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COLLIDING BEAMS COHERENT INSTABILITY

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Beam-beam effects can cause resonant coherent instability of colliding beams when the working point (ν_x, ν_z) is tuned to a machine resonance. The rise time of this instability is proportional to the beam cross-sectional area and increases as a power of the multipole number. The instability can be cured, first, by detuning the frequencies (ν_x, ν_z) from machine resonances and second, by using beams with the same density but different transverse dimensions.

INTRODUCTION

The luminosity of colliding-beams facilities can be limited by the beam-beam interaction. The nature of this limitation has been studied for more than 15 years. Nevertheless, up to now there is no satisfactory theory to describe the mutual influence of colliding beams. Because of the difficulty of solving the self-consistent problem, this subject has been primarily studied in the frame of the "weak-strong beam" approximation. In this approximation, the motion of a particle belonging to the weak beam is traced under distortions from the field of the strong beam. The motion of the strong beam is assumed to be given. As a rule, computer simulation is used to get concrete results. But even this very simple method of calculation is limited by the capabilities of computers. To simplify the calculations, one often uses approximate expressions for the beam field. In large number of iterations, this can cause propagation of errors and distortion of physical patterns. Investigations of the last years with this model have been directed to study stochasticity at the beam-beam interaction.

Computer simulation of the mutual influence in collisions seems to be a more difficult problem, because it requires exact calculation of the fields corresponding to the instantaneous distributions of particles in beams. As the first approximation to solution of the self-consistent beam-beam problem, we consider the problem of collective instability of configuration of beams that would be stationary without collisons. In this approach, the interaction is obviously self-consistent. This allows us to calculate the spectrum of collective oscillations and to give definite instructions to choose the working point at the storage ring. Besides, the calculations of linear theory can be used for construction of a nonlinear theory of the beam-beam interaction (when writing kinetic equations or in computer simulations of strong-strong interaction). The well-known disadvantage of linear theory is that it allows one to calculate only stability thresholds and initial instability growth rates.

Apparently for the first time the problem of coherent instability due to beam-beam effects was studied in Ref. 1 in the context of collective instability of compensated colliding beams.[2] The authors of Ref. 3 tried to describe the instabilities of colliding beams as a two-stream instability in plasma. They did not, however, take into account the basic properties of particle motion in storage rings.

For collective motion, it is very important that particles in the storage ring move close to the closed orbit (with revolution frequency ω_0) and make small oscillations around this orbit with frequencies $\omega_0\nu_x$, $\omega_0\nu_z$, $\omega_0\nu_c$. Thus, without perturbations the normal collective oscillations are described by distribution-function harmonics over phases of particles oscillations

$$f_m \exp(im_x\psi_x + im_z\psi_z + im_c\psi_c). \qquad (1)$$

The integer numbers m_x, m_z and m_c determine the multipolarity of the coherent motion.

The distortion of particle motions by collective fields leads to frequency shift in the unperturbed spectrum $\omega \simeq \omega_0 (m_x\nu_x + m_z\nu_z + m_c\nu_c)$. If this shift is small in comparison with the distance between frequencies in the unperturbed spectrum, the oscillations of Eq. (1) remain stable.[4]

The aim of this paper is to study coherent instabilities of colliding beams due to beam-beam interaction. The methods of linear theory of collective oscillations[4,5] used here allow us to study in the same way the stability of modes with arbitrary multipolarity (betatron, synchrobetatron and others).

Because the beam interaction is conservative, it can lead to dynamical instabilities only when the oscillation frequencies obey some resonant condition. Investigation has shown that for colliding beams there are two kinds of resonances. The first takes place when a normal oscillation of one beam is in resonance with some normal oscillation of the counterrotating beam

$$(m_x\nu_x + m_z\nu_z + m_c\nu_c)_1 - (m_x\nu_x + m_z\nu_z + m_c\nu_c)_2 = 2n. \quad (2)$$

Qualitatively, the behavior of collective motion here is very close to that due to beam interaction with resonant elements of the vacuum chamber.[6] Instability occurs for sum resonances when the detuning from resonance is smaller than the coherent tune shift. The stability conditions do not depend on the sign of detuning. The value of the coherent tune shift is proportional to the linear tune shift due to beam-beam effects. It is necessary to point out that for given detuning, Eq. (2) is valid for any pair of oscillations with appropriate multipole numbers $\{m_1\}$ and $\{m_2\}$.

The second kind of resonance takes place when the frequencies $\{\nu\}$ are tuned to a machine resonance (for simplicity we put $\{\nu_1\} = \{\nu_2\}$)

$$m_x\nu_x + m_z\nu_z + m_c\nu_c = n. \quad (3)$$

Resonances of this kind were discussed in Ref. 1. The specific feature of this resonance is the dependence of stability conditions on the detuning sign. Namely, for counter-charged colliding beams, oscillations will be stable if the working point is placed above the resonance of Eq. (3). Such dependence of stability conditions is specific for two-stream instabilities in plasma.[7] Here Eq. (3) also corresponds to interaction at two streams with velocities ϵ and $-\epsilon$ ($\epsilon = mv - n$) in the space of the phases $\{\psi\}$. For this reason we name this "the resonance two-stream instability."

1. *Basic Equations*

The coherent stability of colliding beams due to beam-beam effects can be investigated using ordinary methods of the linear theory of collective oscillations.[4,5] Here we shall consider the simplest situation when two bunches make head-on collisions at two interaction points.

If the motion of particles near the closed orbit is described by action $I_\alpha = I_\alpha(\mathbf{r}, \mathbf{p}, t)$ and phase $\psi_\alpha = \psi_\alpha(\mathbf{r}, \mathbf{p}, t)$ variables, then the distribution functions of the beams unperturbed by collective fields do not depend on the phase variables

$$f^{(1,2)} = f_0^{(1,2)}(I), \quad (1.1)$$

where the superscripts 1 and 2 denote the beams and I means I_x, I_z, I_c. In this paper, we shall suppose that the canonical transformation from $(\mathbf{r}, \mathbf{p})$ variables to action-phase variables (I, ψ) is given by the equations

$$x, z = a_{x,z}\cos\psi_{x,z}, \quad \theta_{1,2} = \pm\,\omega_s t + \phi\sin\psi_c,$$
$$\mathbf{p}_T = \frac{p_s}{R_0}\frac{d\mathbf{r}_T}{d\theta}, \quad \Delta p = p - p_s\frac{\nu_c}{\alpha}\cos\psi_c,$$
$$\psi_{x,z} = \omega_0(p)\,\nu_{x,z}(p), \quad \psi_c = \omega_s\nu_c$$
$$I_{x,z} = \frac{p_s}{2R_0}(\nu a^2)_{x,z} \quad I_c = R_0 p_s\frac{\nu_c\psi^2}{2\alpha} \quad (1.2)$$

Here $p = M\gamma v$ is the momentum of the particle, θ is its azimuth, the subscript s denotes values on synchronous orbit, ν_x, ν_z, ν_c are the tunes, $2\pi R_0$ is the path length, and α is the momentum compaction factor. In this paper, for simplicity, we assume zero dispersion at interaction points. Non-zero dispersion at interaction points can amplify instability of some synchrobetatron modes. Very close collision effects can give a small relative displacement of the beams at interaction. Investigation of both these effects will be done in another paper. For this reason, we omit in Eq. (1.2) contributions of terms that are proportional to the dispersion function.

The coherent oscillations of beams are described by small non-stationary additions to $f_0^{(1,2)}$

$$f^{(1,2)} = f_0^{(1,2)}(I) + \delta f^{(1,2)}(I, \psi, t).$$

The spectrum of small coherent oscillations can be found by solving the linearized Vlasov equations for the functions $\delta f^{(1,2)}$. In action-phase variables, these equations have the form

$$\left(\frac{\partial}{\partial\theta_s} + \boldsymbol{\nu}\frac{\partial}{\partial\boldsymbol{\psi}}\right)\delta f^{(1)} + \frac{\partial L_{1,2}}{\partial\boldsymbol{\psi}}\cdot\frac{\partial f_0^{(1)}}{\partial\mathbf{I}} = 0,$$
$$\left(\frac{\partial}{\partial\theta_s} + \boldsymbol{\nu}\frac{\partial}{\partial\boldsymbol{\psi}}\right)\delta f^{(2)} + \frac{\partial L_{2,1}}{\partial\boldsymbol{\psi}}\cdot\frac{\partial f_0^{(2)}}{\partial\mathbf{I}} = 0, \qquad (1.3)$$

where

$$\boldsymbol{\nu}\frac{\partial}{\partial\boldsymbol{\psi}} = \nu_x\frac{\partial}{\partial\psi_x} + \nu_z\frac{\partial}{\partial\psi_z} + \nu_c\frac{\partial}{\partial\psi_c},$$

$\theta_s = \omega_s f$ and $L_{1,2}$ (respectively $L_{2,1}$) is the Lagrangian describing interaction of particles belonging to beam 1 with the field induced by beam 2. Neglecting radiation effects,

$$l_b \gg \frac{l_T}{\min\left\{\frac{R_0}{\nu l_T}, \gamma\right\}},$$

and for the ultrarelativistic case ($\gamma \gg 1$) we may write for $L_{1,2}$

$$L_{1,2} = -\frac{2e_1e_2}{\pi v_s}\int\frac{d^2k}{K^2}\cdot\exp(i\mathbf{k}\cdot\mathbf{r}(1))\int d\Gamma_2\delta \times (\phi_1\sin\psi_{c1} - \phi_2\sin\psi_{cz} - 2\omega_s t) \times \exp(-i\mathbf{k}\cdot\mathbf{r}(2))\,\delta f^{(2)}. \qquad (1.4)$$

Here $\mathbf{k}\cdot\mathbf{r} = k_x r_x + k_z z_z$, $d\Gamma$ is the phase-space volume element, l_b is the bunch length, and l_T is the typical transversal dimension of the vacuum chamber.

Using Fourier transformation of $\delta f^{(1,2)}$ over phases,

$$\delta f^{(1,2)} = \sum_m f_m^{(1,2)}(I, \theta_s)\,e^{i\mathbf{m}\cdot\boldsymbol{\psi}},$$

we get from Eq. (1.3) equations for the Fourier amplitudes $f_m^{(1,2)}(I, \theta_s)$

$$\left(\frac{\partial}{\partial\theta_s} + i\mathbf{m}\cdot\boldsymbol{\nu}\right)f_m^{(1)} = i\frac{2e_1e_2}{\pi\nu_s}\mathbf{m}\frac{\partial f_0^{(1)}}{\partial\mathbf{I}} \times \sum_n\frac{e^{-i2n\theta_s}}{2\pi}J_{m_c}(z_1\psi_1) \times \int d\Gamma_2\,e^{-Z_2\phi_2\sin\psi_{c2}}K_m(1/2)\,\delta f^{(2)} \qquad (1.5)$$

and an analogous equation for $f_m^{(2)}(I, \theta_s)$. We define

$$K_m(1/2) = \int\frac{d^2k}{k^2}\times\{\exp(i\mathbf{k}\cdot\mathbf{r}(1))\}_{m_T}\exp(-i\mathbf{k}\cdot\mathbf{r}(2)),$$
$$Z = \mathbf{m}_T\cdot\boldsymbol{\nu}_T + n + \mathbf{m}_T\cdot\frac{d\boldsymbol{\nu}_T}{d\ln\omega_0}. \qquad (1.6)$$

Equation (1.5) shows that $f_m^{(1,2)}$ changes proportionally to the linear beam-beam tune shift

$$\Delta\nu_0 = \frac{Ne_1e_2R_0}{2\pi p\nu\sigma^2\nu},$$

where N is the number of particles in a bunch and σ is the transverse dimension of a bunch.

For orbit stability, $\Delta\nu_0$ should have a small value ($\Delta\nu_0 \ll 1$). Besides, in Ref. 1 it was shown that $\Delta\nu_0 \gg 1$ corresponds to dynamic collective instability of colliding beams, so we shall take $\Delta\nu_0 \ll 1$. Because the interaction of the beams is conservative, instability arises from resonant interaction of particular collective modes. The unperturbed frequencies of collective oscillations ($\Delta\nu_0 \to 0$) $f_m \sim \exp(-i\Delta\cdot\theta_s)$ are

$$\Delta = \mathbf{m}_1\cdot\boldsymbol{\nu}_1 \quad \text{for beam 1}$$
$$\Delta = \mathbf{m}_2\cdot\boldsymbol{\nu}_2 \quad \text{for beam 2} \qquad (1.7)$$

Comparison of Eqs (1.7) and (1.5) gives the resonant condition

$$\mathbf{m}_1\boldsymbol{\nu}_1 - \mathbf{m}_2\cdot\boldsymbol{\nu}_2 = 2n. \qquad (1.8)$$

If for simplicity the bunch length is not too small,

$$l_b \gg \frac{v_T}{v_c}\sigma,$$

and $f_0(I_2, I_c)$ can be factorized. Thus

$$f_0(I_T, I_c) = F_0(I_T)\,\rho(I_c). \tag{1.9}$$

Then for the resonant modes of Eq. (1.8), Eq. (1.5) can be rewritten as

$$f_m{}^{(1,2)} = \Phi_{m_T}(I_T)\rho(I_c)J_{m_c}(z\phi) \times \exp(-i\Delta\cdot\theta_s)$$

$$\Delta_m\Phi_{m_1}(1) = \frac{2e_1e_2}{\pi\nu_s}q_{m_c}(2)\mathbf{m}_{T1} \times \frac{\partial f_0{}^{(1)}}{\partial \mathbf{I}_T}\int d\Gamma_{2T}K(1\mid 2)\Phi_{m_2}(2),$$

$$(\Delta_m - \epsilon)\Phi_{m_2}(2) = \frac{2e_1e_2}{\pi\nu_s}q_{m_c}(1)\,\mathbf{m}_{T2} \times \frac{\partial F^{(2)}}{\partial \mathbf{I}_T}\int d\Gamma_{1T}K(2\mid 1)\Phi_{m_1}(1), \tag{1.10}$$

where

$$q = \int_0^\infty dI_c\rho(I_c)\,J^2_{m_c}(Z\phi_c)\,|_{1,2},$$

$$K(1\mid 2) = \int\frac{d^2k}{k^2}\{e^{i\mathbf{k}\cdot\mathbf{r}(1)}\}_{m_{T1}}\{e^{i\mathbf{k}\cdot\mathbf{r}(2)^*}\}_{m_{T2}},$$

$$\Delta_m = \Delta - \mathbf{m}_1\cdot\boldsymbol{\nu}_1, \tag{1.11}$$

$$\epsilon = \mathbf{m}_2\cdot\boldsymbol{\nu}_2 - \mathbf{m}_1\cdot\boldsymbol{\nu}_1 - 2n.$$

The system of homogeneous integral equations (1.10) determines the spectrum of eigenfrequencies for both betatron ($m_c = 0$) and synchrobetatron ($m_c \neq 0$) collective oscillations of colliding beams. This system vanishes for fast instabilities, when the increment (inverse rise time) ($\sim\Delta\nu_0$) is much larger than the synchrotron tune shift ν_c. In this case, particles cannot displace noticeably in phase ψ_c and one may use a model in which the bunch has a "rigid" azimuthal distribution

$$\rho(I_c) \to \rho(\theta \pm \theta_s).$$

In this model normal oscillations are azimuthal harmonics of $\delta f^{(1,2)4}$

$$\Phi_m \to \sum_{n=-\infty}^{\infty}\Phi_{m,n}\,e^{in(\theta-\theta_s)}.$$

One can see from Eq. (1.5) that the integral equations for resonant modes $\Phi_{m1,n}$ and $\Phi_{m2,n}$ coincides with Eq. (1.10), if one takes the factors $q^{(1,2)}_{m_c}$ to be unity. This is quite obvious: for $\nu_c \ll \Delta\nu_0$, we may omit in the l.h.s. of Eq. (1.10) terms proportional to $m_c\nu_c$. Summation over m_c in the r.h.s. of Eq. (1.10) gives

$$\sum_{m_c} q^{(1,2)}_{m_c} = 1.$$

The equations for $\Phi_{m_1,n}$ are a little simpler than Eq. (1.10), so we consider first the model with "rigid" azimuthal distribution.

2. *One-Dimension Betatron Modes*

We can trace the basic features of the spectrum for the resonances of Eq. (1.8) in a simple model where

1) particles in beams oscillate with the same tunes $\{\nu\}_1 \equiv \{\nu\}_2$;

2) the stationary distribution functions over betatron amplitudes are identical and equal to

$$f_0 = \frac{N R_0}{(2\pi)^2 p\nu_x\sigma^2}\,\delta(I_z)\exp\left(\frac{I_x}{I_0}\right), \tag{2.1}$$

where $I_0 = p\nu_x\sigma^2/R_0$ and σ^2 is the mean-square radial dimension of the beam;

3) the resonance is one-dimensional

$$(m_1 - m_2)\,\nu_x = 2n.$$

In this model, the integral equations for $\phi_{m_1,n}$ and $\phi_{m_2,n}$ become

$$\Delta_m\Phi_m(1) = -2m_1\delta e^{-I/I_0}\int_{-\infty}^{\infty}\frac{dk}{|k|}J_{m_1} \times (ka_1)\int_0^\infty\frac{dI_2}{I0}J_{m_2}(ka_2)\cdot\Phi_m(2)$$

$$(\Delta_m - \epsilon)\Phi_m(2) = -2m_2\delta e^{-I/I_0}\int_{-\infty}^{\infty}\frac{dk}{|k|} \times J_{m_2}(ka_2)\int_0^\infty\frac{dI_1}{I_0}J_{m_1}(ka_1) \times \Phi_m(1). \tag{2.2}$$

We define

$$\epsilon = (m_2 - m_1)\nu_x - 2n$$

and

$$\delta = \frac{Ne_1e_2R_0}{2\pi p v \sigma^2 \nu_x}. \quad (2.2.a)$$

One can see that the kernels of the integral equations (2.2) vanish if $|m_1| + |m_2|$ is odd, so that the oscillations of this type are uncoupled and thus stable. This stability is caused by the orthogonality of the momenta

$$X_m(k) = \int_0^\infty \frac{dI}{I_0} J_m(ka)\Phi_m(I)\frac{1}{\sqrt{|k|}} \quad (2.3)$$

with different parity ($X_m(-k) = (-1)^m X_m(k)$)

Let now $|m_1| + |m_2|$ be even. Using the definition (2.3) we transform Eq. (2.2) into

$$\Delta_m X_{m_1}^{(1)}(k) = -2m_1\delta \int_0^\infty \frac{1}{\sqrt{k_1 k}} e^{-(k^2+k_1^2)/2} \times I_{m_1}(kk_1)\, X_{m_2}(k_1)$$

$$(\Delta_m - \epsilon)X_{m_2}(k) = -2m_2\delta \int_0^\infty \frac{dk_1}{\sqrt{kk_1}} e^{-(k^2+k_1^2)/2} \times I_{m_2}(kk_1)X_{m_1}(k_1). \quad (2.4)$$

Here $I_m(x)$ is the modified Bessel function of order m.

Because of mathematical difficulties, we cannot give a direct solution of Eqs. (2.4). Nevertheless, using general properties of these equations, one can find the stability conditions. We note that the roots of the dispersion equation

$$\Delta_m(\Delta_m - \epsilon) - 4m_1m_2\Lambda^2\delta^2 = 0 \quad (2.5)$$

are determined by the squared eigenvalues of the integral equations

$$\Lambda X_1 = \int_0^\infty \frac{dk_1}{\sqrt{k_1 k}} e^{-(k^2+k_1^2)/2} \times I_{m_1}(kk_1)X_2(k_1) \quad (2.4a)$$

$$\Lambda X_2 = \int_0^\infty \frac{dk_1}{\sqrt{kk_1}} e^{-(k^2+k_1^2)/2} \times I_{m_2}(kk_1)X_1(k_1).$$

If we introduce an operator notation

$$\hat{K}_{1,2}X_{1,2} \equiv \int \frac{dk_1}{\sqrt{kk_1}} \exp\left(\frac{k^2+k_1^2}{2}\right) \times I_{m_{1,2}}(kk_1)X_{1,2}(k_1)$$

and

$$X_{1,2}^2 \equiv \int dk\, X_{1,2}(k)\cdot X_{1,2}(k),$$

we find that

$$\Lambda^2 = \frac{X_1\cdot\hat{K}_2^2\cdot X_1 + X_2\cdot\hat{K}_1^2 X_2}{X_1^2 + X_2^2}$$

are positive numbers because $\hat{K}_1$ and $\hat{K}_2$ are positive-definite operators.

Then Eq. (2.5) gives for Δ

$$\Delta_{1,2} = \frac{\epsilon}{2} \pm \left(\left(\frac{\epsilon}{2}\right)^2 + m_1m_2(2\delta\Lambda)^2\right)^{1/2} \quad (2.6)$$

Instability occurs when

$$m_1\, m_2 < 0$$
$$\frac{\epsilon^2}{4} < -\, m_1\, m_2\,(2\delta\Lambda)^2. \quad (2.7)$$

The first of Eqs. (2.7) means that for instability, oscillations of colliding beams should be in sum resonance. The second gives the width of the resonance.

Analogous dispersion equations and stability criteria were obtained in Ref. 6, when investigating the collective stability of a beam interacting with a high-Q-element of the vacuum chamber. Therefore some results of 6 are valid for resonance instabilities of colliding beams. For instance, damping of particle oscillations can lead to dissipative instability when

$$\frac{\epsilon^2}{4} + m_1m_2(2\delta\Lambda)^2 + \lambda_1\lambda_2 < \frac{\epsilon^2}{4}\left(\frac{\lambda_1 - \lambda_2}{\lambda_1 + \lambda_2}\right)^2. \quad (2.8)$$

Here $\lambda_1 = |m_1|\lambda$, $\lambda_2 = |m_2|\lambda$, where λ is the cooling rate for oscillations. In particular, for colliding electrons and protons ($\lambda_1\cdot\lambda_2 = 0$), the criterion (2.8) will be valid for all sum modes $m_1m_2 < 0$. For e^+e^-, e^-e^- colliding beams, friction can stabilize sum modes with high multipole numbers, i.e.,

$$\lambda^2 + \frac{\epsilon^2}{(|m_1| + |m_2|)^2} > (2\delta\Lambda)^2. \quad (2.9)$$

In constrast with instabilities due to resonance interactions of a beam with its surroundings, the instability rate due to beam-beam effect depends linearly on the beam current

$$Im\,\Delta \simeq \delta\sqrt{|m_1m_2|}\,\Lambda^2_{m_1m_2},\ |\delta| \gg |\epsilon|. \quad (2.10)$$

To evaluate the dependence of $\Lambda_{m_1m_2}$ on multipole numbers, we shall consider the solution of Eq. (2.4) in the short wave length limit $k \gg m = \max(m_1, m_2)$. In the asymptotic region

$$I_m(kk_1) \simeq \left(\frac{2}{\pi kk_1}\right)^{1/2} e^{kk_1},\ k \gg m < 1,$$

the main contribution to the integrals of Eq. (2.4) is from the region $|k - k_1| < 1 \ll k$. Then

$$\Lambda X_1 \simeq \frac{1}{k^2} X_2(k)$$

$$\Lambda X_2 \simeq \frac{1}{k^2} X_1(k), \quad k \gg m \gg 1. \quad k \gg m \gg 1.$$

This yields $\Lambda = \Lambda_{(k)} \simeq 1/k^2$. Extrapolation of $\Lambda(k)$ into the region $k \sim m$ gives

$$\Lambda_m \simeq \frac{1}{m^2}, \quad m = \max(m_1, m_2) > 1. \quad (2.11)$$

Let us write the value at the maximum instability rate. Using Eqs. (2.6), (2.2a) and (2.11), we find

$$\mathrm{Im}\Delta = \frac{Ne_1e_2\beta}{\pi\gamma Mc^2\sigma^2}\frac{1}{m_1}\left(\left|\frac{m_2}{m_1}\right|\right)^{1/2},$$

$$|m_2| < |m_2|. \quad (2.12)$$

The smallest m_1 and m_2 that have the same parity, are $m_2 = 1$ and $m_1 = -3$. For these numbers, Eq. (2.12) gives

$$\mathrm{Im}\Delta = \frac{Ne_1e_2}{\pi\gamma Mc^2\sigma^2}\frac{\beta}{3\sqrt{3}}. \quad (2.13)$$

Here β is the betatron function at the interaction point.

3. *One-Dimensional Excitations But Different Beam Sizes*

In practice there can be a more important situation when beams with approximately the same densities have different transverse dimensions. Let the distribution function be determined by Eq. (2.1), but let the numbers of particles and transverse dimensions of bunches be respectively N_1, N_2 and σ_1, σ_2. For definiteness we take $\sigma_1 \ll \sigma_2$. Then Eqs. (2.4) can be rewritten as

$$\begin{aligned}\Delta_m X_{m_1} &= -2m_1\delta_1\int_0^\infty \frac{dk_1}{\sqrt{kk_1}} \times \exp\left(-\frac{k_1^2 + k^2}{2}\sigma_1^2\right) \times I_{m_1}(kk_1\sigma_1^2)\,X_{m_2}(k_1)\\ (\Delta - \epsilon)X_{m_2} &= -2m_2\delta_2\int_0^\infty \frac{dk_1}{\sqrt{kk_1}} \times \exp\left(-\frac{k^2 + k_1^2}{2}\sigma_2^2\right) \times I_{m_2}(kk_1\sigma_2^2)X_{m_1}(k_1),\end{aligned} \quad (3.1)$$

where $\delta_{1,2}$ can be obtained from δ by replacing (N/σ^2) by $(N/\sigma^2)_{1,2}$. Because of $\sigma_1 \ll \sigma_2$, the functions X_{m_1} and X_{m_2} have quite different scales in k. In the region

$$\frac{|m_1|}{\sigma_1} \gg k > \frac{|m_2|}{\sigma_2},$$

the function X_{m_1} in the second of Eqs. (3.1) is approximately constant and we can write

$$X_{m_2}(k) \simeq \frac{2m_2\sigma_2}{\Delta_m - \epsilon}\frac{X_{m_1}(k)}{k^2\sigma_2^2}; \quad k > \frac{|m_2|}{\sigma_2}. \quad (3.2)$$

Then the first of Eqs. (3.1), after obvious transformations, can be rewritten as

$$\begin{aligned}&\Delta(\Delta - \epsilon)\,X_{m_1}\\ &\simeq 4(m_1\delta_1)(m_2\delta_2)\left(\frac{\sigma_1}{\sigma_2}\right)^2\frac{|m_2|}{\sigma_2}\\ &\times \int_0^\infty \frac{dk_1}{(k_1k)^{3/2}}\exp\left(-\frac{k^2 + k_1^2}{2}\right)\\ &\times I_{m_1}(kk_1)X_{m_1}(k_1).\end{aligned} \quad (3.3)$$

This equation shows that when beams with difference transverse sizes collide, the instability threshold densities of the beams increase (σ_2/σ_1) times. This result arises from the fact that fields acting on particles of the I beam are formed by particles of the II beam placed inside the linear size σ, (and vice versa).

The value of the maximum increment can be evaluated using

$$\mathrm{Im}\Delta \simeq \frac{e_1 e_2 \beta}{\pi\gamma M c^2 \sigma_2^2}\cdot\frac{1}{m_1}\left(\left|\frac{m_2}{m_1}\right| N_1 N_2\right)^{1/2}$$

$$|m_2| < |m_1| \quad (3.4)$$

For the lowest modes, this yields

$$\mathrm{Im}\Delta = \frac{e_1 e_2}{\pi\gamma M c^2 \sigma_2^2}\frac{\beta}{3\sqrt{3}}\sqrt{N_1 N_2}. \quad (3.5)$$

If we take

$$\frac{N_1}{\sigma_1} = \frac{N_2}{\sigma_2},$$

Eq. (3.5) gives

$$\mathrm{Im}\Delta = \frac{N_1 e_1 e_2}{\pi\gamma M c^2 \sigma_1^2}\left(\frac{\sigma_1}{\sigma_2}\right)^{3/2}\frac{\beta}{3\sqrt{3}} = \left(\frac{\sigma_1}{\sigma_2}\right)^{3/2}\mathrm{Im}\Delta(\sigma_1 = \sigma_2), \quad (3.6)$$

where $\mathrm{Im}\Delta(\sigma_1 = \sigma_2)$ is determined by Eq. (2.13).

4. *Two-Dimensional Betatron Excitations*

Now we want to evaluate the excitational spectrum for two-dimensional resonance

$$(m_{x2} - m_{x1})\nu_x + (m_{z2} - m_{z1})\nu_x - 2n = \epsilon < |\delta|. \quad (4.1)$$

We shall consider collisions of round beams with Gaussian distributions in betatron amplitudes.

Repeating calculations of the preceding section with momenta

$$X_m(\mathbf{k}) = \frac{1}{k}\int d\Gamma\Phi_m\,(\exp(i\mathbf{k}\cdot\mathbf{r}))_m{}^*$$

it is easy to find that the eigenfrequencies satisfy the dispersion equation $(\sigma_1 \ll \sigma_2)$

$$\Delta_m(\Delta_m - \epsilon) - \frac{4}{\pi}(\mathbf{m}_1\cdot\boldsymbol{\delta}_1)(\mathbf{m}_2\cdot\boldsymbol{\delta}_2) \times \Lambda^2_{m1,m2}(\sigma_1/\sigma_2)^4 = 0, \quad (4.2)$$

where $\mathbf{m}\cdot\boldsymbol{\delta} = m_x\delta_x + m_z\delta_z$, and the positive numbers $\Lambda^2_{m1,m2}$ are eigenvalues of the integral equation

$$\Lambda^2 X_{m_1} = \frac{2}{\pi}\int\frac{d^2k_1{}^F}{k^2k_1{}^2} \times \frac{I_{m_{x1}}(kk_1)_x\, I_{m_2}(kk_1)_z}{\sqrt{|k_xk_{x1}|\,|k_zk_{z1}|}} \times e^{-(k_1{}^2+k^2)/2}X_{m_1}(\mathbf{k}_1) \quad (4.3)$$

Equation (4.2) shows that instability occurs when

$$(\mathbf{m}_1\boldsymbol{\delta}_1)\,(\mathbf{m}_2\boldsymbol{\delta}_2) < 0 \quad (4.4)$$

and

$$\frac{\epsilon^2}{4} < \frac{4}{\pi^2}\Lambda^2_{m_1m_2}\left(\frac{\sigma_1}{\sigma_2}\right)^4|(\mathbf{m}_1\cdot\boldsymbol{\delta}_1)(\mathbf{m}_2\cdot\boldsymbol{\delta}_2)|. \quad (4.5)$$

The dependence of $\Lambda_{m_1m_2}$ on multipole numbers can be evaluated extrapolating the solution of Eq. (4.3) from the short wavelength region $(|k_x| \gg |m_x|, |k_z| \gg |m_z|)$

$$\Lambda \simeq \frac{1}{|n_xm_z|}\frac{1}{(m_x{}^2 + m_z{}^2)}, \quad |m_x|, |m_z| > 1. \quad (4.6)$$

It is necessary to point out a special attribute of instabilities due to beam-beam effects. The increments are determined by the beam density (N/σ^2) and decrease as a power of the multipole numbers. This behaviour is caused by the fact that the fields responsible for the instability depend on the beam size σ.[1,8] In contrast, the increments of instabilities due to interaction with surrounding electrodes decrease with multipole number with an exponential law[4,5]

$$\mathrm{Im}\Delta \sim N\frac{\sigma^{2m-2}}{[(m-1)\,!]^2}.$$

Notice, that the coherent tune shift in Eq. (4.2) is reduced $(\sigma_1/\sigma_2)^2 \ll 1$ times. The difference from Eq. (3.3) is caused by the fact that the fields responsible for the instability of the beam with distributions (2.1) arise from space charge distributed along a line.

5. *Resonance Two-Stream Instability*

One more kind of collective instability due to beam-beam effects can be caused by tuning of the working point to machine resonances

$$\mathbf{m}\cdot\boldsymbol{\nu} \simeq n \tag{5.1}$$

In this case, even if the coherent tune shift is smaller than the spread in the unperturbed spectrum, the interaction couples modes with $(m_1 n)$ and $(-m_1 - n)$. Provided Eq. (5.1) is valid, these modes became sum resonances, which can cause the instability to appear. In another interpretation, one can say that this instability is due to interaction of two counter-moving streams (with relative velocity 2ϵ, $\epsilon = \mathbf{m}\boldsymbol{\nu} - n$) in the space of betatron phases. Therefore we can call this instability the resonance two-stream instability.

Note that instability of this kind is not specific for colliding beams. Provided Eq. (5.1) is valid interaction of the beam with any surroundings can lead to such two-stream instability.

Let us find the integral equation for collective modes near machine resonances. Taking into account Eqs. (1.5) and (1.6), we have

$$f^{(1)}_{mn} = -\frac{2e_1e_2}{\pi v_s}\cdot\frac{\mathbf{m}_1(\partial f_0(1))/\partial\mathbf{I}}{\Delta - (\mathbf{m}\cdot\boldsymbol{\nu} + n)} \times \int d\Gamma_2 e^{-in\phi_2}\rho_2(\phi_2)f(2)K(1 \mid 2) \tag{5.2}$$

and

$$f^{(2)}_{mn} = -\frac{2e_1e_2}{\pi v_s}\frac{\mathbf{m}_2(\partial f_0(2))/\partial\mathbf{I}}{\Delta - (\mathbf{m}\cdot\boldsymbol{\nu} - n)} \times \int d\Gamma_1 e^{-in\phi_1}\rho_1(\phi_1)f(1)K(2 \mid 1).$$

As formerly, we put $|\delta| \gg \nu_c$; here $\rho(\phi)$ is the phase distribution in the beams, and $\phi_{1,2} = \theta \pm \omega_s t$.

If the beams are short enough ($n\phi_b \ll 1$, $\rho(\phi) \to \delta(\phi)$) and assuming Gaussian distributions in betatron amplitudes, we can transform Eq. (5.2) into equations for the momenta

$$X(k) = \frac{1}{k}\sum_m \int d\Gamma\, f_m{}^{(1,2)}\rho(\phi) \times e^{-in\phi}(e^{-\mathbf{k}\cdot\mathbf{r}})_m{}^*$$

$$X_1 = \frac{2}{\pi}\sum_{m,n}\frac{\mathbf{m}\cdot\boldsymbol{\delta}}{\Delta - (\mathbf{m}\cdot\boldsymbol{\nu} + n)}\int\frac{d^2k_1}{kk_1}q_{m_x} \times (k_x \mid k_{x1})g_{m_z}(k_z \mid k_{z1})X_2(\mathbf{k}_1) \tag{5.3}$$

$$X_2 = \frac{2}{\pi}\sum_{m,n}\frac{\mathbf{m}\cdot\boldsymbol{\delta}}{\Delta - (\mathbf{m}\cdot\boldsymbol{\nu} - n)}\int\frac{d^2k_1}{kk_1}g_{m_x} \times (k_x \mid k_{x1})g_{m_z}(k_z \mid k_{z1})X_1(\mathbf{k}_1).$$

For simplicity, we consider identical unperturbed beams. The kernels $g_m(k \mid k_1)$ are

$$g_m(k \mid k_1) = \exp\left(\frac{k^2 + k_1{}^2}{2}\right) I_{|m|}(k\cdot k_1).$$

Because of the condition (5.1), we can omit nonresonance terms in Eq. (5.3). This yields the dispersion equation

$$1 = \Lambda_m(\mathbf{m}\cdot\boldsymbol{\delta})\left(\frac{1}{\Delta - \epsilon} - \frac{1}{\Delta + \epsilon}\right), \tag{5.4}$$

with $\epsilon = \mathbf{m}\cdot\boldsymbol{\nu} + n$; the Λ_m are the eigenvalues of the integral equation

$$\Lambda_m X(k) = \frac{2}{\pi}\int\frac{d^2k}{kk_1}g_{m_x}(k_x \mid k_{x1}) \times g_{m_z}(k_x \mid k_{x1})X(k_1), \tag{5.5}$$

which are positive numbers because of positive definiteness of the kernel. Equations (5.4) and (5.5) coincide with that obtained and studied in Ref. 1. Note also the coincidence of Eq. (5.4) with the dispersion equation for two-stream instability in plasma physics.

The roots of Eq. (5.4) can be written as

$$\Delta = \pm\sqrt{\epsilon^2 + 2\Lambda_m\epsilon(\mathbf{m}\cdot\boldsymbol{\delta})}. \tag{5.6}$$

This means that oscillations will be unstable if

$$\epsilon\cdot(\mathbf{m}\cdot\boldsymbol{\delta}) < 0 \tag{5.7}$$

and

$$|\epsilon| < 2\Lambda_m|(\mathbf{m}\cdot\boldsymbol{\delta})|. \tag{5.8}$$

Exactly in the resonance $\epsilon = 0$ oscillations are stable (there is no modulation of the relative phase for modes with m and $-m$).

Let us consider the stability conditions (5.7) and (5.8) for the principal modes of oscillations (i.e., for modes with smallest m). We take δ_x and δ_z to be positive, as for e^+e^-, $p\bar{p}$ and e^-p colliding beams. The betatron tunes ν_x and ν_z will be taken to be in the region

$$0 < \nu_x, \nu_z < 1.$$

Analogous to Eq. (4.6), Λ_m can be evaluated as

$$\Lambda_m \simeq \frac{1}{(m_x\cdot m_z)}\cdot\frac{1}{m_x{}^2(\sigma_x/\sigma_x) + m_z{}^2(\sigma_x/\sigma_z)}, \quad |m_x|, |m_z| > 1, \tag{5.9}$$

where σ_x and σ_z are the radial and vertical beam sizes respectively.

In the plane (ν_x, ν_t), the conditions (5.7) and (5.8) determine bands where oscillations are unstable. These bands lie below lines of machine resonances. The widths of one-dimensional resonances decrease as $1/m^2$; i.e., for $m_x = 1$, $\epsilon_0 = 2\delta_0$; $m_x = 2$, $\epsilon_0 = 2\delta_0/4$; $m_x = 3$, $\epsilon_0 = 2\delta_0 9$.

The relative importance of two-dimensional resonances ($m_x m_z \neq 0$) depends on beam parameters. For a ribbon beam ($\sigma_x \gg \sigma_z$ and $\delta_z \gg \delta_x$), the widths are

$$\epsilon_0 \simeq 2\delta_z \left(\frac{\sigma_z}{\sigma_x}\right) \frac{1}{|m_x|\, m_z^2}. \tag{5.10}$$

The broadest are the resonances $\nu_x + \nu_z = n$ and $\nu_z = \nu_x$:

$$\epsilon_0 \simeq 2\delta_z \left(\frac{\sigma_z}{\sigma_x}\right). \tag{5.11}$$

The instability bands for a ribbon beam and multipole numbers below 3 are shown in Fig. 1.

For round beams $\sigma_x = \sigma_z$ and $\delta_x = \delta_z$, and it can be seen from Eqs. (5.4) and (5.6) that there is no instability along $\nu_z = \nu_x$ and the most dangerous two-dimensional resonance is $\nu_z + \nu_x = n$ (see Fig. 2).

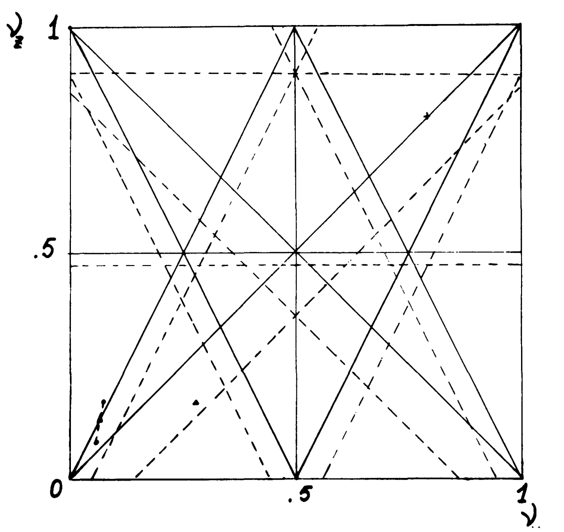

FIGURE 1 The position of instability bands for resonancies with multipole numbers less than 3. Flat beam $\sigma_x \gg \sigma_z$. The value $\Delta\nu_z \alpha(N/\sigma_x\sigma_z)$ is equal $\Delta\nu_z = 0.05$.
—— • working point VEPP-2M,
—— ▴ SPEAR,
—— + DCI.

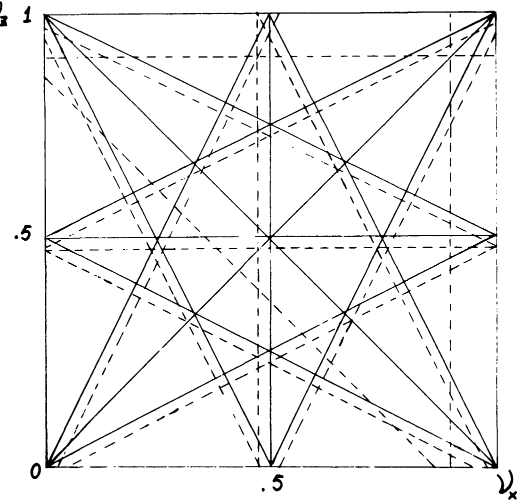

FIGURE 2 The same for round beam $\sigma_x = \sigma_z$, $\Delta\nu_x = \Delta\nu_z = 0.05$.

When beam currents are increased, the lower borders of stopbands move down machine resonances. For round beams, and $\beta_x = \beta_z$ at the interaction point, the working point moves parallel to the resonance $\nu_z = \nu_x$. This can make the region close to $\nu_z = \nu_x$ preferable for the working point.

For flat beams $\sigma_x \gg \sigma_z$, the working point moves mainly along ν_z. This time, placing of the working point as close as possible to the axis ν_z allows one to avoid the dangerous resonances $\nu_x = \nu_z$, $\nu_x + \nu_z = n$ and $\nu_z = 2\nu_x$ (see Fig 1).

Cooling of beams can in principle stabilize the two-stream instability

$$|\mathbf{m}\cdot\boldsymbol{\lambda}| > \Lambda_m |\mathbf{m}\cdot\boldsymbol{\delta}|. \tag{5.12}$$

For real parameters, the condition (5.10) is probably valid for oscillations with high m.

This instability can be weakened by difference in parameters of colliding beams. One of these possibilities was discussed in Sections 3 and 4 and is caused by collisions of the beams with the same density but different lateral sizes. In principle, detuning of frequencies for one beam from machine resonances can increase the thresholds in (5.8) in $(\nu/\delta) \gg 1$ times. This can, however, increase the influence of the resonances (1.8).

6. *Synchrobetatron Modes*

The spectrum of synchrobetatron modes is determined by Eq. (1.10). With zero dispersion at interaction points, Eq. (1.10) differs from the equations studied in Sections 2–5 by factors

$$q_{1,2} = \int_0^\infty dI_c \rho(I_c) J^2_{m_c}(z\phi)\,|_{1,2}.$$

Therefore the increments and stability criteria for synchrobetatron modes can be obtained from appropriate equations of Sections 2–5 by the replacements

$$\delta_1 \to q_2\delta_1, \quad \delta_2 \to q_1\delta_2. \qquad (6.1)$$

For bunches of the same length with Gaussian distributions of synchrotron amplitudes

$$\delta = \frac{1}{2\pi\phi_b{}^2}\exp\left(-\frac{\phi^2}{2\phi_b{}^2}\right),$$

where $\phi_b = l_b/2R_0$, l_b is the bunch length, and q_1 and q_2 are

$$q = \exp(-z^2\phi_0{}^2)I_{m_c}(z^2\phi_b{}^2)$$

$$= \begin{cases} \left(\dfrac{Z\phi_b}{2}\right)^{2m_c}\dfrac{1}{(m_c!)^2}: |Z\phi_b| < |m_c| \\[2ex] \dfrac{1}{Z\phi_b}: \left|\dfrac{Z\phi_b}{m_c}\right| > 1, \end{cases} \qquad (6.2)$$

$$Z = \epsilon_T + \mathbf{m}_T\cdot\frac{d\boldsymbol{\nu}_T}{d\ln\omega_0}. \quad \epsilon_T = \mathbf{M}_T\cdot\boldsymbol{\nu}_T + n.$$

Equation (6.2) shows that the powers of synchrobetatron resonances depend on machine chromaticity. For short beams ($|z\phi_b| \ll m_c$) and given transverse tuning ϵ_T, changing the sign of the chromaticity ($d\nu/d\ln R_0$) can stabilize some modes because this decreases the resonance power.

Now let us write down maximum increments for principal oscillations. For round beams with $\sigma_1 = \sigma_2$, Eqs. (4.2), (4.5), (4.6) and (6.2) give

$$\mathrm{Im}\Delta \simeq \frac{\delta}{9}\cdot\frac{l_b}{2\pi R_0}\left|\epsilon_T{}^{(1)} - \frac{d\nu}{d\ln R_0}\right| \qquad (6.3)$$

for modes (1.8) and $m_{z1} = 1$, $m_{z2} = -3$, $m_{c1} = 1$, $m_{c2} = 0$, $|Z\phi_b| < 1$. For two-stream instability (5.6) and (6.2) yield

$$\mathrm{Im}\Delta \simeq \delta\cdot\frac{l_b}{2\pi R_0}\left|\epsilon_T - \frac{d\nu}{d\ln R_0}\right|,$$

$$m_z = 1,\ m_x = 0.$$

CONCLUSION

Thus the coherent instabilities due to beam-beam effects cover the working plane (ν_x, ν_z) with a network of bands placed near machine resonances where oscillations are unstable. Widths of these bands are proportional to the linear beam-beam tune shift $\Delta\nu_0$ and decrease with multipole numbers $\{m\}$. The frequency spread in the beam $\sim 3/8\, m_T\Delta\nu_0$ increases like m_T and oscillations with high multipole numbers can be stabilized by Landau damping. The stability of the principal modes should be guaranteed by suitable position of the working point.

Among the various configurations of beams practically the most interesting seems in practice to be to collide beams with the same density but different transverse sizes. In this case, the thresholds increase $(\sigma_2/\delta_1)^2 \gg 1$ times (see Section 4). The luminosity for such beams is

$$L \simeq \omega_0\frac{N_1N_2}{(2\pi)^2\sigma_2{}^2}, \quad \sigma_2 \gg \sigma.$$

Taking into account Eq. (4.5), this can be written as

$$L \simeq \omega_0\frac{N_2\epsilon_0\gamma}{2\pi r_0\beta},$$

where ϵ_0 is the distance to the dangerous resonance, $r_0 = e^2/Mc^2$, and β is the betatron function at the interaction point.

For the parameters of UNK[9] and $N_2 = 10^{12}$, $\epsilon_0 = 10^{-2}$, we get

$$L \simeq 3\cdot 10^{31}\,\mathrm{cm}^{-2}\mathrm{sec}^{-1},$$

which is 10 times higher than the luminosity when colliding identical beams.

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