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ON THE STABILITY AND DIAGNOSTICS OF HEAVY IONS IN STORAGE RINGS WITH HIGH PHASE-SPACE DENSITY

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Results for longitudinal stability, beam transfer functions and Schottky noise spectra of coasting beams with high phase-space density are presented, which may serve as a guideline for interpreting experimental observations. Such beams may be obtained by stacking injection from a high-brightness linear accelerator (as anticipated in heavy ion fusion) or by electron cooling. The main feature of heavy-ion beams close to the threshold of stability is a noticeable collective effect, which is related to the high space-charge impedance. Schottky signal spectra can be strongly deformed into a double-peak shape as a result of the two coherent frequencies from the forward- and backward-running beam plasma waves. We particularly emphasize situations where the effective momentum spread is much smaller than predicted by the standard stability criteria. This may either be due to the stabilizing effect of a tail in the momentum distribution or the stabilizing friction force from electron cooling. In the former case we have found a strong asymmetry of the two-bump shape of the Schottky spectrum, which could be a measure for the tail population. A moderately strong friction force also extends the stability region. For very strong cooling, it is possible to restore the Schottky spectrum to a profile like in the low intensity case. Numerical examples are either in dimensionless quantities or applied to U^{92+} at 550 MeV/u, which is a standard case of an ion in the ESR at GSI Darmstadt. It is straightforward to transfer results to other types of ions or energies.

1. INTRODUCTION

The interest in heavy-ion beams of high phase-space density has largely been stimulated by their potential use for heavy-ion inertial fusion (HIF) and recently by several projects of ion storage rings with electron cooling. In HIF it is essential to accumulate a large number of particles and to understand the maximum stable current for a given momentum spread. In contrast with high-energy proton machines, the threshold momentum spread for the microwave instability is determined by the large imaginary part of the coupling impedance due to space charge, rather than by the resistive impedance. In existing measurements at Novosibirsk,[1] CERN[2] and Heidelberg,[3] the effect of this large space-charge coupling impedance has already been verified by a two-bump profile in the Schottky spectrum for a Gaussian-like momentum distribution. The practical consequence for HIF would be an undesirable large momentum spread, unless a "stabilizing tail" in the momentum distribution could help.[4] Such a tail would lead to Landau damping at the coherent frequency of the slow (resistively unstable) wave. This would have an effect on the Schottky signal spectrum and thus on the calculation of a momentum distribution from the measured Schottky spectrum.

Another interesting behavior is expected for a sufficiently low number of particles in the ring, with electron cooling turned on. It is possible that the electron cooling rate directly competes with the microwave instability growth rate and creates stable, cold equilibrium distributions. It is even conceivable that the cooling force suppresses the coherent noise fluctuations, which, however, requires a much stronger cooling force (i.e., cooling e-folding times on the order of the plasma oscillation time rather than the much slower microwave instability e-folding time).

Ideally, if intra-beam scattering is not dominant,[5] one might thus achieve narrow momentum distributions, which are quite different in shape from the observed Schottky spectra. It is thus necessary to incorporate into the calculation of Schottky spectra the collective response of the beam. This is performed in analogy to the dielectric-function approach, well-known in plasma physics,[6] which has been applied more recently to beams. The pioneering work in this respect is by the Novosibirsk group, and the reader is referred to Ref. 1. For a formal derivation of the collective effects on Schottky signals (without cooling) we also refer to Ref. 7.

The present study is oriented towards practical application and in particular the effect of electron cooling. In Section 2 the analytical framework is outlined, following the dielectric approach, along with a simplified collisional term in Vlasov's equation. In Section 3 we present numerical results for beams with stabilizing tails (relevant to HIF), then for equilibria determined by electron cooling, and finally for strong cooling forces and their effect on the noise spectrum. The latter case must depend on the simplifications in the collision term and should therefore be seen only as a qualitative demonstration of the effect of very strong cooling forces.

2. ANALYTICAL FRAMEWORK OF BEAM RESPONSE

In our model, the beam is a collisionless ensemble of particles interacting via the self-consistent electric field, which justifies the use of Vlasov's equation. In addition we need to consider a source term, which is either the rf signal imposed on the beam by a kicker (beam transfer function method), or the noise signal due to the statistical nature of the microscopic particle distribution (Schottky diagnostics). In either case we calculate the response of the particle ensemble on the source term and determine the self-consistent electric field generated by it, which contains the desired information on the distribution function. This will be described in more detail in the next section and in the Appendix.

We consider a beam of heavy ions of charge state q and atomic number A, and start from Vlasov's equation, which we write in the convenient form:

$$\frac{\partial\Psi}{\partial t}+\dot{\tau}\frac{\partial\Psi}{\partial\tau}+\ddot{\tau}\frac{\partial\Psi}{\partial\dot{\tau}}=0 \tag{1}$$

where τ is the time delay of a particle with respect to the reference particle at a position in the ring (see Appendix A) and $\Psi(\tau, \dot{\tau}, t)$ is the distribution function in phase space. Replacing the force term by the electric field and linearizing Vlasov's equation, we obtain for a Fourier component of the perturbed distribution function with frequency Ω and angular harmonic p (see Appendix B):

$$\tilde{f}_p(\dot{\tau}, \Omega) = -i \frac{\eta q c \omega_0}{\gamma_0 \beta_0 A m_0 c^2/e} \tilde{E}_{\|p}(\Omega) \frac{\partial \Psi_0(\omega)/\partial \omega}{\Omega - p\omega} \tag{2}$$

where $\Psi_0(\omega)$ is the unperturbed distribution function of revolution frequencies and $m_0 c^2/e = 931.5\,\mathrm{MeV}/u$.

The perturbed electric field $\tilde{E}_{\|p}$ has two origins. We must distinguish the source component $\tilde{E}^{\mathrm{source}}_{\|p}$, which is either the applied field from a kicker or the field from the statistical current fluctuations, and the field $\tilde{E}^{\mathrm{coll}}_{\|p}$ induced by the beam response $\tilde{I}^{\mathrm{coll}}$. The latter is related to the current modulation via the impedance, which then replaces Maxwell's equations:[8]

$$\begin{aligned} \tilde{E}^{\mathrm{coll}}_{\|p}(\Omega, \theta) &= -\frac{1}{2\pi R} Z_{\|} \tilde{I}^{\mathrm{coll}}(\Omega, \theta) \\ &= -\frac{1}{2\pi R} Z_{\|} \frac{2\pi}{\omega_0} I \int \tilde{f}_p(\dot{\tau}, \Omega)\, d\dot{\tau} e^{i(\Omega t - p\theta)}. \end{aligned} \tag{3}$$

The task is to connect these current modulations with their source terms. In both cases the beam responds on the presence of these source terms with a current modulation, which causes a self-electric field. The problem has to be made self-consistent in that the source field plus the self-electric field determine the current modulation. This is performed by calculating from Eq. (2) the current perturbation (via the appropriate integration over phase space), where one must keep in mind that the electric field perturbation is also a function of the current perturbation. Hence, the beam represents a feedback loop[7] as shown in Fig. 1. It is a familiar concept in plasma physics to express the solution of Vlasov's equation by means of the dielectric function ε, which in turn contains all the information about the beam feedback loop. The result can be written (see

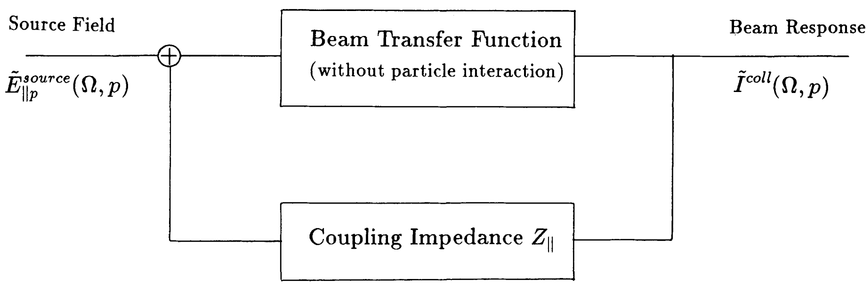

FIGURE 1 Analogy between the beam response to an electric field perturbation $E_{\|p}$ and a feedback loop.

Appendix C) as:

$$\tilde{I}^{\text{coll}} = -i\frac{\eta q I}{\gamma_0\beta_0^2 A m_0 c^2/e}\int\frac{\partial\Psi_0/\partial\omega}{\Omega - p\omega}d\omega\, 2\pi R\tilde{E}_{\parallel p}^{\text{source}} \times \frac{1}{\varepsilon(\Omega, p)}, \tag{4}$$

where the dielectric function is defined as

$$\varepsilon(\Omega, p) = 1 - i\frac{\eta q I}{\gamma_0\beta_0^2 A m_0 c^2/e} Z_{\parallel}\int\frac{\partial\Psi_0/\partial\omega}{\Omega - p\omega}d\omega. \tag{5}$$

For convenience the complex path integral (along the "Landau contour") has been split into a principal value and a residue in Eq. (4)

$$\int\frac{\partial\Psi_0/\partial\omega}{\Omega - p\omega}d\omega = PP\int\frac{\partial\Psi_0/\partial\omega}{\Omega - p\omega}d\omega + i\frac{\pi}{p}\frac{\partial\Psi_0}{\partial\omega}\bigg|_{\omega=\Omega/p} \tag{6}$$

It should be noted here that the condition

$$\varepsilon(\Omega, p) = 0 \tag{7}$$

is the dispersion relation for electrostatic longitudinal modes. For given harmonic p it yields the frequency at which a mode can exist without an external source, i.e., an eigenmode. From Eq. (5), it is obvious that for vanishing impedance or current the effect of polarization of the medium is negligible and we have $\varepsilon = 1$.

2.1. *Beam Transfer Function*

The response is often defined as ratio of the induced beam current modulation over the externally applied voltage $V = 2\pi RE$ (R being the ring radius). By dropping the phase factor we thus obtain a normalized response (see Appendix C.1):

$$r_{\parallel p}(\Omega, p) = -i\frac{\eta q I}{\gamma_0\beta_0^2 A m_0 c^2/e}\int\frac{\partial\Psi_0/\partial\omega}{\Omega - p\omega}d\omega \times \frac{1}{\varepsilon(\Omega, p)}. \tag{8}$$

The beam transfer function is defined as the inverse response[9] according to:

$$\frac{1}{r_{\parallel p}} = i\frac{\gamma_0\beta_0^2 A m_0 c^2/e}{\eta q I}\left(\int\frac{\partial\Psi_0/\partial\omega}{\Omega - p\omega}d\omega\right)^{-1} + Z_{\parallel}, \tag{9}$$

where the first term yields the stability diagram and the second term is the coupling impedance, which gives rise to a shift of the stability curve in the complex plane. The measurement of the transfer function thus allows one, in principle, to determine the impedance if one succeeds in determining the shift of the stability curve. In practice this might be difficult since the location of the origin depends on the distribution function.

If the coupling impedance is determinated, the distribution function can then be calculated from the first term of Eq. (9)—that is to say from the residue of the complex integral of Eq. (6).

2.2. Schottky Noise

Here we employ the same formalism as before, the only difference being that the source term is the current fluctuation of the beam due to the random population in phase space. The fluctuation causes the "incoherent" Schottky noise of low-intensity beams, which results in the well-known expression for the power spectrum of current fluctuations:[10]

$$P_I(\Omega, p) = 2\left(\frac{qe\omega_0}{2\pi}\right)^2 \frac{N}{p}\frac{2\pi}{\omega_0^2}\Psi_0(\Omega/p). \tag{10}$$

The measured spectrum is thus directly proportional to the distribution function. In each Schottky band ($p = 1, 2, \ldots$) the total power is constant, with the amplitude proportional to $1/p$ and the width to p.

For high-intensity beams it is essential to take into account also the self-consistent electric field, which arises as a coherent response to the field from the random fluctuations. This response is very pronounced at frequencies corresponding to the eigenfrequencies of the system. The measured spectrum is then a superposition of the incoherent and the coherent part. With the above formalism one readily finds that the dielectric function gives the total spectrum as explained in Appendix C.2:

$$P_I(\Omega, p) = 2\left(\frac{qe\omega_0}{2\pi}\right)^2 \frac{N}{p}\frac{2\pi}{\omega_0^2}\frac{\Psi_0(\Omega/p)}{|\varepsilon(\Omega, p)|^2}. \tag{11}$$

For $\varepsilon = 1$ we retain the low-intensity result as in Eq. (10), whereas the spectrum is expected to be strongly deformed in the neighbourhood of eigenfrequencies given by Eq. (7).

2.3. Modification with Cooling Force

Electron cooling of the ions can have a twofold effect:

1. The equilibrium distribution is cooled to high phase space density, which is reflected in the above formalism by a changed distribution function.
2. Fluctuations and coherent instabilities are smoothed by collisions with the electrons, which also changes the functional relation between distribution function and response.

We have assumed a relatively simple *ansatz* to dynamically take into account the friction effect of the electron beam on the ions. This is by assuming a "collisional relaxation" towards a given equilibrium distribution, which requires a right-hand-side term in the otherwise collisionless Vlasov's equation. We adopt a modified Boltzmann equation known as the "Krook model":[6]

$$\frac{\partial\Psi}{\partial t} + \dot{\tau}\frac{\partial\Psi}{\partial\tau} + \ddot{\tau}\frac{\partial\Psi}{\partial\dot{\tau}} = -\nu\left(\Psi - \frac{n(\tau, t)}{n_0(\tau)}\Psi_0\right), \tag{12}$$

with

$$n(\tau, t) = \int \Psi(\tau, \dot{\tau}, t)\, d\dot{\tau}. \tag{13}$$

We justify this term by the following considerations:

1. This equation assumes a small deviation from a known equilibrium state Ψ_0. We have not attempted to calculate the equilibrium distribution from the Fokker–Planck equation. This requires consideration of intrabeam scattering in addition to electron cooling, which is beyond an analytical approach. The modified Boltzmann equation thus describes the effect of the cooling force on collective motions, whereas it cannot take into account its influence on the equilibrium distribution.
2. Integrating Eq. (12) over ion velocities leads to:

$$\frac{\partial(V - V_0)}{\partial t} = -\nu(V - V_0). \tag{14}$$

   This linear friction force is appropriate for an ion velocity spread that is small compared with the longitudinal electron velocity spread.[11]
3. Though Eq. (12) is usually applied for collisions between neutral and charged particles, we can use it because the time the electrons pass through the cooling section (~10 ns) is so small that they are not coupling with the collective motion of the ions (with plasma period exceeding ~1 μs). Hence, their role is similar to neutral particles with no collective response (ignoring the Debye shielding of the ions by the electrons).

The collisional relaxation rate ν has the advantage of entering in a rather simple way into the expressions for the beam transfer function, hence also the stability diagram, according to:[12]

$$\tilde{I}^{\text{coll}} = -i \frac{\eta q I}{\gamma_0 \beta_0^2 A m_0 c^2/e} \int \frac{\partial\Psi_0/\partial\omega}{\Omega - p\omega - i\nu}\, d\omega\, 2\pi R \tilde{E}_{\parallel p}^{\text{source}} \times \frac{1}{\varepsilon(\Omega, p, \nu)}, \tag{15}$$

with ε now given by:

$$\begin{aligned}\varepsilon(\Omega, p, \nu) &= 1 - i \frac{\eta q I}{\gamma_0 \beta_0^2 A m_0 c^2/e} Z_\parallel \int \frac{\partial\Psi_0/\partial\omega}{\Omega - p\omega - i\nu}\, d\omega + i\nu \frac{2\pi}{\omega_0^2} \int \frac{\Psi_0(\Omega)}{\Omega - p\omega - i\nu}\, d\omega \\ &= 1 - i \frac{\eta q I}{\gamma_0 \beta_0^2 A m_0 c^2/e} Z_\parallel J(\Omega, p, \nu) + i\nu \frac{2\pi}{\omega_0^2} I(\Omega, p, \nu).\end{aligned} \tag{16}$$

The Schottky power spectrum is modified in the following way:

$$P_I(\Omega, p) = 2\left(\frac{qe\omega_0}{2\pi}\right)^2 \frac{N}{p} \frac{2\pi}{\omega_0^2} \left| \frac{1 + i\nu 2\pi/\omega_0^2\, I(\Omega, p, \nu)}{\varepsilon(\Omega, p, \nu)} \right|^2 \Psi_0(\Omega/p). \tag{17}$$

TABLE I

Typical beam parameters used for numerical calculations

ions	$^{238}U^{92+}$	
energy	556	MeV/u
transition energy γ_t	2.66	
momentum compaction η	−0.244	
number of particles	10^{10}	
average beam radius a_0	0.5	cm
average vacuum chamber radius b_0	6	cm

2.4. Determination of Distribution Function and Impedance (No Cooling Force)

For low-intensity beams far from the stability threshold, the momentum distribution follows directly from the measured Schottky spectrum. This measured distribution can be compared with the momentum distribution from the beam transfer function. This is done by trying different origins in the stability diagram of Eq. (9) until the distribution computed from Eqs. (9) and (6) is close to the measured one. The resulting shift of origin gives the impedance directly.

For cooled beams, the measured Schottky spectrum is also distorted due to coherent response, which leads to a dielectric function different from unity. Since the dielectric function in Eq. (11) is related to the beam transfer function according to Eqs. (5) and (8), we can again, in principle, determine the momentum distribution and impedance from a combined evaluation of the beam transfer function and the Schottky spectrum. Different impedances are tried out until the distribution function computed from Eqs. (9) and (6) is in close agreement with the distribution function computed from Eq. (11) by using ε according to Eqs. (5) and (8).

3. NUMERICAL RESULTS

We present here numerical results of the longitudinal beam transfer function and the Schottky spectrum for a Gaussian beam. We thus consider the normalized distribution function:

$$\Psi_0(\omega) = \frac{\omega_0^2}{2\pi} \frac{1}{\sqrt{\pi}\,\delta\omega} e^{-(\omega-\omega_0)^2/\delta\omega^2}. \tag{18}$$

We adopt here the following notation: $\delta\omega$ is the half width at half maximum and $\Delta\omega$ (or $\Delta P/P$) the full width at half maximum.

Examples are applied to a uranium beam circulating in the ESR[13,14] with the parameters summarized in Table I.

For the ESR, the assumed components of the coupling impedance are:

- space charge impedance:[15]

$$\operatorname{Im}\frac{Z_{\parallel}}{p} = -i\frac{Z_0}{2\beta_0\gamma_0^2}\left(1+2\ln\frac{b_0}{a_0}\right) = -i600\,\Omega,$$

  where $Z_0 = \mu_0 c = 377\Omega$.
- Resistive broadband impedance:

$$\operatorname{Re}\frac{Z_{\parallel}}{p} \simeq 10\Omega.$$

Note that for nonrelativistic energy, the space-charge contribution to the impedance is by far the dominant one.

3.1. *No Cooling Force*

3.1.1. *Beam Transfer Function and Stability Threshold*

The amplitude and phase of the normalized longitudinal transfer function for a beam of low space charge density are shown in Fig. 2. Particles are not correlated and we assume $\varepsilon = 1$. The beam response has the shape of the distribution

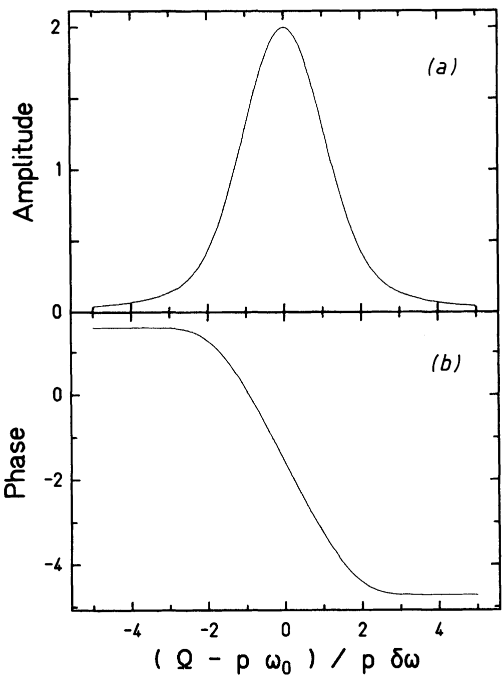

FIGURE 2 Amplitude (a) and phase (b) of the longitudinal beam transfer function for a Gaussian beam.

function and is centered at $\Omega = p\omega_0$. The phase jumps by 2π as the external frequency varies from $p(\omega_0 - \delta\omega)$ to $p(\omega_0 + \delta\omega)$. The real part of the transfer function provides a direct measurement of the distribution function.

The response of a cooled beam is represented in Fig. 3. Corresponding working points[16] in normalized units,

$$U + iV = -\frac{Z_{\parallel}}{p}\frac{2}{\pi}\frac{q}{Am_0c^2/e}\frac{I}{\eta\gamma_0\beta_0^2(\Delta P/P)^2}, \tag{19}$$

are represented in Fig. 4.

The amplitude of the beam response in Fig. 3 is normalized to:

$$\frac{2qI}{(p\pi Am_0c^2/e\gamma_0\beta_0^2(\Delta P/P)^2)}. \tag{20}$$

On the abscissa, we have used the normalized frequency

$$\frac{\Omega - p\omega_0}{p(\delta\omega)_{V=-1}}, \tag{21}$$

where $(\delta\omega)_{V=-1}$ is the reference frequency spread corresponding to the working point $V = -1$. In our case, this scaling factor is

$$(\delta\omega)_{V=-1} = \frac{1}{2}|\eta|\,\omega_0\left(\frac{\Delta P}{P}\right)_{V=-1} = 730\,\text{Hz}. \tag{22}$$

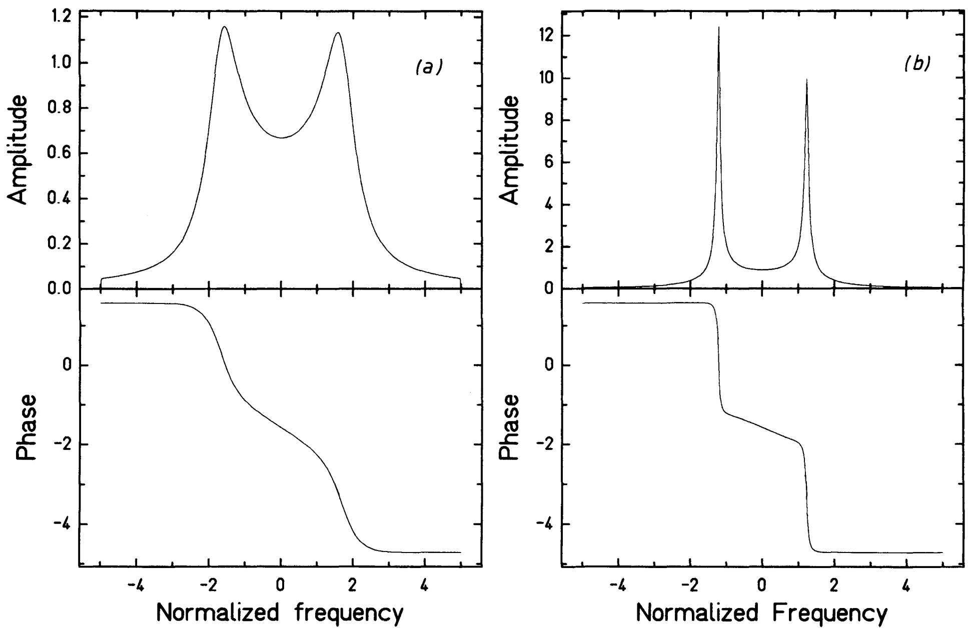

FIGURE 3 Amplitude and phase of the transfer function of a cooled beam. (a) $\Delta P/P = 4.4\,10^{-4}$ (b) $\Delta P/P = 2.2\,10^{-4}$. Corresponding working points are shown in Fig. 4. The normalized frequency is the same for the two curves and is given in Eq. (21). One division represents $730 \times p$ Hz.

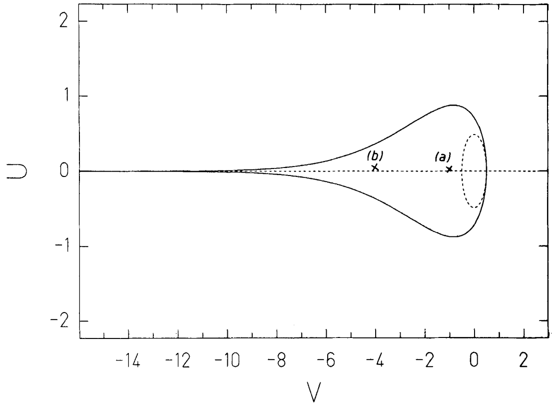

FIGURE 4 Longitudinal stability diagram of a Gaussian beam. The dashed curve represents the Keil–Schnell limit. (a) $(U, V) = (0.01, -1)$, (b) $(U, V) = (0.04, -4)$.

The two curves are centered in $\Omega = p\omega_0$ and one division represents $730 \times p$ Hz.

Two bumps appear in the transfer function. They correspond to the excitation of the slow and fast wave at the coherent beam plasma frequency $\Omega^{\pm}$ by the external electric field. Neglecting momentum spread and assuming that $\operatorname{Im} Z_{\|}/p \gg \operatorname{Re} Z_{\|}/p$, the coherent frequency follows readily from Eq. (5)

$$\Omega^{\pm} = p\omega_0 \pm p\omega_0 \left|\frac{I\eta q}{2\pi A m_0 c^2/e\gamma_0\beta_0^2}\right|^{1/2} \left|\operatorname{Im}\frac{Z_{\|}}{p}\right|^{1/2}. \tag{23}$$

The phase varies by π when the external frequency crosses a peak. This is similar to the response of an oscillator to an external force.

When the beam momentum spread decreases, the amplitude of the maxima increases and their spread shrinks, because the beam is close to the stability threshold. This limit is shown in Fig. 4 for the assumed Gaussian momentum distribution. For completeness we have also plotted the conventional Keil–Schnell limit, which is a circle $|U + iV| = 0.5$. Only the slow wave can give rise to instability, and its response is slightly higher than the one corresponding to the fast wave, which is due to the small resistive impedance.

3.1.2. Schottky Spectrum

Figure 5 shows variations of the band p of the Schottky spectrum for decreasing momentum width.

The Schottky current power is normalized to

$$2(qe\omega_0/2\pi)^2 N/\sqrt{\pi}. \tag{24}$$

The normalized frequency is the same as in Eq. (21).

For large momentum spread (curve a), particle interaction is negligible and the power spectrum represents the distribution function of width $p\delta\omega$. For reduced momentum spread, two peaks appear in the Schottky spectrum corresponding to

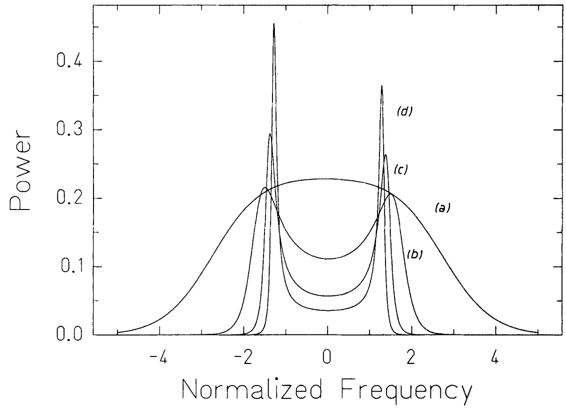

FIGURE 5 Schottky spectrum of a cooled beam. (a) $\Delta P/P = 9.1\,10^{-4}$, (b) $4.3\,10^{-4}$, (c) $3.0\,10^{-4}$, (d) $2.5\,10^{-4}$. Corresponding working points are: (a) $(U, V) = (0.002, -0.2)$, (b) $(0.01, -1)$, (c) $(0.02, -2)$, (d) $(0.03, -3)$.

the excitation of the slow and fast waves by the incoherent beam current fluctuations.

The total power of the considered band shrinks with decreasing momentum spread. Screening of density charge fluctuations is due to the well-known polarisation effect in a plasma.[6] As in dielectrics, if the beam is subject to an electric field perturbation (in this case provided by the beam itself), displacement of ions creates a field in the direction opposite to the perturbing electric field and shields it.

An analytical estimate for the power can be found if the working point lies outside the Keil–Schnell limit. Neglecting the real part of the coupling impedance, the dielectric constant can be approximated by:

$$\varepsilon = 1 - \frac{\Delta\Omega^2}{(\Omega - p\omega_0)^2} - i2\sqrt{\pi}\frac{\Delta\Omega^2}{(p\omega_0)^3}(\Omega - p\omega_0)e^{-(\Omega - p\omega_0)^2/p^2\delta\omega^2}. \tag{25}$$

From Eq. (11), the height and width of the peaks can be approximated by:

$$\begin{aligned} P^p_{\max} &= 2\left(\frac{qe\omega_0}{2\pi}\right)^2 \frac{N}{p}\frac{1}{\sqrt{\pi}\,\delta\omega} e^{\Delta\Omega^2/p^2\delta\omega^2} \frac{1}{[2\sqrt{\pi}\,(\Delta\Omega/p\delta\omega)^3]^2}, \\ \delta\Omega &= \sqrt{\pi}\,\Delta\Omega\left(\frac{\Delta\Omega}{p\delta\omega}\right)^3 e^{-\Delta\Omega^2/p^2\delta\omega^2} \end{aligned} \tag{26}$$

The total power per band is then written as

$$P^p_{\text{tot}} \simeq \frac{2}{\pi}\left(\frac{qe\omega_0}{2\pi}\right)^2 \frac{4\pi^2 A m_0 c^2/e^2\gamma_0\beta_0^2}{\eta q^2\omega_0^3 \operatorname{Im} Z_{\parallel}/p}\,\delta\omega^2, \tag{27}$$

which decreases with momentum spread and is independent of the number of particles. This "signal suppression" phenomenon has been observed at the Novosibirsk NAP ring.[1]

3.1.3. Determination of the Distribution Function

In Fig. 6 we show the first step of the iterative procedure indicated in Section 2.4 to determine the impedance and momentum distribution. We have recalculated the distribution function on one hand from Eq. (9) and on the other hand from Eq. (11), along with Eq. (5), by inserting a guessed impedance, which is 20% larger than the correct impedance. It is seen that for a working point sufficiently far from the stability limit ($V = -0.5$) in particular the Ψ_0 calculated from the beam transfer function is quite close to the correct Ψ_0. Closer to the stability limit ($V = -3$) the deviation is more pronounced. The half-width of the Schottky spectrum evaluation comes out much smaller than the correct one, whereas the beam transfer function evaluation lies between (note that the curves (a), (b), (c) in Fig. 6 have been normalized to equal area corresponding to given intensity). The correct distribution and impedance can be then approached by iterating the impedance until the two curves agree.

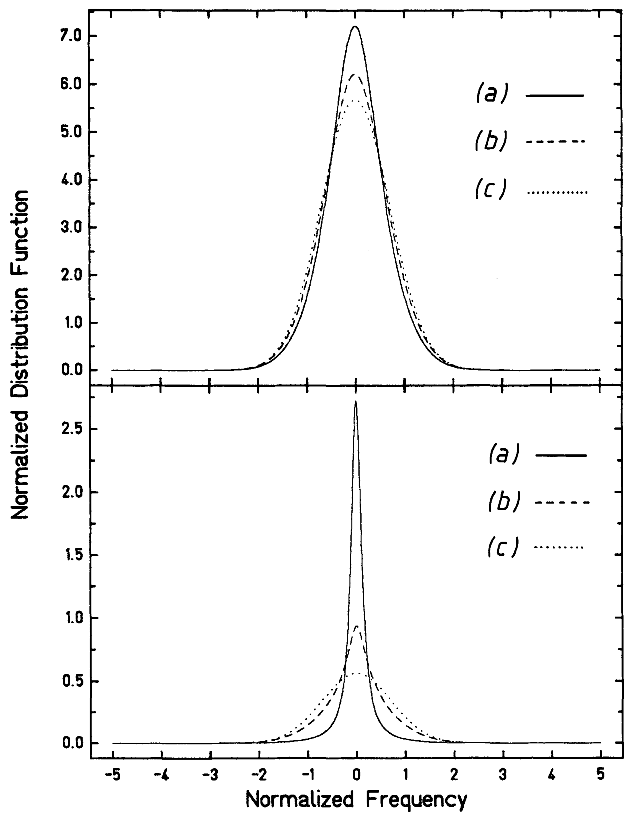

FIGURE 6 Calculation of the distribution function of a cooled beam from the beam transfer function and the Schottky spectrum. The guessed coupling impedance is 20% larger than the correct impedance. (1) $(U, V) = (0.01, -0.5)$ (2) $(U, V) = (0.03, -3)$. (a) (full line) Distribution function computed from the Schottky spectrum, (b) (dashed line) Distribution function computed from the transfer function, (c) (dotted line) True distribution function.

3.2. *Stabilizing Tail*

3.2.1. *Stability Diagram*

It has been shown by computer simulation that a favorable self-stabilizing effect might occur for small momentum width, which is initially subject to the longitudinal microwave instability.[4] According to these calculations the main part of the beam remains at the small momentum width, whereas a small fraction of it develops into a tail extending towards smaller momenta. The fraction of particles in the tail is small only if $|U| \ll |V|$, as is the case for a space-charge-dominated impedance with a small resistive component. The overlap of the tail with the coherent frequency of the unstable slow wave provides the Landau damping necessary for stabilization. If such a tail distribution is confirmed by the experiment, it should manifest itself in the Schottky spectrum and the transfer function.

The stability diagram for a tail distribution is shown in Fig. 7, where the ratio of widths of the tail and the main distribution is 5. The presence of the tail opens the stability limit in the plane $U > 0$. The new stability limit depends on the tail population. With parameters of Table 1, we can see that a beam of momentum spread $\Delta P/P = 6.2\,10^{-5}$, which corresponds to the working point $(U, V) = (1, -50)$, is stabilized if ~2% of the particles compose the tail.

3.2.2. *Schottky Spectrum with Stabilizing Tail*

The overlap of the tail with the slow-wave coherent frequency leads to a strong peak in the Schottky spectrum at precisely this frequency, as shown in Fig. 8. The corresponding peak at the fast-wave coherent frequency disappears with decreasing impulse spread, due to the vanishing number of particles producing noise. It is practically zero for $|V|$ as large as 50, as can be seen in Fig. 8b. A comparison of measured Schottky spectra with calculated ones can be used to determine whether the beam has developed a tail distribution.

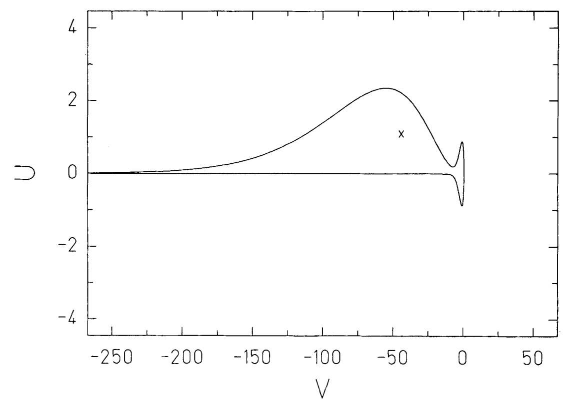

FIGURE 7 Stability diagram of a Gaussian beam with a Gaussian tail towards lower momenta. 2% of the particles compose the tail with half width 5 times that of the main distribution.

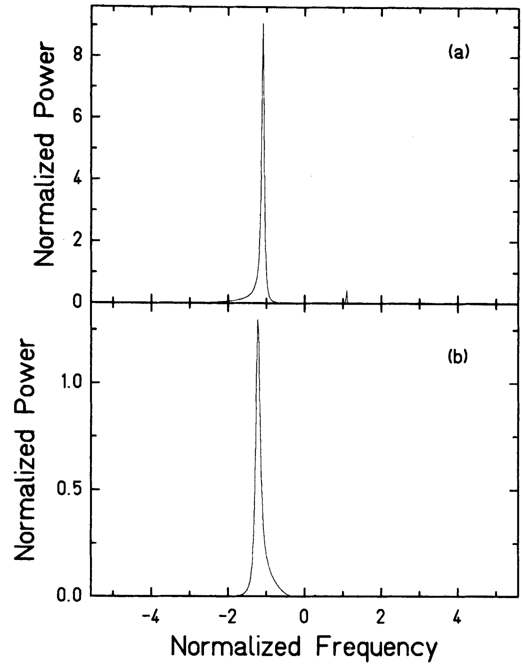

FIGURE 8 Schottky spectrum of a beam with a Gaussian tail towards lower momenta (5 times as broad as main distribution). (a) $\Delta P/P = 1.5\,10^{-4}$, $(U, V) = (0.08, -8)$, (b) $\Delta P/P = 6.2\,10^{-5}$, $(U, V) = (1, -50)$. The intensity is unchanged in both cases.

3.3. Effect of the Cooling Force

3.3.1. Stabilizing effect of strong electron cooling

It is hoped that, with electron cooling, $\Delta P/P$ can become 10^{-5} or less. For the parameters of Table I and a Gaussian momentum distribution, the beam would be subject to the microwave instability. In this section we discuss whether electron cooling can provide a direct dynamic stabilization as an alternative to the previously discussed stabilizing tail.

The variation of the stability limit with growing normalized cooling rate is shown in Fig. 9. The effect of the cooling is to open the stability diagram, which permits the working point to lie outside the stability limits as defined in the absence of cooling. Note that $v/p\delta\omega = 0.1$ is equivalent, in the example of Table I, to a cooling e-folding time of 0.3 ms for $p = 100$ and $\Delta P/P = 2.2\,10^{-4}$. Varying the cooling rate v until the working point coincides with the stability curve, we find that the collective motion is damped if the e-folding time is less than half the growth time:

$$\tau_{\text{cool}} \leq 0.5\ \tau_{\text{inst}}. \tag{28}$$

An expression for the growth rate can be given if $\Delta P/P$ is well below the

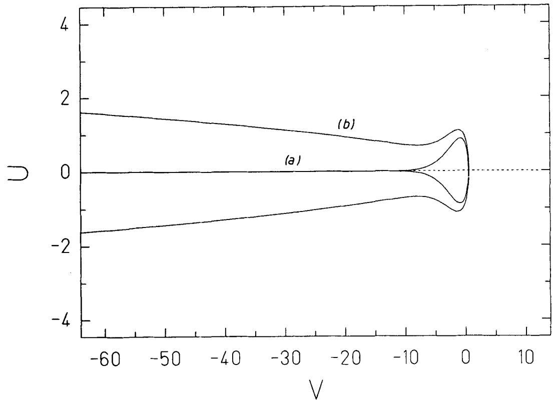

FIGURE 9 Variations of the stability diagram with the cooling rate. (a) $v/p\delta\omega = 0$, (b) $v/p\delta\omega = 0.1$.

threshold value:[16]

$$v_{\text{inst}} = \frac{1}{\tau_{\text{inst}}} = p\omega_0 \left| \frac{I\eta q}{2\pi A m_0 c^2 / e\gamma_0\beta_0^2} \right|^{1/2} \frac{\operatorname{Re} Z_{\|}/p}{2\,|\operatorname{Im} Z_{\|}/p|^{1/2}}. \tag{29}$$

With the parameters of Table I and a broad bandresistive impedance of 10 Ω, the growth time of the instability is:

$$p\tau_{\text{inst}} \simeq 0.2 \text{ s}. \tag{30}$$

The growth time decreases with increasing p, and we require, for stabilization at $p > 100$, a cooling time below 1 ms. Such a cooling time may be difficult to achieve in practice, although it is within the limits of theoretical estimates.[17] However, in view of the scaling $v_{\text{inst}} \sim N^{1/2}$, per Eq. (29), we assume that there is generally a critical intensity below which this stabilization can occur.

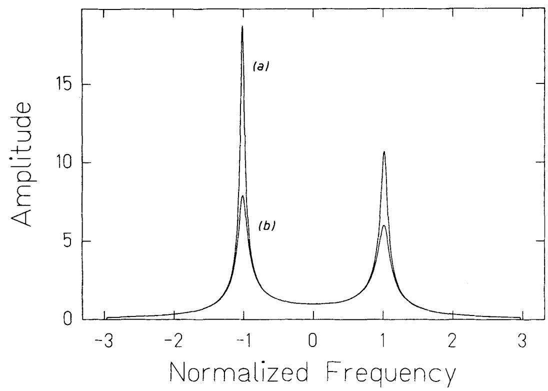

FIGURE 10 Variations of the beam transfer function with growing cooling rate for an amplitude spread $\Delta P/P = 6.2\,10^{-5}$. This corresponds to the working point $(U, V) = (1, -50)$. (a) $v/p\delta\omega = 0.5$, (b) $v/p\delta\omega = 1$.

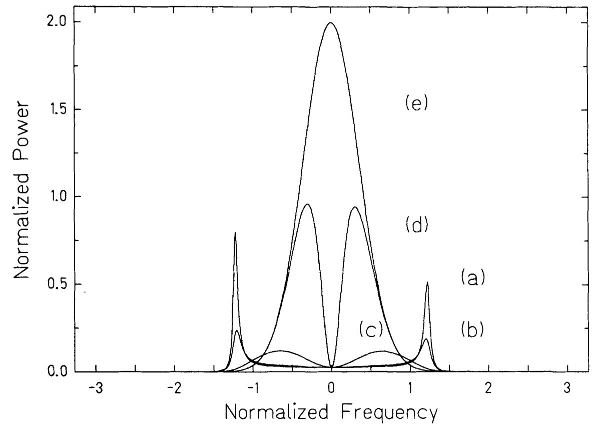

FIGURE 11 Variations of the Schottky spectrum with growing cooling rate for a momentum spread $\Delta P/P = 2.1\,10^{-4}$. This corresponds to the working point $(U, V) = (0.04, -4)$. (a) $v/p\delta\omega = 0$, (b) $v/p\delta\omega = 0.1$, (c) $v/p\delta\omega = 1.$, (d) $v/p\delta\omega = 10$, (e) $v/p\delta\omega = 100$.

3.3.2. Beam Transfer Function

Using Eqs. (15) and (16) we have plotted the beam response with growing cooling rate (Fig. 10). The considered working point $(U, V) = (1, -50)$ lies outside the stability limit in the absence of cooling. The transfer function has the two-bump form again. With increasing cooling force the amplitude of the beam response decreases. For an extremely strong cooling force, transfer-function measurement becomes difficult because of this damping response.

3.3.3. Schottky Spectrum

Variations of the Schottky spectrum with the cooling force are shown in Fig. 11. Without cooling, the spectrum is distorted by the slow and the fast wave. The cooling force damps the collective response. If the cooling rate is sufficiently large, the coherent part of the signal is completely suppressed. Thus, for such a

TABLE II

Cooling times corresponding to the working point $(U, V) = 0.04, -4$ of Fig. 11

Figure	$\Delta P/P$	N	$v/p\delta\omega$	τ_{cool} $(p = 10)$ ms
11.(b)	$2.2\,10^{-4}$	10^{10}	0.1	3
	$2.2\,10^{-5}$	10^{8}		30
11.(c)	$2.2\,10^{-4}$	10^{10}	1	0.3
	$2.2\,10^{-5}$	10^{8}		3
11.(d)	$2.2\,10^{-4}$	10^{10}	10	0.03
	$2.2\,10^{-5}$	10^{8}		0.3
11.(e)	$2.2\,10^{-4}$	10^{10}	100	0.003
	$2.2\,10^{-5}$	10^{8}		0.03

strong cooling force, only the incoherent signal is measured, and the Schottky spectrum has the shape of the distribution function as in the low-intensity case.

Cooling times needed to suppress the collective response depend on the parameter $v/p\delta\omega$ and vary with the number of particles as summarized in Table II. For a given working point, the cooling time has the same $N^{-1/2}$ dependence as the growth time of instability. Obviously, for a given cooling rate, the effect on the Schottky signal is enhanced by choosing a small p.

4. CONCLUSION

We have calculated the beam transfer function and the Schottky spectrum for cooled beams near the stability threshold. Signals are concentrated into two peaks, corresponding to the excitation of the slow and fast coherent plasma waves by the external electric field or by beam current fluctuations. Because of the non-zero real part of the coupling impedance, the response of the slow wave is generally higher than the response of the fast wave. From the deformed beam transfer function, it is not straightforward to extract the distribution function, which is largely influenced by the assumption on the coupling impedance. As the Schottky spectrum is also modified by the coupling impedance, it provides useful supplementary information.

For cooled beams outside the stability threshold, and in the case where space-charge coupling impedance is dominant, a tail can develop at low energy. This has a stabilizing effect, thus providing the necessary Landau damping for the slow plasma wave. This tail manifests itself in the Schottky spectrum and leads to a strong peak at the coherent slow-wave frequency. For a beam far outside the conventional Keil-Schnell limit, only this peak can be measured. The Schottky spectrum can be used as a tool to prove the presence of a tail in further measurements. Electron cooling also damps collective beam motion and stabilizes beams, which otherwise would be unstable. Introducing a simple collision term in the Vlasov equation we have seen that stabilization occurs if the cooling time is less than half the growth time of instability. Beam diagnostics with the transfer function becomes noticeably modified for very strong cooling forces because the beam response is damped. Variations of the Schottky spectrum with increasing cooling force have shown that the suppression of collective beam response transforms the Schottky spectrum to the same shape as in the low-intensity case.

REFERENCES

1. V. Parkhomchuk and D. V. Pestrikov, *Sov. Phys. Tech. Phys.*, **25**(7), July 1980.
2. H. Poth *et al.*, *Z. Phys.* **A332,** 1989, p. 171.
3. M. Steck *et al.*, Proc. 5th Int. Conf. On Electrostatic Acc. and Associated Boosters, Strasbourg–Heidelberg, 1989, to be published in *Nucl. Instr. Meth.*
4. I. Hofmann, *Laser and Part. Beams* (1985), Vol. 3, part. 1, p. 1–8.
5. I. Hofmann, Proc. of 3rd LEAR Workshop, Tignes, 1985, p. 159.

6. Krall, Triverpiece, Principle of Plasma Physic, McGraw-Hill, 1973.
7. S. Chattopadhyay, CERN 84–11, 8 October 1984.
8. L. Palumbo and V. G. Vaccaro, CERN 87-3, 21 April 1987, p. 341.
9. J. Borer and G. Guignard, *IEEE Transactions on Nuclear Science,* Vol. NS-26, No. 3, June 1979.
10. J. Borer, *IX Conf. Hi. En. Acc.,* Stanford 1974.
11. Y. Derbenev and A. N. Skrinsky, *Part. Accel.,* No. 8, p. 1.
12. S. Cocher, *Effets collectifs d'un faisceau refroidi d'ions lourds dans un anneau de stockage,* Thèse soutenue à l'université de Paris XI le 19 mai 1989.
13. B. Franzke, *Nucl. Instr. Meth.,* **B24/25,** 1987 p. 18.
14. B. Franzke, GSI-ESR-TN/86-01.
15. V. H. Neil and A. Sessler, *Review of Scientific Instrument,* Vol. 36, No. 4, p. 429, April 1965.
16. J. L. Laclare, CERN 85–19, 27 November 1985.
17. I. Hofmann, GSI-ESR-87-04 (1987).
18. A. Papoulis, Probability, Random Variables and Stockastic Processes, McGraw-Hill.

APPENDIX A

Notation

Notation is the same as in reference.[16] The reference particle is defined by its constant revolution frequency ω_0 and its revolution period T_0, and a test particle by its revolution frequency ω and its revolution period T.

For a coasting beam, the time delay τ of a test particle with respect to the reference particle is a linear function of time;

$$\tau = \tau_0 + \dot{\tau} t, \tag{31}$$

and $\dot{\tau}$ is given by:

$$\begin{aligned} \dot{\tau} &= \frac{d\tau}{dt} = \frac{\Delta T}{T_0} = -\frac{\Delta\omega}{\omega_0} \\ &= \left(\alpha - \frac{1}{\gamma^2}\right)\frac{\Delta P}{P_0} = \eta\frac{\Delta P}{P_0}, \end{aligned} \tag{32}$$

where α is the momentum compaction and $\Delta P = P - P_0$. When the beam is subjected to a longitudinal force, we can write

$$\ddot{\tau} = \frac{\eta}{P_0}\frac{dP}{dt}. \tag{33}$$

APPENDIX B

Solution of Vlasov's Equation

We consider Vlasov's equation in the form

$$\frac{\partial\Psi}{\partial t} + \dot{\tau}\frac{\partial\Psi}{\partial\tau} + \ddot{\tau}\frac{\partial\Psi}{\partial\dot{\tau}} = 0, \tag{34}$$

where $\Psi(\tau, \dot{\tau}, t)$ is the distribution function of the beam normalized to 1 and $\ddot{\tau}$ is proportional to the longitudinal force:

$$\ddot{\tau} = \frac{\eta}{P_0}\frac{dP}{dt} = \frac{\eta q}{Am_0 c/e\gamma_0\beta_0} E_{\parallel}. \tag{35}$$

To express the beam's response to the electric-field perturbation, we calculate the behavior of a small departure Ψ_1 of the distribution function from the equilibrium state Ψ_0 and write:

$$\Psi(\tau, \dot{\tau}, t) = \Psi_0(\dot{\tau}) + \Psi_1(\tau, \dot{\tau}, t)$$

$$= \Psi_0(\dot{\tau}) + \sum_p \int \tilde{f}_p(\dot{\tau}, \Omega) e^{i(\Omega t - p\theta)} d\Omega. \tag{36}$$

The linearized Vlasov equation is then written as

$$\frac{\partial \Psi_1}{\partial t} + \dot{\tau}\frac{\partial \Psi_1}{\partial \tau} + \ddot{\tau}\frac{\partial \Psi_0}{\partial \dot{\tau}} = 0. \tag{37}$$

Taking its Fourier transformation and noting that

$$E_{\parallel}(\theta, t) = \sum_p \int \tilde{E}_{\parallel p}(\Omega) e^{i(\Omega t - p\theta)}\, d\Omega, \tag{38}$$

we obtain

$$i(\Omega - p\omega_0 + p\omega_0\dot{\tau})\tilde{f}_p(\dot{\tau}, \Omega) + \frac{\eta q e}{\gamma_0 A m_0 c \beta_0} \tilde{E}_{\parallel p}(\Omega) \frac{\partial \Psi_0(\dot{\tau})}{\partial \dot{\tau}} = 0, \tag{39}$$

or

$$\tilde{f}_p(\dot{\tau}, \Omega) = i \frac{\eta q e}{\gamma_0 \beta_0 A m_0 c} \tilde{E}_{\parallel p}(\Omega) \frac{\partial \Psi_0(\dot{\tau})/\partial \dot{\tau}}{\Omega - p\omega_0 + p\omega_0\dot{\tau}}$$

$$= -i \frac{\eta q e \omega_0}{\gamma_0 \beta_0 A m_0 c} \tilde{E}_{\parallel p}(\Omega) \frac{\partial \Psi_0(\omega)/\partial \omega}{\Omega - p\omega}, \tag{40}$$

where we have replaced $\dot{\tau}$ by $-(\omega - \omega_0)/\omega_0$.

APPENDIX C

Beam Response

To calculate the beam response to an electric field perturbation, we use Eq. 40. Here we distinguish the perturbing component of the electric field from the one resulting from the collective response of the beam, and we split the total electric field perturbation into an external part and a collective part:

$$\tilde{E}_{\parallel p} = \tilde{E}_{\parallel p}^{\text{source}} + \tilde{E}_{\parallel p}^{\text{coll}}, \tag{41}$$

where the second term is related to the coupling impedance $Z_{\parallel}$. If $\tilde{I}^{\text{coll}}$ is the collective response of the beam, it can be expressed in the form

$$\tilde{E}_{\|p}^{\text{coll}}(\Omega, \theta) = -\frac{1}{2\pi R} Z_{\|} \tilde{I}^{\text{coll}}(\Omega, \theta), \tag{42}$$

and can be calculated with the help of the distribution function:

$$\tilde{I}^{\text{coll}} = \frac{2\pi}{\omega_0} I \int \tilde{f}_p(\dot{\tau}, \Omega) d\dot{\tau} e^{i(\Omega t - p\theta)}. \tag{43}$$

By performing the integration, it can be written as:

$$\tilde{I}^{\text{coll}} = i \frac{\eta q I}{\gamma_0 \beta_0^2 A m_0 c^2 / e} \int \frac{\partial \Psi_0 / \partial \dot{\tau}}{\Omega - p\omega_0 + p\omega_0 \dot{\tau}} d\dot{\tau}\, 2\pi R \tilde{E}_{\|p}^{\text{source}} \times \frac{1}{\varepsilon(\Omega, p)} \tag{44}$$

or

$$\tilde{I}^{\text{coll}} = -i \frac{\eta q I}{\gamma_0 \beta_0^2 A m_0 c^2 / e} \int \frac{\partial \Psi_0 / \partial \omega}{\Omega - p\omega} d\omega\, 2\pi R \tilde{E}_{\|p}^{\text{source}} \times \frac{1}{\varepsilon(\Omega, p)}, \tag{45}$$

with

$$\varepsilon(\Omega, p) = 1 - i \frac{\eta q I}{\gamma_0 \beta_0^2 A m_0 c^2 / e} Z_{\|} \int \frac{\partial \Psi_0 / \partial \omega}{\Omega - p\omega} d\omega. \tag{46}$$

Using the Landau contour,[6] the integral is written as:

$$\int \frac{\partial \Psi_0 / \partial \omega}{\Omega - p\omega} d\omega = PV \int \frac{\partial \Psi_0 / \partial \omega}{\Omega - p\omega} d\omega + i \frac{\pi}{p} \frac{\partial \Psi_0}{\partial \omega} \bigg|_{\omega = \Omega/p} \tag{47}$$

C.1. BEAM TRANSFER FUNCTION

In the case where the source field is produced by a kicker placed at the azimuth θ_k:

$$\tilde{E}_{\|p}^{\text{source}} = E_0 e^{i(\Omega t - p\theta + p\theta_k)},$$

the beam response measured by a pickup electrode at the azimuth θ_{pu} is:

$$\tilde{I}^{\text{coll}} = -i \frac{\eta q I}{\gamma_0 \beta_0^2 A m_0 c^2 / e} \int \frac{\partial \Psi_0 / \partial \omega}{\Omega - p\omega} d\omega\, 2\pi R E_0 \times \frac{1}{\varepsilon(\Omega, p)} e^{ip(\theta_k - \theta_{\text{pu}})}. \tag{48}$$

C.2. SCHOTTKY NOISE

In the case of Schottky noise the source term comes from the beam itself. Beam current fluctuations $\tilde{I}_{\text{fluct}}$ due to the randomly distributed particles create an electric field related to the coupling impedance by

$$\tilde{E}_{\|p}^{\text{source}} = -\frac{1}{2\pi R} Z_{\|} \tilde{I}_{\text{fluct}}. \tag{49}$$

This field has to be replaced in Eq. (45). The current measured by a pickup electrode at the azimuth θ_{pu} is the sum of beam current fluctuations and the collective response of the beam:

$$\tilde{I}_{pu} = \tilde{I}_{coll} + \tilde{I}_{fluct} \tag{50}$$

$$\tilde{I}_{pu} = i\frac{\eta qI}{\gamma_0\beta_0^2 A m_0 c^2/e}\int \frac{\partial\Psi_0/\partial\omega}{\Omega - p\omega}\,d\omega\, Z_{\|}\tilde{I}_{fluct} \times \frac{1}{\varepsilon(\Omega, p)} + \tilde{I}_{fluct}$$
$$= \frac{\tilde{I}_{fluct}}{\varepsilon(\Omega, p)}. \tag{51}$$

To calculate $\tilde{I}_{fluct}$, we have to know the signal induced by the beam at the pickup electrode. The reference particle passes the pickup at $t = (\theta_p + 2\pi p)/\omega_0$ and the test particle at $t = t_0 + \tau$. The signal induced by the test particle α passing periodically through the PU is:

$$i^\alpha(t) = qe\sum_{p=-\infty}^{+\infty}\delta\left(t - \tau^\alpha - \frac{\theta_p + 2\pi p}{\omega_0}\right)$$
$$= \frac{qe\omega_0}{2\pi}\sum_p e^{ip(\omega^\alpha t - \omega_0\tau_0^\alpha - \theta_p)}, \tag{52}$$

with $\tau^\alpha = \dot{\tau}^\alpha t + \tau_0^\alpha$ and $\omega^\alpha = \omega_0(1 - \dot{\tau}^\alpha)$.

The signal induced by the beam is:

$$I(t) = \sum_{\alpha=1}^{N} i^\alpha(t) = \sum_{\alpha=1}^{N}\sum_p \frac{qe\omega_0}{2\pi}e^{ip(\omega^\alpha t - \omega_0\tau_0^\alpha - \theta_p)}. \tag{53}$$

Particles being randomly distributed in the phase space, τ_0^α can take any value between 0 and $2\pi/\omega_0$. The only contribution to the average beam current comes from $p = 0$ and it follows:

$$\langle I\rangle = \sum_{\alpha=1}^{N}\frac{qe\omega_0}{2\pi} = Nqef_0 = I_0. \tag{54}$$

Current fluctuations are defined as

$$I_{fluct}(t) = I(t) - \langle I\rangle$$
$$= \sum_{\alpha=1}^{N}\sum_{p\neq 0}\frac{qe\omega_0}{2\pi}e^{ip(\omega^\alpha t - \omega_0\tau_0^\alpha - \theta_p)}. \tag{55}$$

The Fourier transformation of the current measured at the PU is:

$$\tilde{I}_{pu}(\Omega) = \sum_{\alpha=1}^{N}\sum_{p\neq 0}\frac{qe\omega_0}{2\pi}\delta(\Omega - p\omega^\alpha)e^{-ip(\omega_0\tau_0^\alpha + \theta_p)}\frac{1}{\varepsilon(\Omega)}. \tag{56}$$

The average value of the signal is zero; the more relevant quantity is its power spectrum. It is defined as the Fourier transform of the autocorrelation function [18]:

$$R(t, t') = \langle I_{pu}(t) I_{pu}^*(t')\rangle. \tag{57}$$

Taking into account the collective response of the beam and using Eq. (51), the autocorrelation function of the signal measured at the pickup can be written as:

$$R(t, t') = \sum_{p\neq 0} \sum_{\alpha=1}^{N} \left(\frac{qe\omega_0}{2\pi}\right)^2 \frac{1}{|\varepsilon(p\omega^\alpha)|^2} e^{ip\omega^\alpha(t-t')}. \tag{58}$$

Its Fourier transform gives the power spectrum of current fluctuations according to

$$P(\Omega) = \sum_{p\neq 0} \sum_{\alpha=1}^{N} \left(\frac{qe\omega_0}{2\pi}\right)^2 \frac{1}{|\varepsilon(p\omega^\alpha)|^2} \delta(\Omega - p\omega^\alpha). \tag{59}$$

Replacing the sum by a integral over the frequency distribution of the beam, it follows that

$$\begin{aligned} P(\omega) &= \sum_{p=1}^{+\infty} 2\left(\frac{qe\omega_0}{2\pi}\right)^2 \frac{N}{p} \frac{2\pi}{\omega_0^2} \frac{\Psi_0(\Omega/p)}{|\varepsilon(\Omega)|^2} \\ &= \sum_{p=1}^{+\infty} P(\Omega, p). \end{aligned} \tag{60}$$