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INTEGRABILITY OF THE TWO-DIMENSIONAL BEAM–BEAM INTERACTION IN A SPECIAL CASE

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It is shown that with the assumption of zero chromaticity, zero dispersion and symmetry between x and y, head-on collisions of bunched beams produce a beam–beam interaction that can be reduced to a one-dimensional weak–strong interaction. The KAM theorem ensures that for small values of the strong beam, the motion is bounded by stable trajectories. There can therefore be no Arnol'd diffusion in these circumstances.

I. INTRODUCTION

It is often stated[1] that a non-autonomous system with two degrees of freedom is affected by Arnol'd diffusion, an instability arising from intersection of nonlinear resonances in four-dimensional phase space. The purpose of this paper is to show that under certain assumptions, the system admits an integral of the motion, a canonical angular momentum p_θ. It is therefore possible to eliminate θ as an ignorable coordinate and reduce the number of degrees of freedom to one. The KAM theorem[2] then ensures that for small values of the beam–beam perturbation the motion is bounded by stable trajectories and does not exhibit Arnol'd diffusion.

II. ASSUMPTIONS

We shall consider here only the so-called "weak–strong" case where a test particle in a "weak" beam crosses periodically a "strong" beam. The motion of the particles in the strong beam is not affected by the presence of the weak beam; therefore their charge distribution can be assumed to be constant in time.

The equations of motion of a test particle in the weak beam are[3]

$$\begin{aligned} x'' + K_x(s)x &= \xi_x F_x(x, y)\delta_p(s) \\ y'' + K_y(s)y &= \xi_y F_y(x, y)\delta_p(s), \end{aligned} \tag{1}$$

where x and y are the displacements of the particle motion from a reference orbit ($x, y = 0$), K_x and K_y are the lattice focusing functions and $\xi_x F_x$, $\xi_y F_y$ represent the interaction with the strong beam, with δ_p a periodic delta-function of period C.

Equations (1) describe the motion of a particle, for instance in the proton-antiproton colliding beam system at Fermilab,[4] if the following precautions are taken:

(i) The two unperturbed betatron tunes ν_x and ν_y do not depend on the particle momentum. This requires chromaticity cancellation in both planes over a reasonable momentum range.

(ii) The lattice parameters at the cross point (α^*, β^*, γ^*) do not depend on the particle momentum, again over some appreciable range. This might require even higher order corrections than those required to flatten the chromaticity.

(iii) The dispersion at the crossing point vanishes over the same momentum range. No condition, though is required on the derivative of the dispersion.

(iv) Both beams are bunched but the interaction is exactly head-on.

(v) The bunch length in the strong beam is small compared with β^*. In this case it is possible to represent the interaction by a lumped kick. That is the interaction has infinitesimally small duration which justifies the periodic delta function on the right-hand side of Eqs. (1).

With these assumptions, which are not too difficult to meet in practice, the interaction between the two beams is independent of the particle momentum and therefore of the phase oscillations. In this case one only requires the integration of the system of Eqs. (1) to calculate the motion of a test particle in the weak beam.

Our approach is static. That is, we are neglecting all sources of noise that would cause the interaction to fluctuate (gas scattering, intrabeam scattering, power-supply noise, etc . . etc . .).

III. THE THEOREM

If the following conditions

$$\nu_x = \nu_y \quad \text{and} \quad \alpha_x^* = \alpha_y^* \tag{2}$$

are satisfied, then there exists an infinite variety of strong-beam charge distributions for which the equations of motion (1) admit at least one integral of motion.

Proof: The interaction force can be derived from a potential function

$$\xi_x F_x(\mathrm{x}, \mathrm{y}) = -\frac{\partial U}{\partial x}$$

and (3)

$$\xi_y F_y(\mathrm{x}, \mathrm{y}) = -\frac{\partial U}{\partial y}.$$

Then Eqs. (1) can be obtained from the Hamiltonian[3]

$$H = \frac{p_x^2 + p_y^2}{2} + \frac{K_x x^2 + K_y y^2}{2} + U(x, y)\delta_p(s). \tag{4}$$

This is a non-autonomous system with two degrees of freedom. The independent variable is s; the canonically conjugate variables are

$$x, p_x = x' \quad \text{and} \quad y, p_y = y'. \tag{5}$$

According to Maxwell's equations, the potential U is related to the charge distribution $\rho(x, y)$ in the strong beam Laplace's equation

$$\nabla^2 U(x, y) = \rho(x, y). \tag{6}$$

Let $\mu_x = 2\pi\nu_x$ and $\mu_y = 2\pi\nu_y$ be the betatron phase advances in the two planes between two consecutive crossings. Whether there is only one or more interactions per revolution is here immaterial, provided that the lattice repeats identically between crossings. In the limit $\xi_x = \xi_y = 0$ the position and angle of the test particle after the n-th crossing are given by

$$x_n = \sqrt{\epsilon_x \beta_x^*}\cos(n\mu_x + \delta_x)$$
$$x'_n = -\sqrt{\frac{\epsilon_x}{\beta_x^*}}\{\alpha_x^* \cos(n\mu_x + \delta_x) + \sin(n\mu_x + \delta_x)\} \tag{7}$$

and similarly for y_n and y_n'. In Eqs. (7), ϵ_x and δ_x are constants of motion. The first, ϵ_x, measures the amplitude of the motion. We carry out the variable transformation

$$(x, x'; y, y') \rightarrow (r, p_r; \theta, p_\theta), \tag{8}$$

with

$$x = \sqrt{r\beta_x^*}\cos\theta \quad \text{and} \quad y = \sqrt{r\beta_y^*}\sin\theta \tag{9}$$

The generating function for this transformation is

$$S(\theta, r; p_x, p_y) = \{xp_x + yp_y\} = x'\sqrt{r\beta_x^*}\cos\theta + y'\sqrt{r\beta_y^*}\sin\theta, \tag{10}$$

from which we derive the new momenta

$$p_\theta = -\frac{\partial S}{\partial \theta} \quad \text{and} \quad p_r = -\frac{\partial S}{\partial r}. \tag{11}$$

In particular from the first, which is an angular momentum,

$$p_\theta = x'\sqrt{r\beta_x^*}\sin\theta - y'\sqrt{r\beta_y^*}\cos\theta = \sqrt{\frac{\beta_x^*}{\beta_y^*}}x'y - \sqrt{\frac{\beta_y^*}{\beta_x^*}}y'x \tag{12}$$

We can estimate p_θ at the interaction point after n crossings in the limit $\xi_x = \xi_y = 0$. By inserting Eqs. (7) and similar equations for y, y' in Eqs. (12) we obtain

$$p_{\theta n} = \sqrt{\epsilon_x \epsilon_y}\sin(\delta_y - \delta_x) \tag{13}$$

which is a constant of the motion.

Let us see now the effect of the kick with ξ_x

and $\xi_y \neq 0$

$$\Delta x_n = 0 \qquad \Delta y_n = 0$$

$$\Delta x_n' = \xi_x F_x(x_n, y_n) \qquad \Delta y_n' = \xi_y F_y(x_n, y_n)$$

We have

$$\begin{aligned}\Delta p_{\theta n} &= \sqrt{\frac{\beta_x^*}{\beta_y^*}}\, y_n \Delta x_n' - \sqrt{\frac{\beta_y^*}{\beta_x^*}}\, x_n \Delta y_n' \\ &= \sqrt{\frac{\beta_x^*}{\beta_y^*}}\, y_n \xi_x F_x(x_n, y_n) \\ &\quad - \sqrt{\frac{\beta_y^*}{\beta_x^*}}\, x_n \xi_y F_y(x_n, y_n) \\ &= -\sqrt{\frac{\beta_x^*}{\beta_y^*}}\, y_n \left(\frac{\partial U}{\partial x}\right)_n \\ &\quad + \sqrt{\frac{\beta_x^*}{\beta_y^*}}\, x_n \left(\frac{\partial U}{\partial y}\right)_n\end{aligned}$$

When the transformation (9) is applied,

$$U = U(x, y) \rightarrow U(r, \theta),$$

and it is obvious that if $\partial U/\partial\theta = 0$, i.e., the potential depends only on the "radial" coordinate r,

$$\frac{\partial U}{\partial x} = \frac{dU}{dr}\frac{\partial r}{\partial x} = 2\frac{x}{\beta_x^*}\frac{dU}{dr}$$

$$\frac{\partial U}{\partial y} = \frac{dU}{dr}\frac{\partial r}{\partial y} = 2\frac{y}{\beta_y^*}\frac{dU}{dr}$$

then $\Delta p_{\theta n} = 0$, that is p_θ remains a constant of motion even with beam–beam interaction.

From (9) we obtain

$$r = \frac{x^2}{\beta_x^*} + \frac{y^2}{\beta_y^*}.$$

If U is a function only of this variable, then we have from Eq. (6) that all the charge distributions satisfying the equation

$$\rho(x, y) = 4\left(\frac{x^2}{\beta_x^{*2}} + \frac{y^2}{\beta_y^{*2}}\right)\frac{d^2U}{dr^2} + 2\left(\frac{1}{\beta_x^*} + \frac{1}{\beta_y^*}\right)\frac{dU}{dr} \qquad (14)$$

with an arbitrary $U = U(r)$, also satisfy the requirements of the theorem.

In particular, if $\beta_x^* = \beta_y^*$ and the strong beam is "round", as it is approximately true for the $\bar{p}p$ project at Fermilab,[4] then a Gaussian charge distribution in the strong beam is consistent with the assumptions of the theorem.

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