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SCALE INVARIANCE IN SUPERSPACE

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A B S T R A C T

We discuss super-Weyl transformations in the context of a Wess-Zumino superspace. It is shown that the component transformations are ordinary scalings, chiral transformations and S supersymmetry and that the physically significant fields are those necessary for superconformal gravity.

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Superconformal gravity has excited considerable interest recently, primarily because the superconformal group may be extended by $U(N)$ internal symmetries, and hence extended versions of the theory may be candidates for a unified treatment of all interactions. The subject has been approached from both a space-time and superspace viewpoint. In the latter case, a detailed treatment of the linearized version has been given [1] and a somewhat different approach to the full theory has also been discussed [2].

We recall that, for ordinary supergravity [*] a tangent space vector V^A to the "manifold" with co-ordinates $Z^M = \{x^m, \theta^\alpha, \bar{\theta}_{\dot{\alpha}}\}$, transforms under its structure group as

$$\delta V^A = V^B L_B{}^A \tag{1}$$

with

$$\begin{aligned} L_{ab} &= - L_{ba} \\ L_\alpha{}^\beta &= -\tfrac{1}{2} (\sigma^{ab})_\alpha{}^\beta L_{ab} \\ L^{\dot{\alpha}}{}_{\dot{\beta}} &= -\tfrac{1}{2} (\bar{\sigma}^{ab})^{\dot{\alpha}}{}_{\dot{\beta}} L_{ab} \end{aligned} \tag{2}$$

i.e., the tangent space group is the super-Lorentz group [3]. In Ref. 2) the structure of this group was amended by additional terms of the form

$$\begin{aligned} L'_a{}^b &= \delta_a^b L \\ L'_\alpha{}^\beta &= \tfrac{1}{2} \delta^\alpha_\beta L \end{aligned} \tag{3}$$

The parameter L here corresponds to superdilations and there are corresponding changes in the superconnection (by the addition of a dilation gauge field) and the supercurvature. Observe that in this approach the proper superconformal part of the superconformal group is not gauged. However, in ordinary space-time a theory has been obtained by gauging the entire superconformal group [4]. Demanding local Q supersymmetry produces the result that the vierbein e_m^a, the gravitino χ_m and the chiral gauge field A_m

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*) Our conventions are essentially those of J. Wess, Ref. 3).

are the independent physical fields. The gauge fields corresponding to proper conformal transformations (K symmetry), Weyl transformations (D symmetry) and S supersymmetry may be eliminated, either through non-propagating equations of motion or by imposing constraints on the group curvature. (The ordinary Lorentz connection is also expressible in terms of the other fields.) Indeed, in a preliminary paper [5] it was shown that gauging the ordinary conformal group leads to scale invariant gravity. Thus the final superconformal action has local A (chiral), S, D and Lorentz symmetries as well as being generally co-ordinate invariant, but only three basic physical fields. In view of these considerations, it seems worthwhile to investigate the possibility that superconformal gravity may be derived from a superspace approach based on the structure group (1) together with appropriately defined superscalings of the supervierbein.

In this note we show that this approach may be carried through with some success. We are able to restrict the basic fields appearing to the right number and also obtain the correct local space-time symmetries. We are able to compare our results with the linearized formulation [1] and obtain agreement there.

We first recall the facts concerning the structure of a superspace [3] based on the structure group (1). Adopting a second-order approach restrictions are imposed on some of the "orthonormal" components of the supertorsion. With the and of the Bianchi identities, the non-vanishing components may be expressed in the form :

$$
\begin{aligned}
T_{\beta\dot{\gamma}}{}^{a} &= 2i(\sigma^{a})_{\beta\dot{\gamma}} \\
T_{\alpha b\gamma} \rightarrow T_{\alpha,\beta\dot{\beta},\gamma} &= \frac{i}{4}\left\{\varepsilon_{\beta\gamma}G_{\alpha\dot{\beta}} - 3\varepsilon_{\alpha\gamma}G_{\beta\dot{\beta}} - 3\varepsilon_{\alpha\beta}G_{\gamma\dot{\beta}}\right\} \\
T_{\alpha b\dot{\gamma}} \rightarrow T_{\alpha,\beta\dot{\beta},\dot{\gamma}} &= -2i\,\varepsilon_{\alpha\beta}\varepsilon_{\dot{\beta}\dot{\gamma}}R^{+} \\
T_{ab\gamma} \rightarrow T_{\alpha\dot{\alpha},\beta\dot{\beta},\gamma} &= -2\varepsilon_{\dot{\alpha}\dot{\beta}}W_{\alpha\beta\gamma} \\
&+\frac{1}{2}\varepsilon_{\alpha\beta}\left\{D_{\dot{\alpha}}G_{\gamma\dot{\beta}} + D_{\dot{\beta}}G_{\gamma\dot{\alpha}}\right\} + \frac{1}{2}\varepsilon_{\dot{\alpha}\dot{\beta}}\left\{\varepsilon_{\beta\gamma}D_{\alpha}R + \varepsilon_{\alpha\gamma}D_{\beta}R\right\}
\end{aligned}
\tag{4}
$$

and complex conjugates. All the components of the supercurvature may then be expressed in terms of three superfields : R, $G_{\alpha\dot{\alpha}}$ (Hermitian) and a totally symmetric $W_{\alpha\beta\gamma}$. In addition one has the following identities

$$
\begin{aligned}
\mathcal{D}_\alpha R^+ &= 0 \\
\mathcal{D}_{\dot{\alpha}} W_{\alpha\beta\gamma} &= 0 \\
\mathcal{D}^\alpha G_{\alpha\dot{\alpha}} &= \mathcal{D}_{\dot{\alpha}} R^+ \\
\mathcal{D}^\alpha W_{\alpha\beta\gamma} &= \frac{i}{2}\mathcal{D}_\beta{}^{\dot{\beta}} G_{\gamma\dot{\beta}} + \frac{i}{2}\mathcal{D}_\gamma{}^{\dot{\beta}} G_{\beta\dot{\beta}}
\end{aligned} \tag{5}
$$

We now define super-Weyl transformations as certain scalings of the supervierbein which preserve the kinematic constraints on the supertorsion. As in the two-dimensional case [6] simple rescalings are too restrictive and one finds a consistent scheme is obtained if

$$
\begin{aligned}
\delta E_M{}^a &= (\Sigma + \Sigma^+) E_M{}^a \\
\delta E_M{}^\alpha &= (2\Sigma^+ - \Sigma) E_M{}^\alpha - \frac{i}{2} E_M{}^a (\bar{\sigma}_a)^{\dot{\alpha}\alpha} \mathcal{D}_{\dot{\alpha}} \Sigma^+ \\
\delta E_M{}^{\dot{\alpha}} &= (2\Sigma - \Sigma^+) E_M{}^{\dot{\alpha}} + \frac{i}{2} E_M{}^a (\bar{\sigma}_a)^{\dot{\alpha}\alpha} \mathcal{D}_\alpha \Sigma
\end{aligned} \tag{6}
$$

where the scaling superparameter Σ is covariantly chiral ;

$$
\mathcal{D}_{\dot{\alpha}} \Sigma = 0 \tag{7}
$$

Using the techniques given in the last paper of Ref. 3) one verifies that the torsion constraints are preserved and that

$$
\begin{aligned}
\delta E &= 2E(\Sigma + \Sigma^+) \\
\delta \Omega_{M,\alpha\beta} &= E_{M\alpha}\mathcal{D}_\beta \Sigma + E_{M\beta}\mathcal{D}_\alpha \Sigma + (\sigma^{ab})_{\alpha\beta} E_{Ma} \mathcal{D}_b(\Sigma + \Sigma^+)
\end{aligned} \tag{8}
$$

whilst the geometrical fields behave as follows :

$$\begin{aligned} \delta R^{+} &= -2(2\Sigma^{+}-\Sigma)R^{+} + \tfrac{1}{4}D_{\alpha}D^{\alpha}\Sigma \\ \delta G_{\alpha\dot\alpha} &= -(\Sigma+\Sigma^{+})G_{\alpha\dot\alpha} + iD_{\alpha\dot\alpha}(\Sigma^{+}-\Sigma) \\ \delta W_{\alpha\beta\gamma} &= -3\Sigma W_{\alpha\beta\gamma} \end{aligned} \tag{9}$$

At first sight these transformations bear little resemblance to the linearized case. However, because of (7) and the identity, valid for any scalar superfield V

$$\{D_{\alpha}D_{\beta}D_{\gamma} + 4R^{+}(\varepsilon_{\alpha\beta}D_{\gamma} - \varepsilon_{\alpha\gamma}D_{\beta})\}V = 0 \tag{10}$$

we may write

$$\Sigma^{+} = (D_{\beta}D^{\beta} + 8R^{+})u^{+} \tag{11}$$

In the linearized limit (R, G, W small) one finds

$$\begin{aligned} \delta R^{+} &= \tfrac{1}{4}D_{\alpha}D^{\alpha}D^{\dot\alpha}D_{\dot\alpha}\,U \\ \delta G_{\alpha\dot\alpha} &= iD_{\alpha\dot\alpha}\{D_{\beta}D^{\beta}U^{+} - D^{\dot\beta}D_{\dot\beta}U\} \\ \delta W_{\alpha\beta\gamma} &= 0 \end{aligned} \tag{12}$$

Hence with

$$U = -2D^{\alpha}\Sigma_{\alpha}$$

we have

$$\delta R^{+} = -\tfrac{1}{2}D_{\alpha}D^{\alpha}D^{\dot\alpha}D_{\dot\alpha}D^{\beta}\Sigma_{\beta}$$

$$\delta G_{\alpha\dot\alpha} = 2iD_{\alpha\dot\alpha}(D^{\dot\beta}D_{\dot\beta}D^{\gamma}\Sigma_{\gamma} - D_{\beta}D^{\beta}D_{\dot\gamma}\bar\Sigma^{\dot\gamma}) \tag{13}$$

in agreement with the linearized results.

As usual we may choose a gauge in which for $\theta, \bar{\theta} = 0$ the supervierbein takes the form [3] :

$$E_m{}^a = e_m{}^a \; ; \; E_m{}^\alpha = \tfrac{1}{2}\chi_m{}^\alpha \; ; \; E_m{}^{\dot\alpha} = \tfrac{1}{2}\bar\chi_m{}^{\dot\alpha}$$

$$E_\mu{}^a = E_\mu{}^{\dot\alpha} = E_{\dot\mu}{}^\alpha = E_{\dot\mu}{}^a = 0$$

$$E_\mu{}^\alpha = \delta_\mu^\alpha \; ; \; E_{\dot\mu}{}^{\dot\alpha} = -\delta_{\dot\mu}^{\dot\alpha} \qquad (14)$$

where $e_m{}^a$ is the space-time vierbein and χ_m the gravitino field. Then utilizing (7) we may write

$$\begin{aligned}\Sigma = {} & f + i\frac{g}{3} + i\theta^\mu \phi_\mu + \tfrac{1}{2}\theta^\mu\theta_\mu F \\ & + i\theta^\mu(\sigma^m)_{\mu\dot\mu}\bar\theta^{\dot\mu}\partial_m\left(f + i\frac{g}{3}\right) \\ & + \tfrac{1}{2}\theta^\mu(\sigma^m)_{\mu\dot\mu}\bar\theta^{\dot\mu}\chi_m{}^\rho\phi_\rho + \cdots\cdots \qquad (15)\end{aligned}$$

In ordinary supergravity, the leading $(\theta, \bar\theta = 0)$ components of R and $G_{\alpha\dot\alpha}$ are the auxiliary fields necessary to close the supersymmetry algebra off-shell [7]. With

$$R^* = A + \ldots$$

and

$$-\tfrac{1}{2}(\bar\sigma_m)^{\dot\alpha\alpha} G_{\alpha\dot\alpha}\Big|_{\theta,\bar\theta=0} = A_m$$

one finds

$$\delta A = -2(f - ig)A + \tfrac{1}{2}F$$

$$\delta A_m = \frac{2}{3}\partial_m g + \tfrac{1}{2}\left(\phi^\mu\chi_{m\mu} + \bar\phi_{\dot\mu}\bar\chi_m{}^{\dot\mu}\right) \qquad (16)$$

Hence in this approach we may use F to set A to zero, whilst the axial vector boson is the chiral gauge field (g being the chiral parameter). For the vierbein and gravitino one finds

$$\delta e_m{}^a = 2 f e_m{}^a$$

$$\delta \chi_m = f \chi_m + g \gamma_5 \chi_m + \gamma_m \phi \qquad (17)$$

where the four-component Majorana spinors are

$$\phi = \begin{pmatrix} \phi_\alpha \\ \bar{\phi}^{\dot\alpha} \end{pmatrix}$$

Thus we may identify f with ordinary Weyl transformations, g with chiral transformations and ϕ with part of S supersymmetry. Furthermore, our transformations for the physical fields are in agreement with those previously obtained modulo conventions [4),7)].

In summary, we feel that this approach to superconformal gravity is an economical one in the sense that there is no need to introduce extra gauge fields or to modify the structure (1) of superspace. We should point out that one of the constraints adopted in the space-time approach, namely the vanishing of the translational gauge curvature is implicit in our choice for the supertorsion (i.e., $T_{bc}{}^a = 0$). It remains to construct an action and show that our approach is indeed equivalent to those given earlier. A possible candidate is

$$S = \int d^8 z \; E \left\{ G^{\alpha\dot\alpha} G_{\alpha\dot\alpha} - 4 R R^+ \right\} \qquad (18)$$

which is readily verified to be invariant (as it is in the linearized case). The simpler action [8)]

$$S = \int d^8 z \; \frac{E}{R} W_{\alpha\beta\gamma} W^{\alpha\beta\gamma} + h.c. \qquad (19)$$

is also invariant under superscalings. It is likely that both actions lead to the same theory and hence imply that the integrands are related by a Gauss-Bonnet theorem appropriate to the topology of superspace. These questions are under investigation and the results will be presented elsewhere [9)].

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