beta^2\ \big|\overset{n}{q-1}\,,\,m\big\rangle\big\langle\overset{n}{q+1}\,,\,m\big|$$
$$+\bar{\beta}^2\wedge\beta^1\ \big|\overset{n}{q+1}\,,\,m\big\rangle\big\langle\overset{n}{q-1}\,,\,m\big|\Big)\Bigg\} \tag{4.16}$$

with $\lambda^{\pm}_{k,l}(n,m) := 0$ for $n<0$.

Upon contracting with (3.22), only the diagonal matrix elements survive in (4.15) and (4.16). It is straightforward to see from (3.66) that the sum of the squares of the Clebsch-Gordan coefficients in each case is independent of $q\in\{-n+2j\}_{j=0}^{n}$. Using the explicit coefficient functions in (3.63) one establishes the identity

$$\lambda^{+}_{k,l}(n-1,m-3)^2 + \lambda^{-}_{k,l}(n+1,m-3)^2 - \tfrac{n+2}{n+1}\,\lambda^{+}_{k,l}(n,m)^2 - \tfrac{n}{n+1}\,\lambda^{-}_{k,l}(n,m)^2 = m \tag{4.17}$$

and we arrive finally at the curvature equations

$$\begin{aligned} g^{a\bar b}\,F^{n,m}_{a\bar b} &= m + \tfrac{n+2}{n+1}\,\lambda^{+}_{k,l}(n,m)^2\ \phi^{+}_{n,m}{}^{\dagger}\,\phi^{+}_{n,m} + \tfrac{n}{n+1}\,\lambda^{-}_{k,l}(n,m)^2\ \phi^{-}_{n,m}{}^{\dagger}\,\phi^{-}_{n,m} \\ &\quad - \lambda^{+}_{k,l}(n-1,m-3)^2\ \phi^{+}_{n-1,m-3}\,\phi^{+}_{n-1,m-3}{}^{\dagger} \\ &\quad - \lambda^{-}_{k,l}(n+1,m-3)^2\ \phi^{-}_{n+1,m-3}\,\phi^{-}_{n+1,m-3}{}^{\dagger}\ . \end{aligned} \tag{4.18}$$

The constant perturbation in (4.18) is just the magnetic charge $m\in\mathbb{Z}$ at a given symmetric vertex $v=(n,m)\in\mathsf{Q}_0(k,l)$. This is a typical feature of quiver vortex equations which will play an important role in the following.

Non-symmetric $\underline{C}^{k,l}$ quiver vortices. Let us now fix a vertex $v=(q,m)_n\in\mathsf{Q}_0(k,l)$ of the non-symmetric $\underline{C}^{k,l}$ quiver. Then (4.8) together with the off-diagonal field strength matrix elements (3.94)–(3.96) yield the bi-covariant Higgs field derivatives

$$\begin{aligned} \partial_{\bar a}\,{}^{n}\phi^{+}_{q,m} + {}_{n\pm1}A^{q+1,m+3}_{\bar a}\ {}^{n}\phi^{+}_{q,m} - {}^{n}\phi^{+}_{q,m}\ {}_{n}A^{q,m}_{\bar a} &= 0\,, \\ \partial_{\bar a}\,{}^{n}\phi^{-}_{q,m} + {}_{n\pm1}A^{q-1,m+3}_{\bar a}\ {}^{n}\phi^{-}_{q,m} - {}^{n}\phi^{-}_{q,m}\ {}_{n}A^{q,m}_{\bar a} &= 0\,, \\ \partial_{\bar a}\,{}^{n}\phi^{0}_{q,m} + {}_{n}A^{q+2,m}_{\bar a}\ {}^{n}\phi^{0}_{q,m} - {}^{n}\phi^{0}_{q,m}\ {}_{n}A^{q,m}_{\bar a} &= 0 \end{aligned} \tag{4.19}$$

plus their hermitean conjugates. From (4.9) and (3.94), along with the explicit forms (3.66) and (3.85)–(3.88), we obtain the linear holomorphic relations

$$\begin{aligned} {}^{n}\phi^{+}_{q,m} - \frac12\,(q-n)\ {}^{n}\phi^{-}_{q+2,m}\,{}^{n}\phi^{0}_{q,m} + \frac12\,(q-n-2)\ {}^{n+1}\phi^{0}_{q-1,m+3}\,{}^{n}\phi^{-}_{q,m} &= 0\,, \\ {}^{n}\phi^{+}_{q,m} - \frac12\,(q+n+2)\ {}^{n}\phi^{-}_{q+2,m}\,{}^{n}\phi^{0}_{q,m} + \frac12\,(q+n)\ {}^{n-1}\phi^{0}_{q-1,m+3}\,{}^{n}\phi^{-}_{q,m} &= 0\ . \end{aligned} \tag{4.20}$$

These equations hold whenever $\lambda^{\pm}_{k,l}(n,m) \neq 0$, and for $q = n$ the second equation is absent. Likewise, from (4.9) and the curvature matrix elements (3.97)–(3.99) we find the quadratic holomorphic relations

$$\begin{aligned} {}^n\phi^+_{q,m}\, {}^{n\pm1}\phi^-_{q+1,m-3} - {}^n\phi^-_{q+2,m}\, {}^{n\pm1}\phi^+_{q+1,m-3} &= 0\,, \\ {}^n\phi^+_{q,m}\, {}^n\phi^0_{q-2,m} - {}^{n\pm1}\phi^0_{q-1,m+3}\, {}^n\phi^+_{q-2,m} &= 0\,, \\ {}^n\phi^-_{q,m}\, {}^n\phi^0_{q-2,m} - {}^{n\pm1}\phi^0_{q-3,m+3}\, {}^n\phi^-_{q-2,m} &= 0\,. \end{aligned} \tag{4.21}$$

Finally, we substitute the diagonal matrix elements (3.93) into (4.6) and use the Kähler form (3.46) on Q_3. Using the explicit coefficient functions (3.63), one finds the identity

$$\tfrac{n-q+2}{n+1}\,\lambda^+_{k,l}(n,m)^2 + \tfrac{n+q}{n+1}\,\lambda^-_{k,l}(n,m)^2 - \tfrac{n-q}{n}\,\lambda^+_{k,l}(n-1,m-3)^2 - \tfrac{n+q+2}{n+2}\,\lambda^-_{k,l}(n+1,m-3)^2 = q-m \tag{4.22}$$

which gives the curvature equations

$$\begin{aligned} g^{a\bar b}\,{}_nF^{q,m}_{a\bar b} = (q+m) &+ \frac{\lambda^+_{k,l}(n,m)^2}{3(n+1)}\Big((n+q+2)\,{}^n\phi^+_{q,m}{}^\dagger\,{}^n\phi^+_{q,m} + 2(n-q+2)\,{}^n\phi^-_{q,m}{}^\dagger\,{}^n\phi^-_{q,m}\Big) \\ &+ \frac{\lambda^-_{k,l}(n,m)^2}{3(n+1)}\Big((n-q)\,{}^n\phi^+_{q,m}{}^\dagger\,{}^n\phi^+_{q,m} + 2(n+q)\,{}^n\phi^-_{q,m}{}^\dagger\,{}^n\phi^-_{q,m}\Big) \\ &- \frac{\lambda^+_{k,l}(n-1,m-3)^2}{3n}\Big((n+q)\,{}^{n-1}\phi^+_{q-1,m-3}\,{}^{n-1}\phi^+_{q-1,m-3}{}^\dagger \\ &\qquad\qquad + 2(n-q)\,{}^{n-1}\phi^-_{q+1,m-3}\,{}^{n-1}\phi^-_{q+1,m-3}{}^\dagger\Big) \\ &- \frac{\lambda^-_{k,l}(n+1,m-3)^2}{3(n+2)}\Big((n-q+2)\,{}^{n+1}\phi^+_{q-1,m-3}\,{}^{n+1}\phi^+_{q-1,m-3}{}^\dagger \\ &\qquad\qquad + 2(n+q+2)\,{}^{n+1}\phi^-_{q+1,m-3}\,{}^{n+1}\phi^-_{q+1,m-3}{}^\dagger\Big) \\ &+ \frac{1}{3}(n-q)(n+q+2)\,{}^n\phi^0_{q,m}{}^\dagger\,{}^n\phi^0_{q,m} - \frac{1}{3}(n+q)(n-q+2)\,{}^n\phi^0_{q-2,m}\,{}^n\phi^0_{q-2,m}{}^\dagger\,. \end{aligned} \tag{4.23}$$

Again the constant perturbation is just the total magnetic charge $(q+m) \in 2\mathbb{Z}$ at the non-symmetric vertex $v = (q,m)_n \in \mathsf{Q}_0(k,l)$. The expressions (4.19)–(4.21) and (4.23) all naturally incorporate the contributions from multiple arrows at a given vertex whenever degenerate weight vectors of the SU(3) representation $\underline{C}^{k,l}$ exist.

4.2 Examples

Let us now look at some explicit instances of the BPS equations from section 4.1 above.

Symmetric $\underline{C}^{1,0}$ quiver vortices. Using (3.31) and abbreviating $\phi := \phi^+_{0,-2}$ again, the DUY equations in this case become vortex equations on M_{2d} given by

$$\begin{aligned} g^{a\bar b}\,F^{1,1}_{a\bar b} &= \mathbf{1}_{p_{1,1}} - \phi\,\phi^\dagger\,, \\ g^{a\bar b}\,F^{0,-2}_{a\bar b} &= -2\big(\mathbf{1}_{p_{0,-2}} - \phi^\dagger\,\phi\big)\,, \\ \partial_{\bar a}\phi + A^{1,1}\,\phi - \phi\,A^{0,-2} &= 0\,. \end{aligned} \tag{4.24}$$

This system is a generalization of the standard holomorphic triple (E_1, E_2, ϕ) [3, 5]. For $d = 2$, it is related to the perturbed Seiberg-Witten monopole equations on the Kähler four-manifold M_4 [6, 12].

Symmetric $\underline{C}^{2,0}$ quiver vortices. Using the Kähler two-form (3.22) along with (3.109)–(3.113) and (3.115)–(3.118), we obtain the DUY equations

$$\begin{aligned} g^{a\bar{b}}\, F^{2,2}_{a\bar{b}} &= 2\big(\mathbf{1}_{p_{2,2}} - \phi^{+}_{1,-1}\, \phi^{+}_{1,-1}{}^{\dagger}\big)\,, \\ g^{a\bar{b}}\, F^{1,-1}_{a\bar{b}} &= -\big(\mathbf{1}_{p_{1,-1}} - 3\phi^{+}_{1,-1}{}^{\dagger}\, \phi^{+}_{1,-1} + 2\phi^{+}_{0,-4}\, \phi^{+}_{0,-4}{}^{\dagger}\big)\,, \\ g^{a\bar{b}}\, F^{0,-4}_{a\bar{b}} &= -4\big(\mathbf{1}_{p_{0,-4}} - \phi^{+}_{0,-4}{}^{\dagger}\, \phi^{+}_{0,-4}\big)\,, \\ \partial_{\bar{a}}\phi^{+}_{1,-1} + A^{2,2}_{\bar{a}}\, \phi^{+}_{1,-1} - \phi^{+}_{1,-1}\, A^{1,-1}_{\bar{a}} &= 0\,, \\ \partial_{\bar{a}}\phi^{+}_{0,-4} + A^{1,-1}_{\bar{a}}\, \phi^{+}_{0,-4} - \phi^{+}_{0,-4}\, A^{0,-4}_{\bar{a}} &= 0\ . \end{aligned} \tag{4.25}$$

This system is an extension of the basic holomorphic chain vortex equations [4, 5].

Symmetric $\underline{C}^{1,1}$ quiver vortices. Using (3.126)–(3.144) we find the curvature equations

$$\begin{aligned} g^{a\bar{b}}\, F^{1,3}_{a\bar{b}} &= 3\left(\mathbf{1}_{p_{1,3}} - \frac{1}{2}\, \phi^{+}_{0,0}\, \phi^{+}_{0,0}{}^{\dagger} - \frac{1}{2}\, \phi^{-}_{2,0}\, \phi^{-}_{2,0}{}^{\dagger}\right)\,, \\ g^{a\bar{b}}\, F^{0,0}_{a\bar{b}} &= 3\big(\phi^{+}_{0,0}{}^{\dagger}\, \phi^{+}_{0,0} - \phi^{-}_{1,-3}\, \phi^{-}_{1,-3}{}^{\dagger}\big)\,, \\ g^{a\bar{b}}\, F^{2,0}_{a\bar{b}} &= \phi^{-}_{2,0}{}^{\dagger}\, \phi^{-}_{2,0} - \phi^{+}_{1,-3}\, \phi^{+}_{1,-3}{}^{\dagger}\,, \\ g^{a\bar{b}}\, F^{1,-3}_{a\bar{b}} &= -3\left(\mathbf{1}_{p_{1,-3}} - \frac{1}{2}\, \phi^{-}_{1,-3}{}^{\dagger}\, \phi^{-}_{1,-3} - \frac{1}{2}\, \phi^{+}_{1,-3}{}^{\dagger}\, \phi^{+}_{1,-3}\right)\,, \end{aligned} \tag{4.26}$$

the bi-covariant Higgs field derivatives

$$\begin{aligned} \partial_{\bar{a}}\phi^{+}_{0,0} + A^{1,3}_{\bar{a}}\, \phi^{+}_{0,0} - \phi^{+}_{0,0}\, A^{0,0}_{\bar{a}} &= 0\,, \\ \partial_{\bar{a}}\phi^{-}_{2,0} + A^{1,3}_{\bar{a}}\, \phi^{-}_{2,0} - \phi^{-}_{2,0}\, A^{2,0}_{\bar{a}} &= 0\,, \\ \partial_{\bar{a}}\phi^{-}_{1,-3} + A^{0,0}_{\bar{a}}\, \phi^{-}_{1,-3} - \phi^{-}_{1,-3}\, A^{1,-3}_{\bar{a}} &= 0\,, \\ \partial_{\bar{a}}\phi^{+}_{1,-3} + A^{2,0}_{\bar{a}}\, \phi^{+}_{1,-3} - \phi^{+}_{1,-3}\, A^{1,-3}_{\bar{a}} &= 0\,, \end{aligned} \tag{4.27}$$

and the holomorphic relation

$$\phi^{+}_{0,0}\, \phi^{-}_{1,-3} - \phi^{-}_{2,0}\, \phi^{+}_{1,-3} = 0\ . \tag{4.28}$$

We refer to this system as a "holomorphic square". It is the basic building block for higher representation symmetric quiver vortices.

Non-symmetric $\underline{C}^{1,0}$ quiver vortices. Using the Kähler form (3.46) on Q_3 and (3.51), the DUY equations reduce on M_{2d} in this case to

$$\begin{aligned} g^{a\bar{b}}\, {}_{1}F^{1,1}_{a\bar{b}} &= 2\left(\mathbf{1}_{{}^{1}p_{1,1}} - \frac{1}{3}\, {}^{0}\phi^{+}_{0,-2}\, {}^{0}\phi^{+}_{0,-2}{}^{\dagger} - \frac{2}{3}\, {}^{1}\phi^{0}_{-1,1}\, {}^{1}\phi^{0}_{-1,1}{}^{\dagger}\right)\,, \\ g^{a\bar{b}}\, {}_{1}F^{-1,1}_{a\bar{b}} &= \frac{4}{3}\, \big({}^{1}\phi^{0}_{-1,1}{}^{\dagger}\, {}^{1}\phi^{0}_{-1,1} - {}^{0}\phi^{-}_{0,-2}\, {}^{0}\phi^{-}_{0,-2}{}^{\dagger}\big)\,, \\ g^{a\bar{b}}\, {}_{0}F^{0,-2}_{a\bar{b}} &= -2\left(\mathbf{1}_{{}^{0}p_{0,-2}} - \frac{1}{3}\, {}^{0}\phi^{+}_{0,-2}{}^{\dagger}\, {}^{0}\phi^{+}_{0,-2} - \frac{2}{3}\, {}^{0}\phi^{-}_{0,-2}{}^{\dagger}\, {}^{0}\phi^{-}_{0,-2}\right)\,, \\ \partial_{\bar{a}}\, {}^{0}\phi^{+}_{0,-2} + {}_{1}A^{1,1}_{\bar{a}}\, {}^{0}\phi^{+}_{0,-2} - {}^{0}\phi^{+}_{0,-2}\, {}_{0}A^{0,-2}_{\bar{a}} &= 0\,, \end{aligned}$$

$$\partial_{\bar a}{}^0\phi^-_{0,-2} + {}_1A^{-1,1}_{\bar a}\;{}^0\phi^-_{0,-2} - {}^0\phi^-_{0,-2}\;{}_0A^{0,-2}_{\bar a} = 0\,,$$
$$\partial_{\bar a}{}^1\phi^0_{-1,1} + {}_1A^{1,1}_{\bar a}\;{}^1\phi^0_{-1,1} - {}^1\phi^0_{-1,1}\;{}_1A^{-1,1}_{\bar a} = 0\,, \tag{4.29}$$

together with the holomorphic relation

$${}^0\phi^+_{0,-2} - {}^1\phi^0_{-1,1}\,{}^0\phi^-_{0,-2} = 0\;. \tag{4.30}$$

We call this system a "holomorphic triangle". It is the basic building block for higher representation non-symmetric quiver vortices.

Non-symmetric $\underline{C}^{2,0}$ quiver vortices. From (3.147)–(3.164) the quiver vortex equations on M_{2d} in this instance are found to comprise the curvature equations

$$\begin{aligned}
g^{a\bar b}\,{}_2F^{2,2}_{a\bar b} &= 4\Big(\mathbf{1}_{{}^2p_{2,2}} - \frac{1}{3}\,{}^1\phi^+_{1,-1}\,{}^1\phi^+_{1,-1}{}^\dagger - \frac{2}{3}\,{}^2\phi^0_{0,2}\,{}^2\phi^0_{0,2}{}^\dagger\Big)\,,\\
g^{a\bar b}\,{}_2F^{0,2}_{a\bar b} &= 2\Big(\mathbf{1}_{{}^2p_{0,2}} - \frac{1}{3}\,{}^1\phi^+_{-1,-1}\,{}^1\phi^+_{-1,-1}{}^\dagger\\
&\qquad - \frac{2}{3}\,{}^1\phi^-_{1,-1}\,{}^1\phi^-_{1,-1}{}^\dagger + \frac{4}{3}\,{}^2\phi^0_{0,2}{}^\dagger\,{}^2\phi^0_{0,2} + \frac{4}{3}\,{}^2\phi^0_{-2,2}\,{}^2\phi^0_{-2,2}{}^\dagger\Big)\,,\\
g^{a\bar b}\,{}_2F^{-2,2}_{a\bar b} &= -\frac{8}{3}\,\big({}^1\phi^-_{-1,-1}\,{}^1\phi^-_{-1,-1}{}^\dagger - {}^2\phi^0_{-2,2}{}^\dagger\,{}^2\phi^0_{-2,2}\big)\,,\\
g^{a\bar b}\,{}_1F^{1,-1}_{a\bar b} &= \frac{4}{3}\,\big({}^1\phi^-_{1,-1}{}^\dagger\,{}^1\phi^-_{1,-1} - {}^1\phi^0_{-1,-1}\,{}^1\phi^0_{-1,-1}{}^\dagger + {}^1\phi^+_{1,-1}{}^\dagger\,{}^1\phi^+_{1,-1} - {}^0\phi^+_{0,-4}\,{}^0\phi^+_{0,-4}{}^\dagger\big)\,,\\
g^{a\bar b}\,{}_1F^{-1,-1}_{a\bar b} &= -2\Big(\mathbf{1}_{{}^1p_{-1,-1}} - \frac{1}{3}\,{}^1\phi^+_{-1,-1}{}^\dagger\,{}^1\phi^+_{-1,-1}\\
&\qquad - \frac{2}{3}\,{}^1\phi^0_{-1,-1}{}^\dagger\,{}^1\phi^0_{-1,-1} - \frac{4}{3}\,{}^1\phi^-_{-1,-1}{}^\dagger\,{}^1\phi^-_{-1,-1} + \frac{4}{3}\,{}^0\phi^-_{0,-4}\,{}^0\phi^-_{0,-4}{}^\dagger\Big)\,,\\
g^{a\bar b}\,{}_0F^{0,-4}_{a\bar b} &= -4\Big(\mathbf{1}_{{}^0p_{0,-4}} - \frac{1}{3}\,{}^0\phi^+_{0,-4}{}^\dagger\,{}^0\phi^+_{0,-4} - \frac{2}{3}\,{}^0\phi^-_{0,-4}{}^\dagger\,{}^0\phi^-_{0,-4}\Big)\,,
\end{aligned} \tag{4.31}$$

the bi-covariant Higgs field derivatives

$$\begin{aligned}
\partial_{\bar a}{}^2\phi^0_{0,2} + {}_2A^{2,2}_{\bar a}\;{}^2\phi^0_{0,2} - {}^2\phi^0_{0,2}\;{}_2A^{0,2}_{\bar a} &= 0\,,\\
\partial_{\bar a}{}^1\phi^+_{1,-1} + {}_2A^{2,2}_{\bar a}\;{}^1\phi^+_{1,-1} - {}^1\phi^+_{1,-1}\;{}_1A^{1,-1}_{\bar a} &= 0\,,\\
\partial_{\bar a}{}^2\phi^0_{-2,2} + {}_2A^{0,2}_{\bar a}\;{}^2\phi^0_{-2,2} - {}^2\phi^0_{-2,2}\;{}_2A^{-2,2}_{\bar a} &= 0\,,\\
\partial_{\bar a}{}^1\phi^-_{1,-1} + {}_2A^{0,2}_{\bar a}\;{}^1\phi^-_{1,-1} - {}^1\phi^-_{1,-1}\;{}_1A^{1,-1}_{\bar a} &= 0\,,\\
\partial_{\bar a}{}^1\phi^+_{-1,-1} + {}_2A^{0,2}_{\bar a}\;{}^1\phi^+_{-1,-1} - {}^1\phi^+_{-1,-1}\;{}_1A^{-1,-1}_{\bar a} &= 0\,,\\
\partial_{\bar a}{}^1\phi^-_{-1,-1} + {}_2A^{-2,2}_{\bar a}\;{}^1\phi^-_{-1,-1} - {}^1\phi^-_{-1,-1}\;{}_1A^{-1,-1}_{\bar a} &= 0\,,\\
\partial_{\bar a}{}^1\phi^0_{-1,-1} + {}_1A^{1,-1}_{\bar a}\;{}^1\phi^0_{-1,-1} - {}^1\phi^0_{-1,-1}\;{}_1A^{-1,-1}_{\bar a} &= 0\,,\\
\partial_{\bar a}{}^0\phi^+_{0,-4} + {}_1A^{1,-1}_{\bar a}\;{}^0\phi^+_{0,-4} - {}^0\phi^+_{0,-4}\;{}_0A^{0,-4}_{\bar a} &= 0\,,\\
\partial_{\bar a}{}^0\phi^-_{0,-4} + {}_1A^{-1,-1}_{\bar a}\;{}^0\phi^-_{0,-4} - {}^0\phi^-_{0,-4}\;{}_0A^{0,-4}_{\bar a} &= 0\,,
\end{aligned} \tag{4.32}$$

the linear holomorphic relations

$$\begin{aligned}{}^{1}\phi^{+}_{1,-1} - {}^{2}\phi^{0}_{0,2}\,{}^{1}\phi^{-}_{1,-1} &= 0\,,\\ {}^{1}\phi^{+}_{-1,-1} + {}^{1}\phi^{-}_{1,-1}\,{}^{1}\phi^{0}_{-1,-1} - 2\,{}^{2}\phi^{0}_{-2,2}\,{}^{1}\phi^{-}_{-1,-1} &= 0\,,\\ {}^{0}\phi^{+}_{0,-4} - {}^{1}\phi^{0}_{-1,-1}\,{}^{0}\phi^{-}_{0,-4} &= 0\,,\end{aligned} \tag{4.33}$$

and the quadratic holomorphic relations

$$\begin{aligned}{}^{2}\phi^{0}_{0,2}\,{}^{1}\phi^{+}_{-1,-1} - {}^{1}\phi^{+}_{1,-1}\,{}^{1}\phi^{0}_{-1,-1} &= 0\,,\\ {}^{1}\phi^{+}_{-1,-1}\,{}^{0}\phi^{-}_{0,-4} - {}^{1}\phi^{-}_{1,-1}\,{}^{0}\phi^{+}_{0,-4} &= 0\,.\end{aligned} \tag{4.34}$$

5. Noncommutative quiver vortices

In this section we shall construct explicit solutions to the nonabelian quiver vortex equations of the previous section. For this, we specialize to the manifold $M_{2d} = \mathbb{C}^d$ with the standard flat Kähler metric $g_{a\bar{b}} = \delta_{ab}$. Although it is known that solutions to the DUY equations in $2d > 4$ dimensions exist [2, 5, 10, 11], they cannot be constructed explicitly as far as we know. To get non-trivial solutions on this space, we will need to apply a noncommutative deformation.

5.1 Noncommutative quiver gauge theory

The Moyal deformation of $\mathbb{C}^d$ is realized by mapping Schwartz functions f on $\mathbb{C}^d$ to compact Weyl operators $\hat{f}$ acting on a separable Hilbert space $\mathcal{H}$. The coordinates z^a and $\bar{z}^{\bar{b}}$ of $\mathbb{C}^d$ are mapped to operators $\hat{z}^a$ and $\hat{\bar{z}}^{\bar{b}}$ subject to the commutation relations

$$\big[\hat{z}^a\,,\,\hat{\bar{z}}^{\bar{b}}\big] = \theta^{a\bar{b}} \tag{5.1}$$

with a constant real-valued antisymmetric matrix $(\theta^{a\bar{b}})$ of maximal rank. All other commutators vanish. We may rotate the coordinates so that $(\theta^{a\bar{b}})$ assumes its canonical form with

$$\theta^{a\bar{b}} = 2\,\delta^{ab}\,\theta^a \tag{5.2}$$

for $\theta^a \in (0,\infty)$, $a = 1,\dots,d$. This defines the noncommutative space $\mathbb{C}^d_\theta$ with isometry group $\mathrm{USp}(d)$ and carrying d commuting copies of the Heisenberg algebra

$$\left[\frac{\hat{z}^a}{\sqrt{2\,\theta^a}}\,,\,\frac{\hat{\bar{z}}^{\bar{b}}}{\sqrt{2\,\theta^b}}\right] = \delta^{ab}\ . \tag{5.3}$$

We will represent this algebra on the standard irreducible Fock module $\mathcal{H}$. Recall [19] that derivatives and integrals of fields on $\mathbb{C}^d$ are mapped according to

$$\widehat{\partial_{\bar{a}} f} \;=\; \theta_{\bar{a}b}\,\big[\hat{z}^b\,,\,\hat{f}\,\big] \qquad \text{and} \qquad \int_{\mathbb{C}^d} \mathrm{d}V\ f \;=\; (2\pi)^d\ \mathrm{Pf}(\theta)\ \mathrm{Tr}_{\mathcal{H}}\big(\hat{f}\,\big)\,, \tag{5.4}$$

where $\theta^{a\bar{b}}\,\theta_{\bar{b}c} = \delta^a{}_c$. For simplicity, we will drop the hats in the notation from now on.

JHEP08(2008)093

To rewrite the nonabelian quiver vortex equations of the previous section using the Moyal deformation, we define the covariant coordinates

$$X^v_a \ := \ A^v_a + \theta_{a\bar b}\,\bar z^{\bar b} \qquad \text{and} \qquad X^v_{\bar a} \ := \ A^v_{\bar a} + \theta_{\bar a b}\,z^{b} \tag{5.5}$$

at each vertex $v \in \mathsf{Q}_0(k,l)$ of the pertinent quiver. Then the components of the field strength tensor can be expressed as

$$F^v_{ab} \ = \ \big[X^v_a\,,\,X^v_b\big] \qquad \text{and} \qquad F^v_{a\bar b} \ = \ \big[X^v_a\,,\,X^v_{\bar b}\big] + \theta_{a\bar b}\ . \tag{5.6}$$

Likewise, the Higgs field gradients $D_{\bar a}\phi_{v,\Phi(v)}$ can be expressed through the $X^v_{\bar a}$ for each arrow $\Phi \in \mathsf{Q}_1(k,l)$. Then the vortex equations reduce to algebraic equations for the collection of operators $\{X^v,\phi_{v,\Phi(v)}\}$ acting on the projective module $\underline{V}^{k,l}\otimes\mathcal{H}$ of rank p over $\mathbb{C}^d_\theta$, where

$$\underline{V}^{k,l} = \bigoplus_{v\in\mathsf{Q}_0(k,l)}\ \underline{V}_{\,v}\otimes\underline{v} \tag{5.7}$$

with $\underline{V}_{\,v}=\mathbb{C}^{p_v}$ the fibre space of the vector bundle $E_{p_v}\to\mathbb{C}^d$. For example, the holomorphicity equations (4.12) become the commutativity equations

$$\big[X^v_a\,,\,X^v_b\big] \ = \ 0 \ = \ \big[X^v_{\bar a}\,,\,X^v_{\bar b}\big]\ . \tag{5.8}$$

Symmetric $\underline{C}^{k,l}$ quiver modules. In the symmetric case, the system of additional algebraic equations at each vertex $v=(n,m)\in\mathsf{Q}_0(k,l)$ reads

$$X^{n\pm1,m+3}_{\bar a}\,\phi^\pm_{n,m} = \phi^\pm_{n,m}\,X^{n,m}_{\bar a}\,, \tag{5.9}$$

$$\phi^+_{n,m}\,\phi^-_{n+1,m-3} = \phi^-_{n+2,m}\,\phi^+_{n+1,m-3}\,, \tag{5.10}$$

$$\begin{aligned}\delta^{ab}\,\big(\big[X^{n,m}_a\,,\,X^{n,m}_{\bar b}\big]+\theta_{a\bar b}\big) = \ & m+\tfrac{n+2}{n+1}\,\lambda^+_{k,l}(n,m)^2\ \phi^+_{n,m}{}^\dagger\,\phi^+_{n,m}+\tfrac{n}{n+1}\,\lambda^-_{k,l}(n,m)^2\ \phi^-_{n,m}{}^\dagger\,\phi^-_{n,m}\\ & -\lambda^+_{k,l}(n-1,m-3)^2\ \phi^+_{n-1,m-3}\,\phi^+_{n-1,m-3}{}^\dagger\\ & -\lambda^-_{k,l}(n+1,m-3)^2\ \phi^-_{n+1,m-3}\,\phi^-_{n+1,m-3}{}^\dagger\ .\end{aligned} \tag{5.11}$$

Non-symmetric $\underline{C}^{k,l}$ quiver modules. In the non-symmetric case, the system of additional algebraic equations at each vertex $v=(q,m)_n\in\mathsf{Q}_0(k,l)$ reads

$${}_{n\pm1}X^{q+1,m+3}_{\bar a}\ {}^n\phi^+_{q,m} = {}^n\phi^+_{q,m}\ {}_nX^{q,m}_{\bar a}\,, \tag{5.12}$$

$${}_{n\pm1}X^{q-1,m+3}_{\bar a}\ {}^n\phi^-_{q,m} = {}^n\phi^-_{q,m}\ {}_nX^{q,m}_{\bar a}\,, \tag{5.13}$$

$${}_{n}X^{q+2,m}_{\bar a}\ {}^n\phi^0_{q,m} = {}^n\phi^0_{q,m}\ {}_nX^{q,m}_{\bar a}\,, \tag{5.14}$$

$$\begin{aligned}{}^n\phi^+_{q,m} = \ & \frac12\,(q-n)\ {}^n\phi^-_{q+2,m}\,{}^n\phi^0_{q,m}\\ & -\frac12\,(q-n-2)\ {}^{n+1}\phi^0_{q-1,m+3}\,{}^n\phi^-_{q,m}\,,\end{aligned} \tag{5.15}$$

$$\begin{aligned}{}^n\phi^+_{q,m} = \ & \frac12\,(q+n+2)\ {}^n\phi^-_{q+2,m}\,{}^n\phi^0_{q,m}\\ & -\frac12\,(q+n)\ {}^{n-1}\phi^0_{q-1,m+3}\,{}^n\phi^-_{q,m}\,,\end{aligned} \tag{5.16}$$

$${}^n\phi^+_{q,m}\ {}^{n\pm1}\phi^-_{q+1,m-3} = {}^n\phi^-_{q+2,m}\ {}^{n\pm1}\phi^+_{q+1,m-3}\,, \tag{5.17}$$

$$ {}^n\phi^+_{q,m}\, {}^n\phi^0_{q-2,m} = {}^{n\pm1}\phi^0_{q-1,m+3}\, {}^n\phi^+_{q-2,m} \,, \tag{5.18} $$

$$ {}^n\phi^-_{q,m}\, {}^n\phi^0_{q-2,m} = {}^{n\pm1}\phi^0_{q-3,m+3}\, {}^n\phi^-_{q-2,m} \,, \tag{5.19} $$

$$ \begin{aligned} \delta^{ab}\left(\big[{}_nX^{q,m}_a\,,\,{}_nX^{q,m}_{\bar b}\big]+\theta_{a\bar b}\right) = \\ & (q+m)+\frac{\lambda^+_{k,l}(n,m)^2}{3(n+1)}\Big((n+q+2)\ {}^n\phi^+_{q,m}{}^{\dagger\, n}\phi^+_{q,m}+2(n-q+2)\ {}^n\phi^-_{q,m}{}^{\dagger\, n}\phi^-_{q,m}\Big) \\ & +\frac{\lambda^-_{k,l}(n,m)^2}{3(n+1)}\Big((n-q)\ {}^n\phi^+_{q,m}{}^{\dagger\, n}\phi^+_{q,m}+2(n+q)\ {}^n\phi^-_{q,m}{}^{\dagger\, n}\phi^-_{q,m}\Big) \\ & -\frac{\lambda^+_{k,l}(n-1,m-3)^2}{3n}\Big((n+q)\ {}^{n-1}\phi^+_{q-1,m-3}\ {}^{n-1}\phi^+_{q-1,m-3}{}^{\dagger} \\ & \qquad\qquad +2(n-q)\ {}^{n-1}\phi^-_{q+1,m-3}\ {}^{n-1}\phi^-_{q+1,m-3}{}^{\dagger}\Big) \\ & -\frac{\lambda^-_{k,l}(n+1,m-3)^2}{3(n+2)}\Big((n-q+2)\ {}^{n+1}\phi^+_{q-1,m-3}\ {}^{n+1}\phi^+_{q-1,m-3}{}^{\dagger} \\ & \qquad\qquad +2(n+q+2)\ {}^{n+1}\phi^-_{q+1,m-3}\ {}^{n+1}\phi^-_{q+1,m-3}{}^{\dagger}\Big) \\ & +\frac{1}{3}\,(n-q)\,(n+q+2)\ {}^n\phi^0_{q,m}{}^{\dagger\, n}\phi^0_{q,m} \\ & -\frac{1}{3}\,(n+q)\,(n-q+2)\ {}^n\phi^0_{q-2,m}\ {}^n\phi^0_{q-2,m}{}^{\dagger}\ . \end{aligned} \tag{5.20} $$

5.2 Finite energy solutions

We will begin by constructing finite energy solutions of Yang-Mills theory on the noncommutative space $X = \mathbb{C}^d_\theta \times G/H$. In the generic (non-BPS) case, the gauge group of the quiver gauge theory of section 5.1 above is

$$ \mathcal{G}^{k,l} = \prod_{v\in\mathsf{Q}_0(k,l)} \mathrm{U}(p_v)\ . \tag{5.21} $$

The corresponding reduction of the Yang-Mills action on X, regarded as an energy functional for static quiver gauge fields on $\mathbb{R}^{0,1}\times\mathbb{C}^d_\theta$ in the temporal gauge, is given by computing

$$ E_{\rm YM} := \frac{1}{4}\ \mathrm{Pf}(2\pi\,\theta) \int_{G/H} \mathrm{Tr}_{\underline{V}^{k,l}\otimes\mathcal{H}}\big(\mathcal{F}\wedge *\mathcal{F}\big)\ . \tag{5.22} $$

Fix an integer r with $0 < r \le p$ and introduce a collection of partial isometries T_v, $v \in \mathsf{Q}_0(k,l)$ on $\mathcal{H}$ realized by $p_v \times r$ Toeplitz operators obeying

$$ T_v^\dagger\, T_v \;=\; \mathbf{1}_r \qquad\text{and}\qquad T_v\, T_v^\dagger \;=\; \mathbf{1}_{p_v} - P_v\,, \tag{5.23} $$

where $P_v = P_v^\dagger = P_v^2$ is a hermitean projector of finite rank

$$ N_v := \mathrm{Tr}_{\underline{V}_v\otimes\mathcal{H}}(P_v)\ . \tag{5.24} $$

Such operators T_v can be constructed explicitly from an SU(3)-equivariant version of the noncommutative ABS construction, analogously to the SU(2) case of [4]. We will return to this point in the next section. We make the ansatz for the gauge connections given by

$$ A^v_a = \theta_{a\bar b}\,\left(T_v\,\bar z^{\bar b}\,T_v^\dagger - \bar z^{\bar b}\,\right), \tag{5.25} $$

which yields the field strength components

$$F^v_{ab} \;=\; 0 \;=\; F^v_{\bar a\bar b} \qquad \text{and} \qquad F^v_{a\bar b} \;=\; \theta_{a\bar b}\, P_v \;=\; \frac{1}{2\theta^a}\,\delta_{ab}\, P_v \tag{5.26}$$

at each vertex $v \in \mathsf{Q}_0(k,l)$. The details of the ansatz for the module morphisms $\phi_{v,\Phi(v)}$ depend on the particular quiver. In the following we will use the projector

$$\mathcal{P} = \sum_{v\in\mathsf{Q}_0(k,l)} P_v \otimes \Pi_v \,, \tag{5.27}$$

where Π_v is the projection of $\underline{C}^{k,l}\big|_H$ onto the irreducible H-module $\underline{v}$.

Symmetric $\underline{C}^{k,l}$ quiver modules. At a given symmetric vertex $v=(n,m)\in\mathsf{Q}_0(k,l)$ we use the partial isometries $T_{n,m}$ above to construct the operators

$$\phi^\pm_{n,m} = T_{n\pm1,m+3}\, T^\dagger_{n,m} \ . \tag{5.28}$$

With the ansatz (5.28), one sees that both the holomorphic relations (5.10) and the non-holomorphic relations

$$\phi^+_{n,m}\,{\phi^-_{n,m}}^\dagger = {\phi^-_{n+1,m+3}}^\dagger\,\phi^+_{n-1,m+3} \tag{5.29}$$

are satisfied. These conditions are necessary to yield a finite Yang-Mills action below. Moreover, with this ansatz one easily checks that the covariant constancy equations (5.9) are satisfied, along with

$$ {\phi^\pm_{n,m}}^\dagger\,\phi^\pm_{n,m} \;=\; \mathbf{1}_{p_{n,m}} - P_{n,m} \;=\; \phi^\pm_{n\mp1,m-3}\,{\phi^\pm_{n\mp1,m-3}}^\dagger \ . \tag{5.30}$$

Thus for our ansatz the off-diagonal field strength components (3.80)–(3.82) all vanish. The reduction of the energy functional (5.22) is therefore given by substituting (3.79) in the basis $\{\beta^i\wedge\bar\beta^j\}$ of $(1,1)$-forms on $\mathbb{C}P^2$ using the Kähler metric given by (3.22) and (4.1) with $g_{a\bar b}=\delta_{ab}$ to get

$$\begin{aligned} E_{\rm YM} \;=\; 2\,{\rm Pf}(2\pi\,\theta)\ {\rm vol}\big(\mathbb{C}P^2\big) \sum_{(n,m)\in\mathsf{Q}_0(k,l)} \Bigg[&(n+1)\sum_{a,b=1}^d {\rm Tr}_{\underline{V}_{\,n,m}\otimes\mathcal{H}}\big|F^{n,m}_{a\bar b}\big|^2 \\ &+ {\rm Tr}_{\underline{V}_{\,n,m}\otimes\underline{(n,m)}\otimes\mathcal{H}}\Big(\big|\mathcal{F}^{n,m\,;\,n,m}_{1\bar1}\big|^2+\big|\mathcal{F}^{n,m\,;\,n,m}_{2\bar2}\big|^2+2\big|\mathcal{F}^{n,m\,;\,n,m}_{1\bar2}\big|^2\Big)\Bigg] \,, \end{aligned} \tag{5.31}$$

where we use the matrix notation $|\mathcal{F}|^2 := \frac12\,(\mathcal{F}^\dagger\,\mathcal{F}+\mathcal{F}\,\mathcal{F}^\dagger)$. We substitute (5.26), (5.30) along with (4.15), (4.16) and the explicit Clebsch-Gordan coefficients (3.66). Using the identity (4.22) along with the actions of the SU(3) generators in (3.60) and (3.62), one finds the curvature components

$$\mathcal{F}_{1\bar1} \;=\; -\frac12\,\mathcal{P}\,(H_{\alpha_1}+H_{\alpha_2}) \qquad \text{and} \qquad \mathcal{F}_{2\bar2} \;=\; \frac12\,\mathcal{P}\,(H_{\alpha_1}-H_{\alpha_2}) \ . \tag{5.32}$$

Using the identity

$$\tfrac1n\,\lambda^+_{k,l}(n-1,m-3)^2+\tfrac{1}{n+2}\,\lambda^-_{k,l}(n+1,m-3)^2-\tfrac{1}{n+1}\,\lambda^+_{k,l}(n,m)^2-\tfrac1n\,\lambda^-_{k,l}(n,m)^2=1 \tag{5.33}$$

along with (3.61), one also finds

$$\mathcal{F}_{1\bar{2}} = -\mathcal{P}\, E_{-\alpha_1} \ . \tag{5.34}$$

It follows that

$$\big|\mathcal{F}_{1\bar{1}}\big|^2 + \big|\mathcal{F}_{2\bar{2}}\big|^2 + 2\big|\mathcal{F}_{1\bar{2}}\big|^2 = \mathcal{P}\, \boldsymbol{C}_2(H)\,, \tag{5.35}$$

where

$$\boldsymbol{C}_2(H) = \big(E_{\alpha_1}\, E_{-\alpha_1} + E_{-\alpha_1}\, E_{\alpha_1} + \frac{1}{2}\, H_{\alpha_1}^2\big) + \frac{1}{2}\, H_{\alpha_2}^2 \tag{5.36}$$

is the quadratic Casimir operator of the holonomy group $H = \mathrm{SU}(2) \times \mathrm{U}(1)$. In the irreducible representation $\underline{(n,m)}$, it acts as the multiplication operator by the eigenvalue $\frac{1}{2}\, n\,(n+2) + \frac{1}{2}\, m^2$, where we have used the fact that the summation range over isospin is symmetric under reflection $q \to -q$.

After tracing over the representation spaces $\underline{(n,m)} \cong \mathbb{C}^{n+1}$ in the last line of (5.31), we arrive finally at the Yang-Mills energy

$$E_{\mathrm{YM}} = 2\ \mathrm{Pf}(2\pi\,\theta)\ \mathrm{vol}\big(\mathbb{C}P^2\big) \sum_{(n,m)\in \mathsf{Q}_0(k,l)} \mathrm{Tr}_{\underline{V}_{n,m}\otimes\mathcal{H}}\big(\varepsilon_{n,m}\ P_{n,m}\big)\,, \tag{5.37}$$

where

$$\varepsilon_{n,m} = (n+1)\left(\frac{1}{4}\sum_{a=1}^{d}\frac{1}{(\theta^a)^2} + \frac{1}{2}\, n\,(n+2) + \frac{1}{2}\, m^2\right)\,. \tag{5.38}$$

The quantity (5.38) is the finite energy density at the symmetric vertex $v = (n,m)$. The θ-dependent term is the Yang-Mills energy of vortices on $\mathbb{C}_\theta^d$ [20], and in the total energy (5.37) it can be interpreted as the tension of N D0-branes bound inside a collection of D($2d$)-branes in the Seiberg-Witten decoupling limit [19], where

$$N := \sum_{(n,m)\in \mathsf{Q}_0(k,l)} (n+1)\, N_{n,m} \ . \tag{5.39}$$

The second term in (5.38) is the angular momentum contribution from the isospin $I = \frac{n}{2}$ of the instanton gauge potential $B_{n,m}$. The third term is recognized as the Yang-Mills energy of a monopole on the projective plane $\mathbb{C}P^2$ of magnetic charge $m \in \mathbb{Z}$ (in suitable area units for the embedded two-cycle $\mathbb{C}P^1 \subset \mathbb{C}P^2$). The charge one SU(2) instanton on $\mathbb{C}P^2$ also contributes overall multiplicity factors $(n+1)$ corresponding to the dimension of the irreducible SU(2) representation it lives in at the vertex (n,m). The correlation between monopole and instanton quantum numbers is a consequence of the fact that the instanton bundle can be realized as the SU(2)-bundle

$$\mathcal{I} = S^5 \times_\rho \mathrm{SU}(2) \tag{5.40}$$

associated to the Hopf bundle (3.21) by the diagonal representation $\rho : \mathrm{U}(1) \to \mathrm{SU}(2)$.

Note that in contrast to the SU(2)-equivariant quiver gauge theories based on the symmetric space $\mathbb{C}P^1$ [4, 13], the ansatz (5.28) automatically yields a finite Yang-Mills energy, without the need of multiplying them by suitable coefficient functions on $\mathsf{Q}_0(k,l)$. We are free to multiply the field configurations (5.28) by arbitrary complex numbers of modulus one, which are functions on $\mathsf{Q}_0(k,l)$. By working in the basis generated by the canonical one-forms β^i and $\bar{\beta}^i$ on $\mathbb{C}P^2$, one can straightforwardly adapt the proof of [4] to show that our ansatz solves the full Yang-Mills equations (4.4) on $X = \mathbb{C}_\theta^d \times \mathbb{C}P^2$.

JHEP08(2008)093

Non-symmetric $\underline{C}^{k,l}$ quiver modules. At a non-symmetric vertex $v = (q,m)_n \in \mathsf{Q}_0(k,l)$, we use the partial isometries ${}^nT_{q,m}$ above to construct the operators

$$
\begin{aligned}
{}^n\phi^{+}_{q,m} &= {}^{n\pm1}T_{q+1,m+3}\,{}^nT_{q,m}{}^{\dagger}\,, \qquad & {}^n\phi^{-}_{q,m} &= {}^{n\pm1}T_{q-1,m+3}\,{}^nT_{q,m}{}^{\dagger}\,, \\
{}^n\phi^{0}_{q,m} &= {}^{n}T_{q+2,m}\,{}^nT_{q,m}{}^{\dagger}\,, & &
\end{aligned} \tag{5.41}
$$

where the values $n\pm1$ depend on the particular weight vectors $(q,m)_n$ as in (2.21). With this ansatz one has the linear holomorphic relations

$$
{}^n\phi^{+}_{q,m} = {}^n\phi^{-}_{q+2,m}\,{}^n\phi^{0}_{q,m} \qquad \text{and} \qquad {}^n\phi^{+}_{q,m} = {}^{n\pm1}\phi^{0}_{q-1,m+3}\,{}^n\phi^{-}_{q,m} \tag{5.42}
$$

which together imply (5.15) and (5.16), and one also has the quadratic holomorphic relations (5.17)–(5.19). Moreover, one easily checks the linear non-holomorphic relations

$$
\begin{aligned}
{}^n\phi^{-}_{q,m} &= {}^n\phi^{+}_{q-2,m}\,{}^n\phi^{0}_{q-2,m}{}^{\dagger} \qquad \text{and} \qquad & {}^n\phi^{-}_{q,m} &= {}^{n\pm1}\phi^{0}_{q-1,m+3}{}^{\dagger}\,{}^n\phi^{+}_{q,m}\,, \\
{}^n\phi^{0}_{q,m} &= {}^n\phi^{-}_{q+2,m}{}^{\dagger}\,{}^n\phi^{+}_{q,m} \qquad \text{and} \qquad & {}^n\phi^{0}_{q,m} &= {}^{n\pm1}\phi^{+}_{q+1,m-3}\,{}^{n\pm1}\phi^{-}_{q+1,m-3}{}^{\dagger}
\end{aligned} \tag{5.43}
$$

as well as the quadratic non-holomorphic relations

$$
\begin{aligned}
{}^n\phi^{-}_{q,m}\,{}^n\phi^{0}_{q,m}{}^{\dagger} &= {}^{n\pm1}\phi^{0}_{q-1,m+3}{}^{\dagger}\,{}^n\phi^{-}_{q+2,m}\,, \\
{}^{n\pm1}\phi^{+}_{q,m}\,{}^{n\pm1}\phi^{-}_{q,m}{}^{\dagger} &= {}^n\phi^{-}_{q+1,m+3}{}^{\dagger}\,{}^n\phi^{+}_{q-1,m+3}\,, \\
{}^n\phi^{+}_{q,m}\,{}^n\phi^{0}_{q,m}{}^{\dagger} &= {}^{n\pm1}\phi^{0}_{q+1,m+3}{}^{\dagger}\,{}^n\phi^{+}_{q+2,m}\,.
\end{aligned} \tag{5.44}
$$

As the covariant constancy equations (5.12)–(5.14) are also easily verified, it follows again that for our ansatz all off-diagonal field strength matrix elements (3.94)–(3.102) vanish.

The reduction of the Yang-Mills energy functional (5.22) is given by substituting (3.93) in the basis $\{\gamma^a\wedge\bar\gamma^b\}$ of $(1,1)$-forms on Q_3 and using the Kähler metric given by (3.46) to get

$$
\begin{aligned}
E_{\rm YM} = 2\ \mathrm{Pf}(2\pi\,\theta)\ \mathrm{vol}(Q_3) \sum_{(q,m)_n\in\mathsf{Q}_0(k,l)} \mathrm{Tr}_{{}^n\underline{V}_{q,m}\otimes\underline{(q,m)_n}\otimes\mathcal{H}} \Bigg[\sum_{a,b=1}^{d} \big|{}_nF^{q,m}_{a\bar b}\big|^2 \\
+\frac{4}{3}\left(\frac{1}{4}\big|{}_n\mathcal{F}^{q,m\,;\,q,m}_{1\bar1}\big|^2 + \big|{}_n\mathcal{F}^{q,m\,;\,q,m}_{2\bar2}\big|^2 + \big|{}_n\mathcal{F}^{q,m\,;\,q,m}_{3\bar3}\big|^2\right)\Bigg]\,.
\end{aligned} \tag{5.45}
$$

We use the explicit expressions (5.26), (3.85)–(3.88) and (3.66), along with the projector identities

$$
\begin{aligned}
\mathbf{1}_{{}^np_{q,m}} - {}^n\phi^{\pm}_{q,m}{}^{\dagger}\,{}^n\phi^{\pm}_{q,m} = {}^nP_{q,m} &= \mathbf{1}_{{}^np_{q,m}} - {}^n\phi^{0}_{q,m}{}^{\dagger}\,{}^n\phi^{0}_{q,m}\,, \\
\mathbf{1}_{{}^np_{q,m}} - {}^{n\pm1}\phi^{+}_{q-1,m-3}\,{}^{n\pm1}\phi^{+}_{q-1,m-3}{}^{\dagger} = {}^nP_{q,m} &= \mathbf{1}_{{}^np_{q,m}} - {}^{n\pm1}\phi^{-}_{q+1,m-3}\,{}^{n\pm1}\phi^{-}_{q+1,m-3}{}^{\dagger}\,, \\
\mathbf{1}_{{}^np_{q,m}} - {}^n\phi^{0}_{q-2,m}\,{}^n\phi^{0}_{q-2,m}{}^{\dagger} = {}^nP_{q,m}\,. &
\end{aligned} \tag{5.46}
$$

Using (3.60), (3.62) and the identity (4.22), we find the curvature components

$$
\mathcal{F}_{1\bar1} = -\frac{1}{2}\,\mathcal{P}\,(H_{\alpha_1}+H_{\alpha_2})\ , \qquad \mathcal{F}_{2\bar2} = \frac{1}{2}\,\mathcal{P}\,(H_{\alpha_1}-H_{\alpha_2}) \qquad \text{and} \qquad \mathcal{F}_{3\bar3} = -\mathcal{P}\,H_{\alpha_1}\ . \tag{5.47}
$$

In this way we arrive finally at the Yang-Mills energy

$$E_{\rm YM} = \frac{1}{2}\,\mathrm{Pf}(2\pi\,\theta)\;\mathrm{vol}(Q_3)\sum_{(q,m)_n\in\mathsf{Q}_0(k,l)}\mathrm{Tr}_{{}^n\underline{V}_{q,m}\otimes\mathcal{H}}\big(\varepsilon_{q,m}\;{}^nP_{q,m}\big)\;, \tag{5.48}$$

where

$$\varepsilon_{q,m} = \sum_{a=1}^{d}\frac{1}{(\theta^a)^2} + \frac{4}{3}\left(q^2 + q_U^2 + \frac{1}{4}\,q_V^2\right) \tag{5.49}$$

is the finite energy density at the non-symmetric vertex $v = (q,m)_n$, and we have introduced the additional charges

$$q_U \;:=\; -\frac{1}{2}\,(q-m) \qquad \text{and} \qquad q_V \;:=\; \frac{1}{2}\,(q+m)\;. \tag{5.50}$$

The meaning of the new charges (5.50) can be understood from the concept of "U-spin", an electromagnetic analog of isospin. Recall that the quantum number $q\in\mathbb{Z}$ is twice the third component of isospin in the subgroup $\mathrm{SU}(2)\subset\mathrm{SU}(3)$ generated by $E_{\pm\,\alpha_1}, H_{\alpha_1}$, with E_{α_1} acting by shifts $q\mapsto q+2$ and leaving the hypercharge quantum number m invariant. The U-spin subgroup of SU(3) is the SU(2) subgroup generated by the operators $E_{\pm\,\alpha_2}, \frac{1}{4}\,(H_{\alpha_2}-H_{\alpha_1})$. All states in a U-spin multiplet have the same electric charge eigenvalue $Y_U := -\frac{1}{2}\,(q+\frac{m}{3})$, the U-spin analog of hypercharge. Thus the magnetic charge $q_U\in\mathbb{Z}$ is twice the third component of U-spin. The Weyl subgroup $S_3\subset\mathrm{SU}(3)$ takes isospin into U-spin, and the explicit transformation of states in the Biedenharn basis for $\underline{C}^{k,l}$ can be found in [18]. The operator E_{α_2} shifts $q_U\mapsto q_U+2$ and leaves $m_U := 3Y_U$ invariant. Likewise, the "V-spin" subgroup $\mathrm{SU}(2)\subset\mathrm{SU}(3)$ is generated by $E_{\pm\,(\alpha_1+\alpha_2)}, \frac{1}{4}\,(H_{\alpha_1}+H_{\alpha_2})$, with associated quantum numbers $q_V\in\mathbb{Z}$ shifted by $q_V\mapsto q_V+2$ and $m_V := \frac{3}{2}\,(q-\frac{m}{3})$ invariant under the action of $E_{\alpha_1+\alpha_2}$.

Thus the vertex energy (5.49) contains that of *three* non-interacting charges on Q_3, one for each arrow of the non-symmetric quiver. The sum of these charges is the total magnetic charge

$$q + q_U + q_V = q + m \tag{5.51}$$

at the vertex $v=(q,m)_n$. Note that the energy density is independent of the degeneracy label n. The energy of the V-spin charge is down by $\frac{1}{4}$ due to the area of the embedded two-cycle $\mathbb{C}P^1$ dual to the (1,1)-form $\gamma^1\wedge\bar\gamma^1$ on Q_3 (see (3.46)). All of this is qualitatively similar to the quiver energies associated to the symmetric space $\mathbb{C}P^1\times\mathbb{C}P^1$ [13], which also carry abelian vertex charges but only two arrows per vertex. For the present solutions the noncommutative vortex number is now

$$N = \sum_{(q,m)_n\in\mathsf{Q}_0(k,l)} {}^nN_{q,m}\;, \tag{5.52}$$

and the verification of the Yang-Mills equations on $X=\mathbb{C}^d_\theta\times Q_3$ proceeds exactly as outlined before in the basis generated by the canonical one-forms γ^a and $\bar\gamma^a$ on Q_3. As expected from the construction of section 3.2, the sum over monopole charges q at fixed isospin n, mimicking the integration over the $\mathbb{C}P^1$ fibres of the twistor bundle (3.32), maps the non-symmetric energy density (5.49) onto the symmetric one (5.38) via the summation identity

$$\sum_{q\in\{-n+2j\}_{j=0}^{n}} q^2 = \frac{1}{3}\, n\,(n+1)\,(n+2)\ . \tag{5.53}$$

5.3 BPS solutions

The solutions obtained thus far are generically unstable non-BPS solutions of the Yang-Mills equations on $X = \mathbb{C}_\theta^d \times G/H$, due to the presence of non-trivial vacua at more than one quiver vertex in $\mathsf{Q}_0(k,l)$. Using the identities (4.17) and (4.22) it is easy to see from the projector equations (5.26), (5.30) and (5.46) that the ansätze of section 5.2 above are incompatible with the BPS equations (5.11) and (5.20) if more than one projector P_v is non-zero. To obtain non-trivial BPS solutions we need a more specialized ansatz inside our previous ones. Let us set $p_v = r$ for all $v \neq v_0$, where $v_0 \in \mathsf{Q}_0(k,l)$ is a distinguished vertex of the pertinent quiver. A natural choice for v_0 comes from the choice of trivial SU(2) spins $j_\pm = 0$ in (3.67) and (3.69), for which $n = q = 0$ and $m = -2(k-l)$. The gauge group is now

$$\mathcal{G}_{\rm BPS}^{k,l} = {\rm U}(p_{v_0}) \times {\rm U}(r)^{\#\mathsf{Q}_0(k,l)-1}\ , \tag{5.54}$$

where the number of quiver vertices $\#\mathsf{Q}_0(k,l) = (k+1)\,(l+1)$ in the symmetric case while $\#\mathsf{Q}_0(k,l) = d^{k,l}$ in the non-symmetric case is the dimension (2.18) of the irreducible SU(3) representation $\underline{C}^{k,l}$.

At each vertex $v \neq v_0$, we take the vacuum solution with $A_a^v = 0$ and trivial module morphisms $\phi_{v,\Phi(v)} = \mathbf{1}_r$. At $v = v_0$ we take the partial isometry ansätze of section 5.2 above. In both symmetric and non-symmetric cases, the vortex number of this solution is $N = N_{v_0}$, and the noncommutative vortex equations require a non-trivial relation

$$\sum_{a=1}^{d} \frac{1}{\theta^a} = -4(k-l) \tag{5.55}$$

between the noncommutative geometry and the geometry of the coset space G/H. By putting $P_v = 0$ for all $v \neq v_0$ in (5.37) and (5.48), from (5.55) one finds that the BPS energies are proportional to the vortex number N with proportionality constant dependent only on the geometry of the noncommutative space $X = \mathbb{C}_\theta^d \times G/H$. Note that the constraint (5.55) requires $k < l$. For $k \geq l$ one can straightforwardly modify the constructions of this section with some $\theta^a < 0$.

6. Topological charges

The class of solutions constructed in the previous section only exist when the system is subjected to a noncommutative deformation. In this final section we will compute their topological charges in order to further unravel their physical meaning. We will do this in three independent but equivalent ways. Firstly, we shall calculate the instanton charges directly in the original Yang-Mills gauge theory on $X = \mathbb{C}_\theta^d \times G/H$. Secondly, we will demonstrate how the topological charges can be elegantly understood in terms of the K-theory class determined by the partial isometries which parametrize the noncommutative instantons. Finally, we will show that these charges also coincide with the Euler characteristic of a certain complex which is canonically associated to the vortex solutions of the quiver gauge theory on $M_{2d} = \mathbb{C}^d_\theta$. These latter two constructions illustrate a novel interpretation of our classical field configurations in terms of D-branes.

Although our treatment of the DUY equations in general dimensions primarily relates to the stability of holomorphic vector bundles over Kähler manifolds, for appropriate values of d one has contact with the D-brane picture in bosonic or supersymmetric string theory. The standard interpretation of the noncommutative configurations at each quiver vertex is that of an unstable system of D0-branes inside D($2d$)-branes (this is done in detail in [20]), in the presence of a constant B-field background and in the Seiberg-Witten scaling limit. This is done by calculating both the tension (first term in (5.38) and (5.49)) and the spectrum of small fluctuations about each instanton configuration, in a (boundary) conformal field theory analysis. The difference in our case is that a system of D-branes excited at only a single vertex is stable on its own (consistently with what is found in section 5.3). This is due to the monopole/instanton fields carried by the branes via the dimensional reduction, whose topological charges prevent a decay to the closed-string vacuum. It is a generalization of the flux stabilization mechanism of Bachas-Douglas-Schweigert [21].

The D-brane interpretation works in principle both in bosonic string theory and in type IIA superstring theory. The latter is probably more desirable, as only in that case is there a notion of Ramond-Ramond charge, as is implicit in our interpretation. As for the fermionic sector, we are following the standard prescription in obtaining BPS solutions — we set all fermionic and auxiliary fields to zero. There is no problem in adding in fermions, supersymmetric or not. The D-branes are thought to wrap topologically trivial worldvolumes $\mathbb{R}^{2d}$ (only for those and not for general Kähler manifolds we put forward the D-brane interpretation) in a *constant* B-field background, hence the Freed-Witten anomaly automatically vanishes.

6.1 Chern-Weil invariants

We will first present a field theoretic derivation of the topological charges of the multi-instanton solutions in the original Yang-Mills gauge theory on $X = \mathbb{C}^d_\theta \times G/H$. Given the G-equivariant vector bundle (3.56) over the Kähler manifold X, we may construct various topological charges classifying our gauge field configurations by taking products in complimentary degrees of the Kähler two-form (4.2) and of the curvature two-form (3.30) representing the usual Chern characteristic classes of the bundle $\mathcal{E}^{k,l}$. For each $j = 1, \dots, d + \frac{d_H}{2}$, we define Chern-Weil topological invariants of $\mathcal{E}^{k,l}$ by

$$\mathrm{Ch}_j\left(\mathcal{E}^{k,l}\right) = \frac{1}{j!}\,\left(\frac{\mathrm{i}}{2\pi}\right)^j\,\mathrm{Pf}(2\pi\,\theta)\,\int_{G/H}\mathrm{Tr}_{\underline{V}^{k,l}\otimes\mathcal{H}}\left(\mathcal{F}^j\right)\wedge\frac{\Omega^{d+d_H/2-j}}{(d+d_H/2-j)!}\ . \tag{6.1}$$

The quantity $\mathrm{Ch}_1(\mathcal{E}^{k,l})$ is proportional to the degree of the bundle $\mathcal{E}^{k,l}$, while $\mathrm{Ch}_2(\mathcal{E}^{k,l})$ is proportional to the Yang-Mills action (5.22) in the BPS limit [11, 13]. Many of these charges will in fact vanish for topological reasons.

JHEP08(2008)093

In this section we will consider only the top degree $j = d + \frac{d_H}{2}$ Chern number $\mathrm{Ch}_{d+d_H/2}(\mathcal{E}^{k,l})$, and refer to this as the instanton charge Q for brevity. Explicitly, one has

$$Q := \frac{1}{\big(d+d_H/2\big)!}\left(\frac{\mathrm{i}}{2\pi}\right)^{d+d_H/2}\ \mathrm{Pf}(2\pi\,\theta)\ \int_{G/H}\ \mathrm{Tr}_{\underline{V}^{k,l}\otimes\mathcal{H}}\big(\,\underbrace{\mathcal{F}\wedge\cdots\wedge\mathcal{F}}_{d+d_H/2}\,\big)\ . \tag{6.2}$$

To compute the integral (6.2), we note that generally for the gauge field configurations (5.25), (5.26), the non-vanishing components of the field strength tensor (3.30) on X along $\mathbb{C}^d_\theta$ are given by

$$\mathcal{F}_{2a-1\ 2a} \ = \ 2\,\mathrm{i}\,\mathcal{F}_{a\bar a} \ = \ -\frac{\mathrm{i}}{\theta^a}\,\mathcal{P}\ , \tag{6.3}$$

where $\mathcal{P}$ is the projector (5.27). The remaining details of the computation depend explicitly on the particular quiver.

Symmetric $\underline{C}^{k,l}$ quiver charges. By working as before in the basis $\{\beta^i\wedge\bar\beta^j\}$ of $(1,1)$-forms on $\mathbb{C}P^2$, the instanton density in (6.2) may be calculated as

$$\begin{aligned}\tfrac{1}{(d+2)!}\,\epsilon^{\mu_1\cdots\mu_{2d+4}}\,\mathcal{F}_{\mu_1\mu_2}\cdots\mathcal{F}_{\mu_{2d+3}\mu_{2d+4}} &= (-\,\mathrm{i})^2\,\big(\mathcal{F}_{12}\,\mathcal{F}_{34}\cdots\mathcal{F}_{2d-1\ 2d}\big)\,\big(\mathcal{F}_{1\bar1}\,\mathcal{F}_{2\bar2}-|\mathcal{F}_{1\bar2}|^2\big)\\ &= \left(\frac{(-\,\mathrm{i})^{d+2}}{\mathrm{Pf}(\theta)}\right)\,\mathcal{P}\,\left(-\,\frac14\,\big(H_{\alpha_1}^2-H_{\alpha_2}^2\big)-\frac12\,\big(E_{\alpha_1}\,E_{-\alpha_1}+E_{-\alpha_1}\,E_{\alpha_1}\big)\right)\end{aligned} \tag{6.4}$$

after substituting the field strength components (6.3), (5.32) and (5.34). After tracing over the representation spaces $\underline{(n,m)}\cong\mathbb{C}^{n+1}$ using (3.60)–(3.62) and symmetry of the isospin summation over q, one finds

$$Q = \left(\frac{1}{2\pi}\right)^2\,\mathrm{vol}\big(\mathbb{C}P^2\big)\ \sum_{(n,m)\in\mathsf{Q}_0(k,l)}\ \frac14\,(n+1)\,\big(m^2-n\,(n+2)\big)\ \mathrm{Tr}_{\underline{V}_{\,n,m}\otimes\mathcal{H}}\big(P_{n,m}\big)\ , \tag{6.5}$$

where $\mathrm{vol}(\mathbb{C}P^2) := \int_{\mathbb{C}P^2}\beta^1\wedge\bar\beta^1\wedge\beta^2\wedge\bar\beta^2$. The normalization can be fixed by recalling from section 3.1 that the Kähler two-form (3.22) on $\mathbb{C}P^2$ determines the generator $[\eta]$ of the integer cohomology ring $\mathrm{H}^{2\bullet}(\mathbb{C}P^2;\mathbb{Z})$ through

$$\eta = \frac{\omega_{\mathbb{C}P^2}}{2\pi}\ , \tag{6.6}$$

with the intersection numbers

$$\int_{\mathbb{C}P^1}\eta \ = \ 1 \ = \ \int_{\mathbb{C}P^2}\eta\wedge\eta \tag{6.7}$$

for any linearly embedded projective line $\mathbb{C}P^1\subset\mathbb{C}P^2$. This fixes $\mathrm{vol}(\mathbb{C}P^2) = 2(2\pi)^2$, and the topological charge is finally given by

$$Q = \sum_{(n,m)\in\mathsf{Q}_0(k,l)}\ \frac12\,(n+1)\,\big(m^2-n\,(n+2)\big)\,N_{n,m}\ . \tag{6.8}$$

Note that (6.8) is indeed integer-valued as (n,m) have the same parity.

JHEP08(2008)093

Non-symmetric $\underline{C}^{k,l}$ quiver charges. In the G-equivariant basis of $(1,1)$-forms on Q_3, the instanton density in (6.2) is given by

$$\begin{aligned} \tfrac{1}{(d+3)!}\,\epsilon^{\mu_1\cdots\mu_{2d+6}}\,\mathcal{F}_{\mu_1\mu_2}\cdots\mathcal{F}_{\mu_{2d+5}\mu_{2d+6}} &= (-\,\mathrm{i})^3\,\big(\mathcal{F}_{12}\,\mathcal{F}_{34}\cdots\mathcal{F}_{2d-1\ 2d}\big)\,\big(\mathcal{F}_{1\bar1}\,\mathcal{F}_{2\bar2}\,\mathcal{F}_{3\bar3}\big) \\ &= \left(\frac{(-\,\mathrm{i})^{d+3}}{\mathrm{Pf}(\theta)}\right)\,\mathcal{P}\,\left(\frac14\,\big(H_{\alpha_1}^2-H_{\alpha_2}^2\big)\,H_{\alpha_1}\right) \end{aligned} \tag{6.9}$$

after substituting the field strength components (6.3) and (5.47). After tracing using (3.60), (3.62) and (5.50), one finds

$$Q=\left(\frac{1}{2\pi}\right)^3\,\mathrm{vol}(Q_3)\sum_{(q,m)_n\in\mathsf{Q}_0(k,l)} q\,q_U\,q_V\ \mathrm{Tr}_{{}^n\underline{V}_{q,m}\otimes\mathcal{H}}\big({}^nP_{q,m}\big)\ , \tag{6.10}$$

where

$$\mathrm{vol}(Q_3):=\int_{Q_3}\ \bigwedge_{a=1}^{3}\ \gamma^a\wedge\bar\gamma^a\ . \tag{6.11}$$

The normalization can again be determined by examining the generators of the integer cohomology ring of the space Q_3.

For this, it is convenient to use the description of Q_3 as the twistor fibration (3.32). Since this is a sphere bundle, we may write down the corresponding Gysin long exact sequence relating the cohomology groups of Q_3 and $\mathbb{C}P^2$. Since the cohomology of the projective plane is concentrated in even degree, it implies that all odd degree cohomology groups of Q_3 vanish and the even degree ones are determined by the short exact sequences

$$0\ \longrightarrow\ \mathrm{H}^{2j}\big(\mathbb{C}P^2\,;\,\mathbb{Z}\big)\ \xrightarrow{\ \pi^*\ }\ \mathrm{H}^{2j}(Q_3;\mathbb{Z})\ \xrightarrow{\ \pi_*\ }\ \mathrm{H}^{2j-2}\big(\mathbb{C}P^2\,;\,\mathbb{Z}\big)\ \longrightarrow\ 0 \tag{6.12}$$

for $j=0,1,2,3$, where π^* is the usual pullback and π_* is integration along the $\mathbb{C}P^1$ fibre. Setting $j=3$ shows that the generator $[\tau]$ of $\mathrm{H}^6(Q_3;\mathbb{Z})\cong\mathrm{H}^4(\mathbb{C}P^2;\mathbb{Z})$ is given in terms of the generating element $[\eta]$ of the cohomology ring of $\mathbb{C}P^2$ through

$$\pi_*(\tau)=\eta\wedge\eta\,, \tag{6.13}$$

with

$$\int_{Q_3}\tau=1\ . \tag{6.14}$$

Recall from section 3.1 that $\mathrm{H}^2(Q_3;\mathbb{Z})=\mathbb{Z}[\sigma_1]\oplus\mathbb{Z}[\sigma_2]$ where

$$\sigma_i=\tfrac{\mathrm{i}}{2\pi}\,g_i\ . \tag{6.15}$$

Setting $j=1$ in (6.12) shows that a consistent choice of basis is given by setting $\pi^*(\eta)=\sigma_1$, $\pi_*(\sigma_1)=0$ and $\pi_*(\sigma_2)=1$, so that π_* can be represented by integration over the two-cycle $\mathbb{C}P^1_{(2)}\subset Q_3$. Then from (6.12) with $j=2$ it follows that $\pi^*(\eta\wedge\eta)=\sigma_1\wedge\sigma_1$ and $\sigma_1\wedge\sigma_2$ generate $\mathrm{H}^4(Q_3;\mathbb{Z})\cong\mathbb{Z}\oplus\mathbb{Z}$, with $\pi_*(\sigma_1\wedge\sigma_1)=0$ and $\pi_*(\sigma_1\wedge\sigma_2)=\eta$. Hence by (6.13) we have

$$\tau=\sigma_1\wedge\sigma_1\wedge\sigma_2\ . \tag{6.16}$$

On substituting (6.15), (3.44), and (3.40) into (6.16) and (6.14), we fix the normalization (6.11) as $\mathrm{vol}(Q_3) = 2(2\pi)^3$. For the topological charge (6.10) we thus find

$$Q = -2 \sum_{(q,m)_n \in \mathsf{Q}_0(k,l)} q\, q_U\, q_V\ {}^nN_{q,m} = \sum_{(q,m)_n \in \mathsf{Q}_0(k,l)} \frac{1}{2}\, q \left(q^2 - m^2\right)\ {}^nN_{q,m}\ . \tag{6.17}$$

Now $Q \in 2\mathbb{Z}$ since the integers $(q,m)_n$ have the same parity.

6.2 K-theory charges

We will now describe a natural interpretation of the instanton charges (6.8) and (6.17) in terms of equivariant K-theory and brane-antibrane annihilation on the manifold M_{2d}, using the explicit solutions we have constructed in section 5.2. The main physical idea is that the instantons on $X = \mathbb{C}^d_\theta \times G/H$ can be interpreted as a system of p coincident $\mathrm{D}(2d + d_H)$-branes wrapping the coset space G/H, with the G-equivariance condition splitting the rank p as in (3.54) and wrapping G/H with instanton and monopole fields. After dimensional reduction, we are left with an equivalent system of $\mathrm{D}(2d)$ branes and antibranes carrying the appropriate topological quantum numbers, which stabilize the marginally bound space-filling brane configurations. On the subset of branes at vertex $v \in \mathsf{Q}_0(k,l)$ there lives a Chan-Paton gauge potential $A^v \in \mathrm{End}(E_{p_v})$, and neighbouring subsets are connected by Higgs fields $\phi_{v,\Phi(v)} \in \mathrm{Hom}(E_{p_v}, E_{p_{\Phi(v)}})$ corresponding to massless open string excitations. The BPS vortex configurations of section 5.3 are stable bound states of M D0-branes inside the system of $\mathrm{D}(2d)$-branes, where $M = (\#\mathsf{Q}_0(k,l) - 1)\, N$. We will now explain how a particular K-theory construction naturally leads to this physical picture.

The noncommutative instantons are classified by the G-equivariant K-theory group of X which may be computed via the equivariant excision theorem to get

$$\mathrm{K}_G(X) = \mathrm{K}_G\big(G \times_H M_{2d}\big) \cong \mathrm{K}_H(M_{2d})\ . \tag{6.18}$$

Since H acts trivially on M_{2d}, this group reduces to the product

$$\mathrm{K}_G(X) \cong \mathrm{K}(M_{2d}) \otimes \mathrm{R}(H) \tag{6.19}$$

of the ordinary K-theory of M_{2d} with the representation ring of the subgroup H. The K-theory charge is thus computed by finding appropriate representatives for each of these factors in the quiver gauge theory.

We start with the second factor. Collapsing M_{2d} to a point in (6.19) shows that

$$\mathrm{R}(H) \cong \mathrm{K}_G\big(G/H\big)\ . \tag{6.20}$$

It follows that the classes in $\mathrm{R}(H)$ may be constructed from appropriate representatives of the homogeneous vector bundles (2.1), and (6.19) is equivalent to the isotopical bundle decompositions given by (3.53) and (3.56). Under the isomorphism (3.57), we can use the index class of the equivariant Dirac operator $\not{D}$ in the background of quiver gauge fields on the homogeneous space G/H which live in the second component of the subspaces (3.58). Denote by $\not{D}_v$, $v \in \mathsf{Q}_0(k,l)$ the spin^c Dirac operator in the background G-equivariant gauge fields corresponding to the irreducible H-module $\underline{v}$ (see e.g. [22]–[24] for explicit constructions). In a suitable basis, there are chiral decompositions

$$\not{D} \;=\; \sum_{v\in\mathsf{Q}_0(k,l)} \not{D}_v\otimes\Pi_v \;=\; \sum_{v\in\mathsf{Q}_0(k,l)} \begin{pmatrix} 0 & \not{D}_v^+ \\ \not{D}_v^- & 0 \end{pmatrix}\otimes\Pi_v \tag{6.21}$$

into twisted Dolbeault-Dirac operators $\not{D}_v^\pm$.

For each vertex v, there is an index class

$$\underline{\mathrm{index}}(\not{D}_v) := \ker\big(\not{D}_v^+\big)\ominus\ker\big(\not{D}_v^-\big) \tag{6.22}$$

in the G-equivariant K-theory of G/H, whose virtual dimension is the index of the Dirac operator $\not{D}_v$. In analogy to the quiver gauge theories based on the symmetric space $\mathbb{C}P^1$ [4, 13], we will call the gauge field excitations on M_{2d} associated to vertex v a *brane* if $\ker(\not{D}_v^-)=\{0\}$ and an *antibrane* if $\ker(\not{D}_v^+)=\{0\}$. This associates K-theory charges on G/H to D-brane charges on M_{2d}. Note that these are *not* the same as the topological charges associated to the gauge field configurations of the homogeneous vector bundle (2.1). By the Atiyah-Singer index theorem

$$\mathrm{index}(\not{D}_v) = \int_{G/H} \mathrm{ch}(\mathcal{V}_v\otimes\mathcal{L}_c)\wedge\widehat{A}\big(G/H\big)\,, \tag{6.23}$$

where the complex line bundle $\mathcal{L}_c\to G/H$ determines the spin^c structure. Hence the K-theory charge generally couples topological charges with both the U(1) charge $c_1(\mathcal{L}_c)$ of the spin^c fermion and the Pontrjagin numbers of the tangent bundle to the coset space G/H. Such curvature couplings are a standard feature of D-brane charge. (In the $\mathbb{C}P^1$ case, the two types of charges agree as the index of the Dirac operator coincides with the first Chern number of the m-monopole line bundle over $\mathbb{C}P^1$.) The class (6.22) represents the symmetric spinors which survive the G-invariant dimensional reduction from X to M_{2d} in the G-equivariant ABS construction of K-theory classes on $\mathbb{R}^{2d}$ [4, 13]. We will denote the respective disjoint vertex subsets corresponding to branes and antibranes by $\mathsf{Q}_0^\pm\subset\mathsf{Q}_0(k,l)$.

Let us now describe the first factor in (6.19). The Toeplitz operators T_v obeying (5.23) determine an index class

$$\underline{\mathrm{index}}\big(T_v^\dagger\big) \;:=\; \ker\big(T_v^\dagger\big)\ominus\ker(T_v) \;=\; \ker\big(T_v^\dagger\big) \;\cong\; \mathbb{C}^{N_v} \tag{6.24}$$

in the K-theory group $\mathrm{K}(\mathbb{C}^d_\theta)$. The rank of the corresponding projector P_v is the index of $T_v^\dagger$, $N_v=\mathrm{index}\big(T_v^\dagger\big)$, and P_v projects the noncommutative quiver bundle $\underline{V}^{k,l}\otimes\mathcal{H}$ onto the finite-dimensional quiver module

$$\underline{T} := \bigoplus_{v\in\mathsf{Q}_0(k,l)} \underline{\mathrm{index}}\big(T_v^\dagger\big)\ . \tag{6.25}$$

Using the chirality grading introduced above, we define a $\mathbb{Z}_2$-grading of the fibre space (5.7) by

$$\underline{V}^{k,l} \;=\; \underline{V}_+\oplus\underline{V}_- \qquad \text{with} \quad \underline{V}_\pm \;:=\; \bigoplus_{v\in\mathsf{Q}_0^\pm} \underline{V}_v\otimes\underline{v}\ . \tag{6.26}$$

Using the explicit instanton solutions of section 5.2, we will demonstrate how to construct odd operators

$$\boldsymbol{T} \,:\, \underline{V}_{+} \otimes \mathcal{H} \longrightarrow \underline{V}_{-} \otimes \mathcal{H} \qquad \text{with} \quad \boldsymbol{T}^2 = 0 \tag{6.27}$$

from the Toeplitz operators T_v.

This defines a two-term complex corresponding to the basic brane-antibrane system with tachyon field $\boldsymbol{T}$. Its cohomology is a representative of the K-theory Euler class generating $\mathrm{K}(\mathbb{C}_\theta^d)$. The single brane-antibrane system is obtained by a "folding" of the component branes and antibranes at the vertices of the quiver, as dictated by G-equivariance. A description of the moduli involved in this folding process will be given in section 6.3 below. The tachyon field (6.27) has an isotopical decomposition $\boldsymbol{T} = \sum_{v\in\mathsf{Q}_0(k,l)} \boldsymbol{T}_v \otimes \Pi_v$, and putting everything together the virtual module

$$\underline{\boldsymbol{\mathcal{T}}} := \bigoplus_{v\in\mathsf{Q}_0(k,l)} \underline{\mathrm{index}}(\boldsymbol{T}_v) \otimes \ker(\not{\mathcal{D}}_v) \tag{6.28}$$

is the K-theory class in (6.19) we are looking for. We will see that the associated *K-theory* charge on M_{2d}, i.e. the virtual dimension $\mathcal{Q}$ of this module, is canonically related to the *topological* charge Q on X computed in section 6.1 above, i.e. the topological charges of the instanton gauge fields before the dimensional reduction.

The crux of the construction, in addition to an explicit determination of the Dirac index, is thus an explicit model for the tachyon operator (6.27). It naturally appears in a graded connection formalism [4, 13] which is a rewriting of the equivariant gauge theory on X as an ordinary Yang-Mills gauge theory on the corresponding quiver bundle over M_{2d}, appropriate to its interpretation in terms of brane-antibrane systems on M_{2d}. With respect to the $\mathbb{Z}_2$-grading (6.26), the even "diagonal" parts of the graded connections are built from the gauge connection one-forms as

$$\boldsymbol{A} = \sum_{v\in\mathsf{Q}_0(k,l)} A^v \otimes \Pi_v \tag{6.29}$$

in $\bigoplus_{v\in\mathsf{Q}_0(k,l)} \Omega^{0,1}(\mathrm{End}(E_{p_v}))$, along with similar formulas for the equivariant gauge potentials in $\bigoplus_{v\in\mathsf{Q}_0(k,l)} \Omega^{0,1}(\mathrm{End}(\mathcal{V}_v))^G$. The odd zero-form components of the graded connections determine the tachyon fields (6.27) and are associated with the "off-diagonal" subspace of $\Omega^{0,1}(\mathrm{End}(E^{k,l}))^H$ given by

$$\bigoplus_{v\in\mathsf{Q}_0(k,l)} \ \bigoplus_{\Phi\in\mathsf{Q}_1(k,l)} \Omega^0\big(\mathrm{Hom}(E_{p_v}, E_{p_{\Phi(v)}})\big) \otimes \mathrm{Hom}_H\big(\,\underline{v}\,,\,\underline{\Phi(v)}\,\big)\,, \tag{6.30}$$

where we have used H-equivariance. The details of the construction again depend on the particular quiver.

Symmetric $\underline{\boldsymbol{C}}^{\boldsymbol{k,l}}$ quiver charges. By Künneth's theorem, the representation ring of the holonomy group $H = \mathrm{SU}(2)\times\mathrm{U}(1)$ is the product $\mathrm{R}(H) = \mathrm{R}(\mathrm{SU}(2))\otimes\mathrm{R}(\mathrm{U}(1))$. As in the $\mathbb{C}P^1$ cases, using the isomorphism (6.20) we can identify the representation ring of $\mathrm{U}(1)$ with the formal Laurent polynomial ring $\mathbb{Z}[\mathcal{L},\mathcal{L}^\vee\,]$ generated by classes of the monopole line bundle over $\mathbb{C}P^2$. The class of $\mathcal{L}$ is also tied to the (reduced) K-theory of the projective plane itself, which is generated by $\mathcal{L}$ and $\mathcal{L}\otimes\mathcal{L}$. By constructing irreducible representations of SU(2) in the standard way through symmetrizations of the fundamental representation **2**, there is an isomorphism $\mathrm{R(SU(2))} \cong \mathbb{Z}[\mathbf{2}]$ which under (6.20) can be identified with the formal polynomial ring in classes of the SU(2) instanton bundle $\mathcal{I} \to \mathbb{C}P^2$. The representation ring may thus be presented as

$$\mathrm{R}(H) \cong \mathbb{Z}[\mathcal{I}] \otimes \mathbb{Z}\big[\mathcal{L}\,,\,\mathcal{L}^\vee\,\big]\ . \tag{6.31}$$

We will represent classes in (6.31) by using the Dirac index on $\mathbb{C}P^2$. As we now demonstrate, the index of the spin^c Dirac operator on $\mathbb{C}P^2$ in the background instanton and monopole fields associated to the representation $\underline{(n,m)}$ of H is given by

$$\mathrm{index}(\not{D}_{n,m}) \ = \ \frac{1}{8}\,(n+1)\,(m+n+1)\,(m-n-1) \ = \ \frac{1}{8}\,(n+1)\,\big(m^2-n\,(n+2)-1\big)\ . \tag{6.32}$$

The Dirac spectrum was determined in [24] in the following way. By identifying the Lie algebra $\mathfrak{g}^{\mathbb{C}} = \mathrm{sl}(3,\mathbb{C})$ with the Lie algebra of complex left-invariant vector fields on the group G, we get a natural action of the universal enveloping algebra $\mathcal{U}(\mathfrak{g}^{\mathbb{C}})$ on $\mathrm{C}^\infty(G)$. The quadratic Casimir element

$$\boldsymbol{C}_2 = \sum_{\alpha\in\Delta^+}\,\big(E_\alpha\,E_{-\alpha} + E_{-\alpha}\,E_\alpha\big) + \frac{1}{2}\,\big(H_{\alpha_1}^2 + H_{\alpha_2}^2\big) \tag{6.33}$$

thereby induces an invariant second order differential operator, which coincides with the metric Laplace-Beltrami operator on $\mathbb{C}P^2$. Up to curvature terms, this operator coincides with the square of the Dirac operator. Its spectrum is given by the quadratic Casimir eigenvalues

$$C_2(k,l) = \frac{1}{3}\,\big(k\,(k+3) + l\,(l+3) + k\,l\big) \tag{6.34}$$

in the irreducible SU(3) representation $\underline{C}^{k,l}$. After coupling to the background gauge fields, the twisted Laplace-Beltrami operator can be expressed as a difference of the quadratic Casimir operators (6.33) and (5.36).

Using the construction of section 2.2, the Dirac kernels may thus be determined by the G-module $\underline{C}^{k_{n,m},l_{n,m}}$ which minimizes the Casimir invariant (6.34), subject to the constraints (3.69) that $\underline{C}^{k_{n,m},l_{n,m}}$ contain the representation $\underline{(n,m)}$ of $H = \mathrm{SU}(2)\times\mathrm{U}(1)$. In [24] it was shown that the chiral case $\ker(\not{D}^-_{n,m}) = \{0\}$ corresponds to the configurations with $m^2 > n^2$, for which $\ker(\not{D}^+_{n,m})$ is isomorphic to the SU(3) representation $\underline{C}^{k,l}$ having $|m| = k+2l$, $n = k$ in (2.20). The dimension (2.18) of this module coincides with the index (6.32) after the shift $m \to m - c_1(T\mathbb{C}P^2)$, where $c_1(T\mathbb{C}P^2) = 3$ is the first Chern number of the tangent bundle over $\mathbb{C}P^2$. This shift arises from the fact that the chiral symmetric spinors couple only to the U(1) part of the spin^c connection from which the Dirac operator is constructed [22, 23]. The antichiral case $\ker(\not{D}^+_{n,m}) = \{0\}$ corresponds to $m^2 \le n^2$, for which $\ker(\not{D}^-_{n,m})$ is isomorphic to the module $\underline{C}^{k,l}$ with $|m| = k-l$, $n = k+l$ in (2.20). The corresponding dimension (2.18) coincides with minus the index (6.32) after the shift $n \to n-1$. This shift accounts for the fact that the antichiral symmetric spinors couple only to the SU(2) part of the spin^c connection. Note that because of these shifts, the index (6.32) is generically fractional. Nevertheless, it will turn out to be the one appropriate to our K-theory construction.

Next we turn to the construction of the tachyon field (6.27). For this, it is useful to recall the mapping onto the SU(2) spins $j_\pm = j_\pm(n,m)$ defined in (3.67). The shifts along vertices induced by the arrows of the symmetric quiver in these variables take the simple forms

$$\begin{aligned} j_+(n+1,m+3) &= j_+(n,m)+\frac{1}{2} \qquad \text{and} \qquad & j_+(n-1,m+3) &= j_+(n,m)\,, \\ j_-(n-1,m+3) &= j_-(n,m)-\frac{1}{2} \qquad \text{and} \qquad & j_-(n+1,m+3) &= j_-(n,m)\,. \end{aligned} \tag{6.35}$$

The vertex redefinition (3.67) thus orients the symmetric quiver diagram onto a rectangular quiver of the same type as those which arise for the symmetric space $\mathbb{C}P^1\times\mathbb{C}P^1$. This will enable us to adapt some of the constructions of [13] to the present case.

Using the Dirac operator analysis above, we decompose the vertex set $\mathsf{Q}_0(k,l)$ into the disjoint subsets

$$\mathsf{Q}_0^+ = \big\{(n,m)\in\mathsf{Q}_0(k,l)\,\big|\,m^2>n^2\big\} \quad \text{and} \quad \mathsf{Q}_0^- = \big\{(n,m)\in\mathsf{Q}_0(k,l)\,\big|\,m^2\le n^2\big\} \tag{6.36}$$

corresponding respectively to positive and negative values of the index (6.32). Using the SU(2) spin variables in (3.69), these subsets further split into disjoint unions $\mathsf{Q}_0^\pm = \mathsf{Q}_0^{\pm+}\sqcup\mathsf{Q}_0^{\pm-}$ with

$$\begin{aligned} \mathsf{Q}_0^{+\pm} &= \big\{(n,m)\in\mathsf{Q}_0^+\,\big|\,2j_\pm - j_\mp < \pm\,\tfrac{k-l}{2}\big\}\,, \\ \mathsf{Q}_0^{-\pm} &= \big\{(n,m)\in\mathsf{Q}_0^-\,\big|\,j_\pm - 2j_\mp \le \pm\,\tfrac{k-l}{2} \quad \text{and} \quad j_\pm - j_\mp \ge \pm\,\tfrac{k-l}{3}\big\}\,. \end{aligned} \tag{6.37}$$

Using these decompositions we can define a bi-grading of the fibre space (6.26) by

$$\underline{V}_\pm = \underline{V}_{\pm+}\oplus\underline{V}_{\pm-} \qquad \text{with} \quad \underline{V}_{\pm\bullet} = \bigoplus_{(n,m)\in\mathsf{Q}_0^{\pm\bullet}} \underline{V}_{n,m}\otimes\underline{(n,m)}\,. \tag{6.38}$$

Given the operators (5.28), we define morphisms in (6.30) by

$$\phi^\pm := \sum_{(n,m)\in\mathsf{Q}_0(k,l)} \phi^\pm_{n,m}\otimes\Bigg(\sum_{q\in\{-n+2j\}_{j=0}^n}\Big(\big|\overset{n\pm1}{q-1}\,,\,m+3\big\rangle\big\langle\overset{n}{q}\,,\,m\big|+\big|\overset{n\pm1}{q+1}\,,\,m+3\big\rangle\big\langle\overset{n}{q}\,,\,m\big|\Big)\Bigg)\,. \tag{6.39}$$

Using (6.35) along with finite-dimensionality of the path algebra associated to the given symmetric quiver, we find the generic nilpotency conditions

$$\begin{aligned} \big(\phi^+\big)^i &\ne 0\,,\quad i=1,\dots,k \qquad \text{and} \qquad & \big(\phi^+\big)^{k+1} &= 0\,, \\ \big(\phi^-\big)^j &\ne 0\,,\quad j=1,\dots,l \qquad \text{and} \qquad & \big(\phi^-\big)^{l+1} &= 0\,. \end{aligned} \tag{6.40}$$

The operators (6.39) are thus naturally associated with the zero-form components of a $\mathbb{Z}_{k+1}\times\mathbb{Z}_{l+1}$-graded connection. From the holomorphic relations (5.10) it follows that they obey the commutation relation

$$\big[\phi^+\,,\,\phi^-\big]=0\,, \tag{6.41}$$

and therefore generate a p-dimensional representation of the two-dimensional abelian algebra $\mathfrak{u}$. In addition, from the non-holomorphic relations (5.29) one finds the commutativity condition

$$\big[\boldsymbol{\phi}^+ \,,\, \boldsymbol{\phi}^{-\,\dagger}\big] = 0 \tag{6.42}$$

along with hermitean conjugates.

For any positive integer s, we use (5.23) and (5.28) to derive the identities

$$\begin{aligned}\left(\boldsymbol{\phi}^\pm\right)^s = \sum_{(n,m)\in\mathsf{Q}_0(k,l)} T_{n\pm s,m+3s}\, T^\dagger_{n,m} \otimes \Bigg(\sum_{q\in\{-n+2j\}_{j=0}^n} \Big(\big|\overset{n\pm s}{q-s}\,,\, m+3s\big\rangle\big\langle\overset{n}{q}\,,\, m\big| \qquad &(6.43)\\ + s\sum_{j=1}^{s-1} \big|\overset{n\pm s}{q-s+2j}\,,\, m+3s\big\rangle\big\langle\overset{n}{q}\,,\, m\big| + \big|\overset{n\pm s}{q+s}\,,\, m+3s\big\rangle\big\langle\overset{n}{q}\,,\, m\big|\Big)\Bigg) \,.\end{aligned}$$

Using these formulas we now introduce the operators

$$\boldsymbol{\mu}^+ \ := \ \left(\boldsymbol{\phi}^+\right)^{[\frac{k}{2}]+1} \qquad \text{and} \qquad \boldsymbol{\mu}^- \ := \ \left(\boldsymbol{\phi}^-\right)^{[\frac{l}{2}]+1} \tag{6.44}$$

where $[x]$ denotes the integer part of $x\in\mathbb{R}$. With respect to the $\mathbb{Z}_2\times\mathbb{Z}_2$-grading in (6.38), using (6.40), (6.43) and (6.35) one can straightforwardly show that they are odd holomorphic maps

$$\begin{aligned}\boldsymbol{\mu}^+ \ &: \underline{V}_{++}\otimes\mathcal{H} \ \longrightarrow \ \underline{V}_{--}\otimes\mathcal{H} \qquad \text{with} \quad \left(\boldsymbol{\mu}^+\right)^2 \ = \ 0\,,\\ \boldsymbol{\mu}^- \ &: \underline{V}_{-+}\otimes\mathcal{H} \ \longrightarrow \ \underline{V}_{+-}\otimes\mathcal{H} \qquad \text{with} \quad \left(\boldsymbol{\mu}^-\right)^2 \ = \ 0\,.\end{aligned} \tag{6.45}$$

The operators (6.44) thus produce the desired bi-complex of noncommutative tachyon fields between branes and antibranes.

It is instructive at this stage to look at the limiting case $l=0$. Then the quiver collapses to a holomorphic chain with $k+1$ vertices. One has $j_-=0$, $n=2j_+$ and $m=3n-2k$. In this case $m-n=2(n-k)\le 0$ at each vertex $v=(n,m)$, while $m+n=2(2n-k)$. The branes are located at vertices $v=(n,m)$ with $n=0,1,\dots,\big[\frac{k}{2}\big]-1$ along the chain, while the antibranes have $n=\big[\frac{k}{2}\big],\big[\frac{k}{2}\big]+1,\dots,k$. The tachyon field $\boldsymbol{\mu}^+$ shifts $m+n$ by $4\big[\frac{k}{2}\big]+4$ and hence takes branes into antibranes. A completely analogous picture holds for $k=0$, whereby $j_+=0$, $n=2j_-$ and $m=-3n+2l$, with the tachyon field $\boldsymbol{\mu}^-$ taking antibranes into branes. In this case $m+n\ge 0$ while $m-n$ is positive for $0<n<\big[\frac{l}{2}\big]$ and negative for $\big[\frac{l}{2}\big]\le n\le l$. These chains are qualitatively the same as the brane-antibrane systems associated to the symmetric space $\mathbb{C}P^1$ [4]. However, the symmetric quiver diagram associated to a generic module $\underline{C}^{k,l}$ is not simply the product of two chains as in the $\mathbb{C}P^1\times\mathbb{C}P^1$ case [13].

In contrast to the quiver gauge theories associated to the symmetric space $\mathbb{C}P^1$, the tachyon fields (6.45) do not map between branes of equal and opposite K-theory charges (6.32). On isotopical components, their non-trivial kernels and cokernels are given by the finite-dimensional vector spaces

$$\begin{aligned}\ker\big(\boldsymbol{\mu}^+_{n,m}\big) \ &= \ \mathrm{im}\big(P_{n,m}\big) \qquad \text{and} \qquad \ker\big(\boldsymbol{\mu}^{+\,\dagger}_{n,m}\big) \ = \ \mathrm{im}\big(P_{n+[\frac{k}{2}]+1,m+3[\frac{k}{2}]+3}\big)\,,\\ \ker\big(\boldsymbol{\mu}^-_{n,m}\big) \ &= \ \mathrm{im}\big(P_{n,m}\big) \qquad \text{and} \qquad \ker\big(\boldsymbol{\mu}^{-\,\dagger}_{n,m}\big) \ = \ \mathrm{im}\big(P_{n-[\frac{l}{2}]-1,m+3[\frac{l}{2}]+3}\big)\,.\end{aligned} \tag{6.46}$$

Denote by $\boldsymbol{\mu}^{\pm}_{\alpha\beta}$, $\alpha,\beta=\pm$, the restrictions of the operators (6.44) to $\underline{V}_{\alpha\beta}$. Following [13], with respect to the $\mathbb{Z}_2$-grading (6.26) and a suitable basis for the vector space $\underline{V}^{k,l}$, we define the operator

$$\boldsymbol{T} := \boldsymbol{\mu}^{+}_{++} \oplus \boldsymbol{\mu}^{-}_{+-}{}^{\dagger} \ . \tag{6.47}$$

It is an odd map (6.27) which produces the appropriate two-term complex representing the brane-antibrane system with tachyon field (6.47).

To incorporate the twistings by the instanton and monopole bundles, we use the ABS construction to extend the tachyon field (6.47) to the operator

$$\boldsymbol{\mathcal{T}} := \boldsymbol{T}\otimes\mathbf{1} \, : \underline{\mathcal{C}}_{+} \ \longrightarrow \ \underline{\mathcal{C}}_{-} \ , \tag{6.48}$$

where

$$\underline{\mathcal{C}}_{\pm} \ := \bigoplus_{(n,m)\in\mathsf{Q}_0^{\pm}} \big(\underline{V}_{n,m}\otimes\mathcal{H}\big)\otimes\underline{C}_{\pm}^{k_{n,m},l_{n,m}} \qquad \text{with} \quad \underline{C}_{\pm}^{k_{n,m},l_{n,m}} \ = \ \ker\big(\not{D}^{\pm}_{n,m}\big) \ . \tag{6.49}$$

From (6.32) and (6.46) it follows that the K-theory charge represented as the index of the tachyon field (6.48) is given by

$$\begin{aligned} \mathcal{Q} := \mathrm{index}(\boldsymbol{\mathcal{T}}) \ &= \ \dim\ker(\boldsymbol{\mathcal{T}}) - \dim\ker\big(\boldsymbol{\mathcal{T}}^{\dagger}\big) \\ &= \sum_{(n,m)\in\mathsf{Q}_0^{++}} \frac{1}{8}\,(n+1)\,\big(m^2-n\,(n+2)-1\big) \\ &\quad\times\Big[\big(N_{n,m}+N_{n+[\frac{k}{2}]-[\frac{l}{2}],m+3[\frac{l}{2}]+3[\frac{k}{2}]+6}\big)-\big(N_{n+[\frac{k}{2}]+1,m+3[\frac{k}{2}]+3}+N_{n-[\frac{l}{2}],m+3[\frac{l}{2}]+3}\big)\Big] \ , \end{aligned} \tag{6.50}$$

where we have used the commutativity relations (6.41) and (6.42). This charge is related to the instanton charge (6.8) through

$$Q = 4\mathcal{Q} + \frac{1}{2}\,N \ , \tag{6.51}$$

where N is the noncommutative vortex number (5.39).

Non-symmetric $\underline{C}^{k,l}$ quiver charges. The representation ring of the maximal torus T is a product $\mathrm{R}(T)=\mathrm{R}(\mathrm{U}(1))\otimes\mathrm{R}(\mathrm{U}(1))$, which can be identified with the formal Laurent polynomial ring

$$\mathrm{R}(T)\cong\mathbb{Z}\big[\mathcal{L}_{(1)}\,,\,\mathcal{L}_{(1)}^{\vee}\big]\otimes\mathbb{Z}\big[\mathcal{L}_{(2)}\,,\,\mathcal{L}_{(2)}^{\vee}\big] \tag{6.52}$$

in the two monopole line bundles $\mathcal{L}_{(i)}\to Q_3$. The (reduced) K-theory of the space Q_3 is generated by $\mathcal{L}_{(i)}$, $\mathcal{L}_{(1)}\otimes\mathcal{L}_{(i)}$ and $\mathcal{L}_{(1)}\otimes\mathcal{L}_{(1)}\otimes\mathcal{L}_{(2)}$ with $i=1,2$. For the Dirac index in (6.52), we will now show that the index of the spin^c Dirac operator on Q_3 in the background monopole fields corresponding to the irreducible representation $\underline{(q,m)_n}$ of T is given by

$$\mathrm{index}(\not{D}_{q,m}) = \frac{1}{8}\,q\,\big(m^2-q^2\big) \ . \tag{6.53}$$

The Dirac spectrum in this case was also computed in [24] from the natural spin^c connection on Q_3 with torsion, by using exactly the same technique as in the symmetric case above.

The chiral case $\ker(\not{D}^-_{q,m}) = \{0\}$ corresponds to background gauge field configurations on Q_3 with $q^2 \geq m^2$, for which $\ker(\not{D}^+_{q,m})$ is isomorphic to the G-module $\underline{C}^{k,l}$ having $|q| = k + l$, $|m| = k - l$. The corresponding dimension (2.18) coincides with minus the index (6.53) after shifting $q \to q \pm 2$. The antichiral case $\ker(\not{D}^+_{q,m}) = \{0\}$ corresponds to $q^2 \leq m^2$, for which the Dirac kernel $\ker(\not{D}^-_{q,m})$ is isomorphic to the SU(3) representation $\underline{C}^{k,l}$ with $|q| = k$, $|m| = k + 2l$. The corresponding dimension (2.18) agrees with (6.53) after the shifts $q \to q \pm 1$, $m \to m \pm 3$. In both cases the shifts account for the contributions of the intrinsic spin^c fermion to the U(1) monopole charges.

We correspondingly decompose the vertex set $\mathsf{Q}_0(k,l)$ into disjoint subsets

$$\mathsf{Q}_0^+ = \big\{(q,m)_n \in \mathsf{Q}_0(k,l) \bigm| q^2 > m^2\big\} \qquad \text{and} \qquad \mathsf{Q}_0^- = \big\{(q,m)_n \in \mathsf{Q}_0(k,l) \bigm| q^2 \leq m^2\big\} . \tag{6.54}$$

Using the U-spin and V-spin charge variables (5.50), we further decompose these subsets into $\mathsf{Q}_0^\pm = \mathsf{Q}_0^{\pm +} \sqcup \mathsf{Q}_0^{\pm -}$ with

$$\begin{aligned} \mathsf{Q}_0^{++} &= \big\{(q,m)_n \in \mathsf{Q}_0^+ \big| q_U < 0\big\} && \text{and} \quad \mathsf{Q}_0^{+-} = \big\{(q,m)_n \in \mathsf{Q}_0^+ \big| q_V < 0\big\} , \\ \mathsf{Q}_0^{-+} &= \big\{(q,m)_n \in \mathsf{Q}_0^- \big| q_U \geq 0 \,, q > 0\big\} && \text{and} \quad \mathsf{Q}_0^{--} = \big\{(q,m)_n \in \mathsf{Q}_0^- \big| q_V \geq 0 \,, q \leq 0\big\} . \end{aligned} \tag{6.55}$$

The corresponding bi-grading of the fibre space (6.26) is then given by

$$\underline{V}_\pm = \underline{V}_{\pm +} \oplus \underline{V}_{\pm -} \qquad \text{with} \quad \underline{V}_{\pm \bullet} = \bigoplus_{(q,m)_n \in \mathsf{Q}_0^{\pm \bullet}} {}^n\underline{V}_{q,m} \otimes \underline{(q,m)_n} \, . \tag{6.56}$$

Given the operators (5.41), we define morphisms in (6.30) by

$$\begin{aligned} \boldsymbol{\phi}^\pm &:= \sum_{(q,m)_n \in \mathsf{Q}_0(k,l)} \Big({}^{n-1}T_{q\pm 1, m+3} \, {}^nT_{q,m}{}^\dagger \otimes \big|\overset{n-1}{q \pm 1}\,,\, m+3\big\rangle\big\langle \overset{n}{q}\,,\, m\big| \\ &\qquad\qquad + {}^{n+1}T_{q\pm 1, m+3} \, {}^nT_{q,m}{}^\dagger \otimes \big|\overset{n+1}{q \pm 1}\,,\, m+3\big\rangle\big\langle \overset{n}{q}\,,\, m\big| \Big) \, , \\ \boldsymbol{\phi}^0 &:= \sum_{(q,m)_n \in \mathsf{Q}_0(k,l)} {}^nT_{q+2,m} \, {}^nT_{q,m}{}^\dagger \otimes \big|\overset{n}{q + 2}\,,\, m\big\rangle\big\langle \overset{n}{q}\,,\, m\big| \, . \end{aligned} \tag{6.57}$$

From the quadratic holomorphic relations (5.17)–(5.19) it follows that these $p \times p$ matrix-valued operators satisfy the commutativity equations

$$\big[\boldsymbol{\phi}^+ \,,\, \boldsymbol{\phi}^-\big] = 0 \qquad \text{and} \qquad \big[\boldsymbol{\phi}^\pm \,,\, \boldsymbol{\phi}^0\big] = 0 \, , \tag{6.58}$$

while from the non-holomorphic relations (5.44) one finds

$$\big[\boldsymbol{\phi}^+ \,,\, \boldsymbol{\phi}^{-\,\dagger}\big] = 0 \qquad \text{and} \qquad \big[\boldsymbol{\phi}^\pm \,,\, \boldsymbol{\phi}^{0\,\dagger}\big] = 0 \, , \tag{6.59}$$

plus hermitean conjugates. From the linear holomorphic relations (5.42) one has

$$\boldsymbol{\phi}^+ = \boldsymbol{\phi}^- \, \boldsymbol{\phi}^0 = \boldsymbol{\phi}^0 \, \boldsymbol{\phi}^- \, , \tag{6.60}$$

while from (5.43) it follows that

$$\boldsymbol{\phi}^- = \boldsymbol{\phi}^+ \, \boldsymbol{\phi}^{0\,\dagger} = \boldsymbol{\phi}^{0\,\dagger} \, \boldsymbol{\phi}^+ \qquad \text{and} \qquad \boldsymbol{\phi}^0 = \boldsymbol{\phi}^{-\,\dagger} \, \boldsymbol{\phi}^+ = \boldsymbol{\phi}^+ \, \boldsymbol{\phi}^{-\,\dagger} \, , \tag{6.61}$$

JHEP08(2008)093

along with hermitean conjugates.

From the isospin range (2.22) it follows that the operators $\boldsymbol{\phi}^0$ obey the generic nilpotency conditions

$$\big(\boldsymbol{\phi}^0\big)^i \neq 0\,, \quad i = 1,\dots,k+l+1 \qquad \text{and} \qquad \big(\boldsymbol{\phi}^0\big)^{k+l+2} = 0\,. \tag{6.62}$$

From the relations (6.58)–(6.61) it then also follows that

$$\big(\boldsymbol{\phi}^\pm\big)^i \neq 0\,, \quad i = 1,\dots,k+l+1 \qquad \text{and} \qquad \big(\boldsymbol{\phi}^\pm\big)^{k+l+2} = 0\,. \tag{6.63}$$

Generic non-vanishing powers of the operators (6.57) are readily computed with the results

$$\begin{aligned}\big(\boldsymbol{\phi}^\pm\big)^s &= \sum_{(q,m)_n\in\mathsf{Q}_0(k,l)} \Big({}^{n-s}T_{q\pm s,m+3s}\,{}^nT_{q,m}{}^\dagger\otimes\big|\overset{n-s}{q\pm s}\,,\,m+3s\big\rangle\big\langle\overset{n}{q}\,,\,m\big| \\ &\qquad\qquad + s\sum_{j=1}^{s-1}{}^{n-s+2j}T_{q\pm s,m+3s}\,{}^nT_{q,m}{}^\dagger\otimes\big|\overset{n-s+2j}{q\pm s}\,,\,m+3s\big\rangle\big\langle\overset{n}{q}\,,\,m\big| \\ &\qquad\qquad + {}^{n+s}T_{q\pm s,m+3s}\,{}^nT_{q,m}{}^\dagger\otimes\big|\overset{n+s}{q\pm s}\,,\,m+3s\big\rangle\big\langle\overset{n}{q}\,,\,m\big|\Big)\,, \\ \big(\boldsymbol{\phi}^0\big)^s &= \sum_{(q,m)_n\in\mathsf{Q}_0(k,l)} {}^nT_{q+2s,m}\,{}^nT_{q,m}{}^\dagger\otimes\big|\overset{n}{q+2s}\,,\,m\big\rangle\big\langle\overset{n}{q}\,,\,m\big|\,,\end{aligned} \tag{6.64}$$

for any positive integer s.

From (6.60) and (6.61) it follows that only two of the operators in (6.57) are independent. We will work with $\boldsymbol{\phi}^\pm$ and use the second relation in (6.61) to generate $\boldsymbol{\phi}^0$. The operators $\boldsymbol{\phi}^\pm$ thus form the odd zero-form components of a $\mathbb{Z}_{k+l+2}\times\mathbb{Z}_{k+l+2}$-graded connection. Using (6.64) we then define the bi-tachyon fields

$$\boldsymbol{\mu}^\pm := \big(\boldsymbol{\phi}^\pm\big)^{[\frac{k+l+2}{2}+1]+1}\,. \tag{6.65}$$

Similarly to (6.45), they are odd holomorphic maps

$$\begin{aligned}\boldsymbol{\mu}^+ &: \underline{V}_{++}\otimes\mathcal{H} \longrightarrow \underline{V}_{--}\otimes\mathcal{H} \qquad \text{with} \qquad \big(\boldsymbol{\mu}^+\big)^2 = 0\,, \\ \boldsymbol{\mu}^- &: \underline{V}_{+-}\otimes\mathcal{H} \longrightarrow \underline{V}_{-+}\otimes\mathcal{H} \qquad \text{with} \qquad \big(\boldsymbol{\mu}^-\big)^2 = 0\end{aligned} \tag{6.66}$$

whose isotopical components have non-trivial kernels and cokernels given by

$$\ker\big({}^n\boldsymbol{\mu}^\pm_{q,m}\big) = \mathrm{im}\big({}^nP_{q,m}\big) \qquad \text{and} \qquad \ker\big({}^n\boldsymbol{\mu}^\pm_{q,m}{}^\dagger\big) = \bigoplus_{j=0}^{s}\mathrm{im}\big({}^{n-s+2j}P_{q\pm s,m+3s}\big)\,, \tag{6.67}$$

with $s := \big[\frac{k+l+2}{2}\big]+1$. Note that the cokernel in (6.67) naturally takes into account the degeneracies of weight vectors, i.e. the multiple arrows in the quiver diagram.

The tachyon field

$$\boldsymbol{T} := \boldsymbol{\mu}^+_{++}\oplus\boldsymbol{\mu}^-_{+-} \tag{6.68}$$

then yields the requisite two-term complex (6.27). We extend it as in (6.48) using the noncommutative ABS spaces where

$$\underline{\mathcal{C}}_\pm := \bigoplus_{(q,m)_n\in\mathsf{Q}_0^\pm}\big({}^n\underline{V}_{q,m}\otimes\mathcal{H}\big)\otimes\underline{C}_\pm^{k_{q,m},l_{q,m}} \qquad \text{with} \qquad \underline{C}_\pm^{k_{q,m},l_{q,m}} = \ker\big(\not{D}^\pm_{q,m}\big)\,. \tag{6.69}$$

Using (6.53), (6.58), (6.59) and (6.67) we compute the index $\mathcal{Q}$ of the tachyon field (6.68) as before to get

$$\mathcal{Q} = \sum_{(q,m)_n \in \mathsf{Q}_0^{++}} \frac{1}{8}\, q\left(q^2 - m^2\right) \Big[\left({}^nN_{q,m} + {}^nN_{q,m+6s}\right) - \left({}^nN_{q+s,m+3s} + {}^nN_{q-s,m+3s}\right)\Big] \ . \tag{6.70}$$

This charge is related to the instanton charge (6.17) as

$$Q = 4\mathcal{Q} \ . \tag{6.71}$$

6.3 Euler-Ringel characters

The category of representations of the quiver with relations $(\mathsf{Q}(k,l), \mathsf{R}(k,l))$ provides a complete framework for understanding our instanton solutions. It gives a more detailed picture of the dynamics, particularly of how the original configuration on X folds itself into branes and antibranes on M_{2d} within the category of quiver modules. Following [13], we start from the instanton module (6.25) over $(\mathsf{Q}(k,l), \mathsf{R}(k,l))$ and its projective Ringel resolution. Since there are no relations among our relations, this leads to the exact sequence

$$\begin{aligned} 0 \longrightarrow \bigoplus_{v\in\mathsf{Q}_0(k,l)} \bigoplus_{\rho\in\mathsf{R}(k,l)} \underline{\mathcal{P}}_{\rho(v)} \otimes \ker\left(T_v^\dagger\right) \longrightarrow \bigoplus_{v\in\mathsf{Q}_0(k,l)} \bigoplus_{\Phi\in\mathsf{Q}_1(k,l)} \underline{\mathcal{P}}_{\Phi(v)} \otimes \ker\left(T_v^\dagger\right) \longrightarrow \\ \longrightarrow \bigoplus_{v\in\mathsf{Q}_0(k,l)} \underline{\mathcal{P}}_v \otimes \ker\left(T_v^\dagger\right) \longrightarrow \underline{T} \longrightarrow 0 \end{aligned} \tag{6.72}$$

where $\underline{\mathcal{P}}_v$ is the quiver representation defined as the subspace of the path algebra generated by all paths which start from vertex $v \in \mathsf{Q}_0(k,l)$. The first sum in (6.72) runs through the holomorphic relations of the quiver which are indexed by paths starting at vertex v and ending at vertex $\rho(v)$.

Let

$$\underline{W} = \bigoplus_{v\in\mathsf{Q}_0(k,l)} \underline{W}_v \tag{6.73}$$

be the canonical representation of $(\mathsf{Q}(k,l), \mathsf{R}(k,l))$ determined by the "folding" of K-theory charges in the equivariant ABS construction. We regard it as an element of the representation ring of the quiver. It will be determined explicitly below in terms of the Dirac kernels $\underline{C}^{k_v,l_v} = \ker(\not{D}_v)$ of section 6.2 above. The module $\underline{W}$ represents the coupling of K-theory charges on G/H to the instanton modules $\underline{T}$.

Applying the covariant functor $\mathrm{Hom}(-\,,\underline{W})$ to the projective resolution (6.72) yields the complex

$$\begin{aligned} 0 \longrightarrow \mathrm{Hom}\big(\underline{T}\,,\underline{W}\big) \longrightarrow \bigoplus_{v\in\mathsf{Q}_0(k,l)} \mathrm{Hom}\big(\ker(T_v^\dagger)\,,\underline{W}_v\big) \longrightarrow \\ \longrightarrow \bigoplus_{v\in\mathsf{Q}_0(k,l)} \bigoplus_{\Phi\in\mathsf{Q}_1(k,l)} \mathrm{Hom}\big(\ker(T_v^\dagger)\,,\underline{W}_{\Phi(v)}\big) \longrightarrow \\ \longrightarrow \bigoplus_{v\in\mathsf{Q}_0(k,l)} \bigoplus_{\rho\in\mathsf{R}(k,l)} \mathrm{Hom}\big(\ker(T_v^\dagger)\,,\underline{W}_{\rho(v)}\big) \longrightarrow \mathrm{Ext}^2\big(\underline{T}\,,\underline{W}\big) \longrightarrow 0\ . \end{aligned} \tag{6.74}$$

We define $\mathrm{Ext}^p(\underline{T}\,,\,\underline{W}\,)$ to be the p-th cohomology group of this complex. For $p = 0$, the group $\mathrm{Ext}^0(\underline{T}\,,\,\underline{W}\,) = \mathrm{Hom}(\underline{T}\,,\,\underline{W}\,)$ corresponds to the vertices $\mathsf{Q}_0(k,l)$ and classifies partial gauge symmetries of the instanton system. The group $\mathrm{Ext}^1(\underline{T}\,,\,\underline{W}\,) = \mathrm{Ext}(\underline{T}\,,\,\underline{W}\,)$ corresponds to the arrows $\mathsf{Q}_1(k,l)$ and classifies deformations of $\underline{T} \oplus \underline{W}$ which describe bound states of the constituent D-branes arising from partial gauge symmetries. The group $\mathrm{Ext}^2(\underline{T}\,,\,\underline{W}\,)$ corresponds to the relations $\mathsf{R}(k,l)$.

We now compute the relative Euler-Ringel form on the representation ring of the quiver. Using (6.74) we find

$$
\begin{aligned}
\chi\big(\underline{T}\,,\,\underline{W}\,\big) &:= \sum_{p\geq 0} (-1)^p \ \dim \mathrm{Ext}^p\big(\underline{T}\,,\,\underline{W}\,\big) \\
&= \sum_{v\in\mathsf{Q}_0(k,l)} \Bigg(\dim \mathrm{Hom}\big(\ker(T_v^\dagger)\,,\,\underline{W}_{\,v}\big) + \sum_{\rho\in\mathsf{R}(k,l)} \dim \mathrm{Hom}\big(\ker(T_v^\dagger)\,,\,\underline{W}_{\,\rho(v)}\big) \\
&\qquad\qquad - \sum_{\Phi\in\mathsf{Q}_1(k,l)} \dim \mathrm{Hom}\big(\ker(T_v^\dagger)\,,\,\underline{W}_{\,\Phi(v)}\big)\Bigg) \\
&= \sum_{v\in\mathsf{Q}_0(k,l)} N_v \left(w_v + \sum_{\rho\in\mathsf{R}(k,l)} w_{\rho(v)} - \sum_{\Phi\in\mathsf{Q}_1(k,l)} w_{\Phi(v)} \right) ,
\end{aligned}
\tag{6.75}
$$

where $w_v := \dim(\underline{W}_{\,v})$. We will define the virtual representation (6.73) such that the virtual dimensions w_v, $v \in \mathsf{Q}_0(k,l)$ obey the linear inhomogeneous recursion relations

$$
w_v + \sum_{\rho\in\mathsf{R}(k,l)} w_{\rho(v)} - \sum_{\Phi\in\mathsf{Q}_1(k,l)} w_{\Phi(v)} = \mathrm{index}(\not{D}_v)\,, \tag{6.76}
$$

together with vanishing conditions at the boundaries of the pertinent quiver. Then the character (6.75) coincides with the topological charges computed previously. Let us now turn to the explicit constructions of the modules $\underline{W}$.

Symmetric $\underline{C}^{k,l}$ quiver charges. The symmetric quiver diagram has only one holomorphic relation ρ given by (4.14), which takes a vertex $v = (n,m)$ to $\rho(v) = (n,m+6)$, and two arrows $\Phi^\pm$ taking $v = (n,m)$ to $\Phi^\pm(v) = (n\pm 1, m+3)$. We thus need to solve the four-term linear inhomogeneous recursion relation

$$
w_{n,m} + w_{n,m+6} - w_{n+1,m+3} - w_{n-1,m+3} = \mathrm{index}(\not{D}_{n,m})\,, \tag{6.77}
$$

where the index is given by (6.32). By introducing the SU(2) spin variables $i := -2j_+$ and $\alpha := 2j_-$, using (6.35) one finds that the left-hand side of (6.77) is the same as that which arises for the symmetric $\mathbb{C}P^1 \times \mathbb{C}P^1$ quiver [13]. Rewriting the solution of that case in terms of our variables, one has

$$
w_{n,m} = \sum_{n'=0}^{n-2} \ \sum_{\substack{j=0 \\ m'=-2(k-l)+3j}}^{2(j_+-j_-)} \mathrm{index}(\not{D}_{n',m'})\ . \tag{6.78}
$$

These numbers correspond to the virtual dimensions of the SU(3) representations

$$\underline{W}_{n,m} = \bigoplus_{n'=0}^{n-2} \bigoplus_{\substack{j=0 \\ m'=-2(k-l)+3j}}^{2(j_+-j_-)} \underline{\mathrm{index}}(\not{D}_{n',m'}) \ . \tag{6.79}$$

This produces a non-decreasing sequence of representations $\{\underline{W}_{n,m}\}$ as we move along the quiver of constituent D-branes, such that the G-module $\underline{W}_{n,m}$ gives extensions of the instanton and monopole fields carried by the elementary brane state at vertex $v = (n,m) \in \mathsf{Q}_0(k,l)$. Thus all in all, aside from a few details arising from the nonabelian nature of the holonomy group $H = \mathrm{SU}(2)\times\mathrm{U}(1)$ in this case, the physics of the $\mathbb{C}P^2$ quiver gauge theory is qualitatively similar to that of the $\mathbb{C}P^1 \times \mathbb{C}P^1$ quiver gauge theory.

Non-symmetric $\underline{C}^{k,l}$ quiver charges. The non-symmetric quiver diagram has several holomorphic relations. From (4.20) we obtain a linear relation $\rho \in \mathsf{R}(k,l)$ taking vertex $v = (q,m)_n$ into $\rho(v) = (q+1,m+3)_{n\pm1}$. From (4.21) we obtain the three respective quadratic holomorphic relations ρ^{+-}, $\rho^{\pm0}$ taking $v = (q,m)_n$ into $\rho^{+-}(v) = (q,m+6)_{n\pm1}$, $\rho^{+0}(v) = (q+3,m+3)_{n\pm1}$ and $\rho^{-0}(v) = (q+1,m+3)_{n\pm1}$. There are three arrows Φ^0, $\Phi^\pm$ taking v into $\Phi^0(v) = (q+2,m)_n$, $\Phi^+(v) = (q+1,m+3)_{n\pm1}$ and $\Phi^-(v) = (q-1,m+3)_{n\pm1}$. Since the index (6.53) in this case is independent of the degeneracy label n, we will suppose that the same is true of the virtual dimensions ${}^n w_{q,m} = w_{q,m}$. Then (6.76) simplifies to the six-term recursion relation

$$w_{q,m} + w_{q+1,m+3} + w_{q,m+6} + w_{q+3,m+3} - w_{q+2,m} - w_{q-1,m+3} = \mathrm{index}(\not{D}_{q,m}) \ . \tag{6.80}$$

We see therefore that the number of independent terms in (6.76) is equal to the dimension d_H of the coset space G/H.

We have not succeeded in finding an instructive and compact explicit solution to the system (6.80). Nevertheless, we may prove the existence of a unique solution as follows. Let us recall the U-spin and V-spin electric charge eigenvalues $u := Y_U = -\frac{1}{2}\,(q+\frac{m}{3})$ and $v := Y_V = \frac{1}{2}\,(q-\frac{m}{3})$ from section 5.2. Setting $w_{q(u,v),m(u,v)} =: a_{u,v}$, we may rewrite the recurrence relations (6.80) in the form

$$a_{u,v} = a_{u,v-1} + a_{u-1,v+1} - a_{u-1,v} - a_{u-1,v-1} - a_{u-2,v+1} + b_{u,v} \tag{6.81}$$

where

$$b_{u,v} = \frac{1}{2}\,(v-u)\,(u+2v)\,(2u+v) \ . \tag{6.82}$$

It is now straightforward to see that (6.81) satisfies the hypotheses of Theorem 5 in [25], from which we deduce the existence of a unique solution $a_{u,v}$ to (6.81). The sequence $a_{u,v}$ can in this way be evaluated inductively, or alternatively via the kernel method which yields some solution of the recursion relation. Several properties of this solution can be deduced from the results of [25]. It describes the moduli of the folding of branes and antibranes in this case. Note that the index (6.82) changes sign when one interchanges U-spin and V-spin electric charges.

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