Particle Accelerators, 1993, Vol. 42(3–4), 171–179
Reprints available directly from the publisher
Photocopying permitted by license only

APPLICATION OF DIFFERENTIAL ALGEBRA DA TOOLS TO MAGNETIC FIELD CALCULATIONS*

JAMES B. MURPHY[1] and MICHAEL F. REUSCH[2]

[1]*National Synchrotron Light Source, Brookhaven National Laboratory, Upton, NY 11973*
[2]*Grumman Space Systems, 4 Independence Way, Princeton, NJ 08540*

(Received 1 July 1992; in final form 24 June 1993)

The calculation of the magnetic field for an arbitrary-geometry air-core magnet is performed by numerous computer codes. However, the accelerator scientist is often interested in the multipole content of the fields, which can be difficult to compute accurately and to high orders for arbitrary coil geometry. Applying the tools of differential algebra, a computer code has been devised that can compute arbitrary-order multipoles and the changes in these multipoles as a function of the coil locations for air-core magnets.

KEY WORDS: Electromagnetic Field Calculations, Magnetic Fields, Magnets: Air-Core

1 INTRODUCTION

Compact electron storage rings with superconducting dipole magnets are being built as sources of soft x-rays for x-ray lithography.[1] In several of these designs the dipole magnets are of the coil-dominated air-core variety. For example, the SXLS ring, under construction at BNL, contains two 180° air-core superconducting dipoles with a nominal field of 3.87 T. Because the machine is so compact, with a circumference of only 8.5 meters, the dipoles must be "combined-function" magnets that contain a gradient for vertical focusing and a sextupole component for chromaticity correction. The field is generated by a set of coils shown schematically in Figure 1.

The magnetic field in an air-core magnet is given by the Biot-Savart Law,[2]

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int d^3r' \frac{\vec{J}(\vec{r'}) \times (\vec{r} - \vec{r'})}{|\vec{r} - \vec{r'}|^3} \tag{1}$$

where $\vec{J}(\vec{r})$ is the current density in the conductor and $\vec{r}$ is the observation point.

* Work performed under the auspices of the U.S. Department of Energy and funded by the Defense Advanced Research Projects Administration

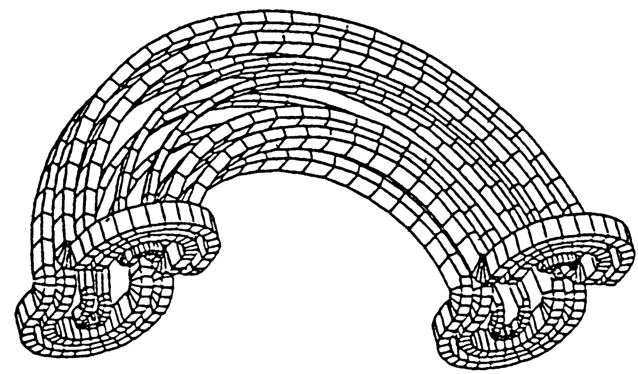

FIGURE 1: Coil geometry for 180° air core superconducting dipole magnet

For a given coil configuration, the field can in principle be computed to the limits of the computer's precision, since there is no cumbersome mesh required as in an iron dominated magnet. Numerous computer codes[3–5] exist to carry out air-core magnetic field calculations. The difficulty remains in obtaining the correct multipole coefficients and determining the sensitivity of these to the coil locations.

2 THREE-DIMENSIONAL NONISOMAGNETIC FIELDS AND MULTIPOLE COEFFICIENTS

For the complicated coil geometry given in Figure 1, the magnetic field is truly three-dimensional, the curvature of the magnet is large (ρ = 60 cm) and, for an air-core magnet, the field is nonisomagnetic. The general expression for the midplane symmetric magnetic field in curvilinear coordinates (x, y, s) is given by Brown and Servranckx[6]. For the present purpose it suffices to write out the expression only to second order in the transverse coordinates x and y,

$$B_x(x,y,s) = A_{11}y + A_{12}xy + \cdots \tag{2a}$$

$$B_y(x,y,s) = A_{10} + A_{11}x + \frac{1}{2!}A_{12}x^2 + \frac{1}{2!}A_{30}y^2 + \cdots \tag{2b}$$

$$B_s(x,y,s) = \frac{1}{(1+hx)}\left[\dot{A}_{10}y + \cdots\right. \tag{2c}$$

where $A_{1,n}(s) \equiv \partial^n B_y(s)/\partial x^n$, $A_{30}(s) \equiv -\ddot{A}_{10} - A_{12} - hA_{11}$, $\dot{A}_{10} \equiv dA_{10}/ds$ and $h \equiv 1/\rho(s)$.

The coefficients $A_{1,n}(s) = \partial^n B_y(s)/\partial x^n$, are the multipole coefficients widely used in accelerator physics; $A_{10} \equiv$ Dipole, $A_{11} \equiv$ Quadrupole and $A_{12} \equiv$ Sextupole, etc. There are two other types of terms in Equation (2): nonisomagnetic terms, for example, $\dot{A}_{10}$; and combined function feed-up terms such as hA_{11}.

Several methods to obtain the multipole coefficients have been used in the past:

1. Numerical finite differencing to compute derivatives.
2. Fitting of the pointwise field data by polynomials in x and/or y.
3. Fast Fourier Transform (FFT) of the field on a circle surrounding the reference trajectory.

Each of these methods suffers from drawbacks for nonisomagnetic combined-function bending magnets.

Numerical finite differencing yields unacceptable results for all but the lowest order multipole coefficients.

Typical problems encountered with polynomial fitting of data are as follows:

How many data points are required?

Over what range in x or y should the fit be applied? And,

What is the maximum order of the polynomial to be used?

Finally, the FFT method, which is best suited for straight multipole elements such as quadrupoles or sextupoles, runs into trouble with the nonisomagnetic and combined-function feed-up terms.

3 APPLICATION OF DA TOOLS TO FIELD COMPUTATIONS

What is really desired is to be able to obtain a Taylor series expansion of equation (1) in x, y and s directly. This is precisely what the DA software of Berz[7–8] can do. In this way the multipole coefficients, the nonisomagnetic terms, and the combined-function feed-up terms are obtained without any of the problems associated with the above methods. We have written a computer code to perform these magnetic field calculations and added DA capabilities to be able to compute the desired multipoles throughout the magnet $[A_{1,n}(s)]$ and the changes in these multipoles due to positional errors in the coil placement[9].

The coils in the SXLS air-core superconducting magnet are constructed out of arcs or straight sections of rectangular cross section. For a magnet constructed of arcs of rectangular cross section with a uniform current density J_i, the vertical field in the midplane is given as a single numerical integral over the angular extent of the coil, and the radial and vertical integrals in Equation (1) can be carried out analytically:

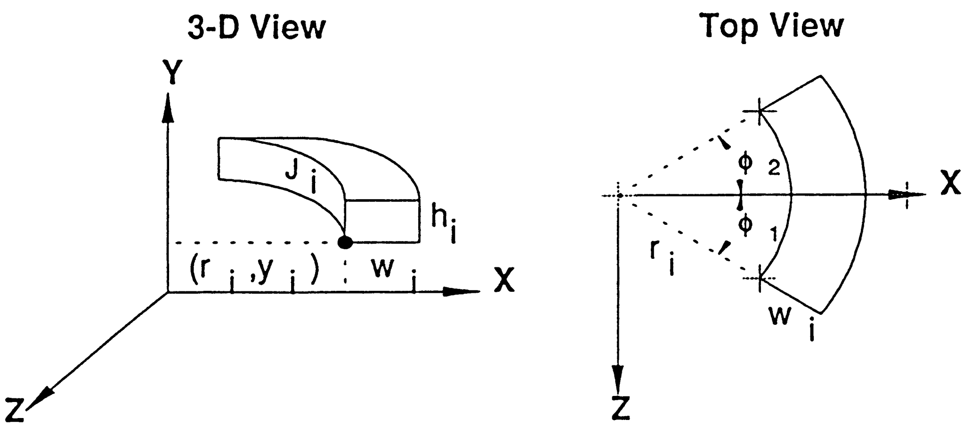

FIGURE 2: Geometry of arc of rectangular cross section

$$B_y(x) = \frac{\mu_o}{4\pi} \sum_{i=1}^{\text{coils}} J_i \int_{\phi_1}^{\phi_2} d\phi \left[y \ln \left(\frac{\sqrt{r^2 - 2rx\cos\phi + y^2 + x^2}}{x} + \frac{r}{x} - \cos\phi \right) \right. \quad (3)$$

$$- \frac{x}{2} \cos\phi \ln \left(\frac{\sqrt{r^2 - 2rx\cos\phi + y^2 + x^2} + y}{\sqrt{r^2 - 2rx\cos\phi + y^2 + x^2} - y} \right)$$

$$\left. - x \sin\phi \tan^{-1} \left(\frac{y(r - x\cos\phi)}{x \sin\phi \sqrt{r^2 - 2rx\cos\phi + y^2 + x^2}} \right) \right]_{r=r_i, y=y_i}^{r=r_i+w, y=y_i+h}$$

where r_i and y_i are the lower left hand coordinates of the coil location, w_i and h_i are the width and height of the coils, respectively (see Figure 2). For a midplane symmetric magnet the above sum need only be done over the top plane coils and the result is simply doubled. A simpler expression, where all the integrals can be done analytically, for a straight section of rectangular cross section is given in Reference 3.

From the point of view of the DA tools, the above expression is simply an analytic function of several variables:

$$B_y(x, r_1, y_1) = \frac{\mu_o}{4\pi} \sum_{i=1}^{\text{coils}} J_i \int_{\phi_1}^{\phi_2} d\phi F(\phi, x, r_1, y_1). \quad (4)$$

The DA tools provide for arbitrary-order differentiation with respect to any of the variables (ϕ, x_i, r_i, y_i). It must be noted that the derivatives are *not* computed as numerical finite difference quotients, which would give very poor accuracy.

4 MULTIPOLE COEFFICIENTS

The DA package can compute the Taylor coefficients, $\partial_x^n B_y(s)$, defined as follows,

$$B_y(x,s) = B_y(s) + x \cdot \partial_x B_y(s) + \frac{x^2}{2!} \cdot \partial_x^2 B_y(s) + \frac{x^3}{3!} \cdot \partial_x^3 B_y(s) + \cdot, \qquad (5)$$

where the $\partial_x^n B_y(s)$ are evaluated on the reference orbit and x is the deviation from the reference orbit in the radial direction. The Taylor coefficients are simply the multipole coefficients widely used in accelerator physics codes. The nonisomagnetic terms in equation (2) are also directly computed as $\partial_s^m \partial_x^n B_y(s)$ for arbitrary m and n. For a midplane symmetric magnet the other two field components, B_x and B_s, can be obtained from the $\partial_s^m \partial_x^n B_y(s)$ or an expression such as equation (3) can be used with the DA tools to produce these multipole components.

5 COIL POSITION TOLERANCES

The DA tools can also be used to compute the effects of coil positioning errors on the various multipole by producing a Taylor series of $\vec{B}$ with respect to the position of each of the coils, r_i and y_i, which serve as limits to the integrals in Equation (3). Such a tool is very valuable to set limits on the coil positioning errors during construction of the magnet. Table 1 shows some of the possible coefficients.

6 APPLICATION TO AIR-CORE SUPERCONDUCTING DIPOLE MAGNETS

The DA-equipped version of the Biot-Savart was used to layout the set of coils shown in Figure 1 for the superconducting dipole magnet for the SXLS storage ring at BNL. Figure 3 displays the calculated multipole components through the decapole term for half of the symmetric magnet, the center of the magnet is taken as $s = 0$. Near the ends of the magnets ($s \approx 1$ meter) where the field is highly nonisomagnetic the DA-equipped version of the Biot-Savart law is particularly adept at extracting the multipole coefficients as evidenced by the smooth, continuous behavior shown in Figure 3.

TABLE 1: Effects of Coil Positions on Multipoles

Derivative	Tolerance Coefficients
$\partial_{r_1} B_y(s)$	Change of dipole field with r_1
$\partial_{y_1} \partial_x^1 B_y(s)$	Change of quadrupole with y_1
$\partial_{y_1} \partial_x^3 B_y(s)$	Change of octupole with y_1

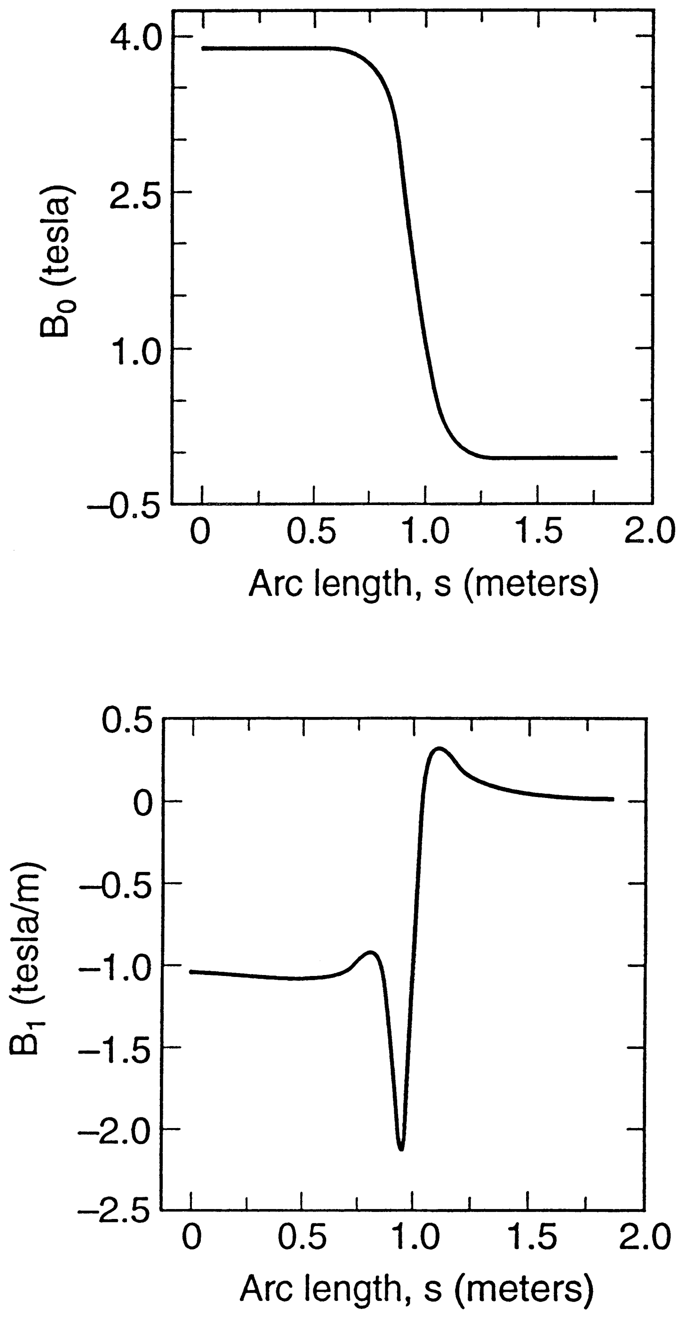

FIGURE 3: Multiple coefficients $B(x,s)=\sum B_n(s)\ x^n/n!$, vs. arc length for half of a symmetric 180° magnet (center of magnet is at $S=0$, $\rho=0.6037$ m).

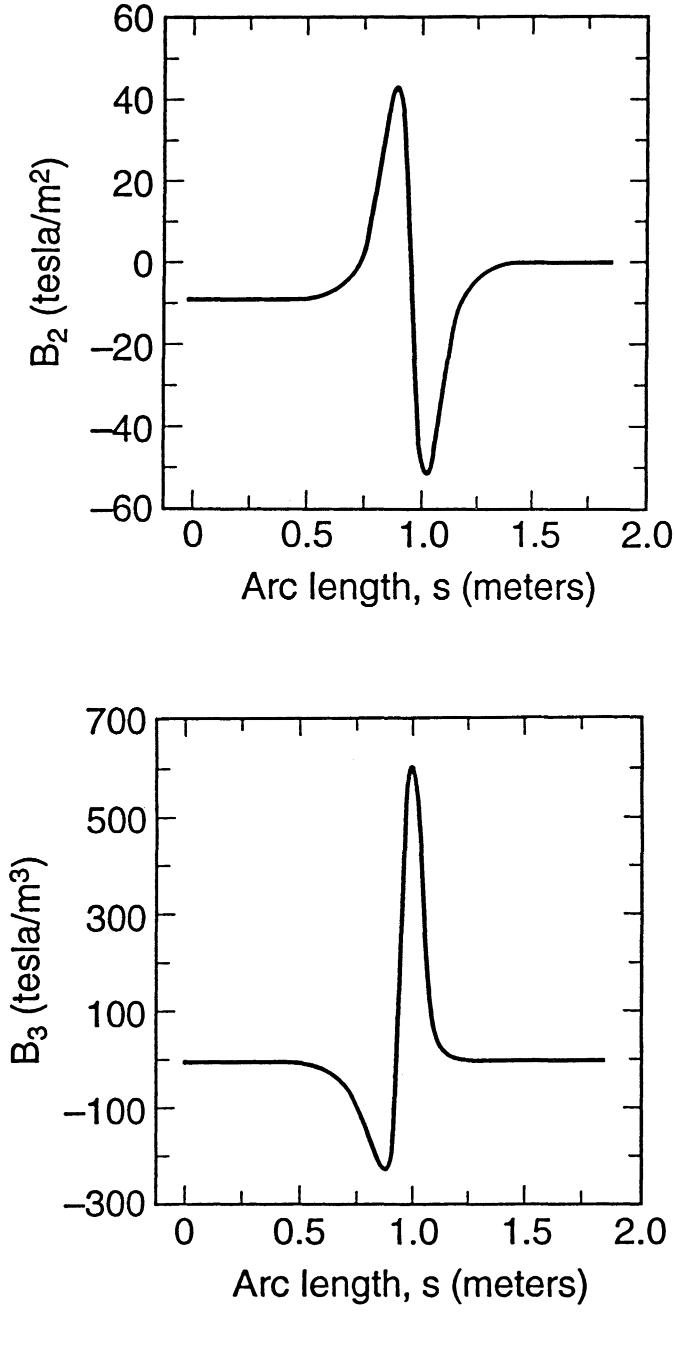

FIGURE 3: (Continued)

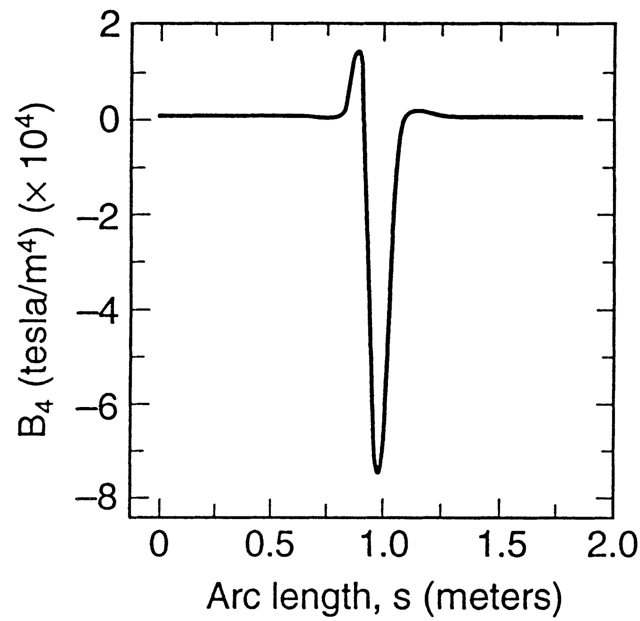

FIGURE 3: (Continued)

In Table 2 the multipole coefficients at the center of the magnet are given as well as the changes in these coefficients due to a 10-mil (.254-mm) deviation of the coils from their ideal locations. The deviations were taken in a direction to maximize the change to each of the multipoles independently and, in this case, to maintain midplane symmetry.

TABLE 2: Ideal Multipole Coefficients and Tolerances at the Magnet Center

Multipole	Ideal $A_{1i}(0)$	ΔA_{1i} Coil Position
A_{10} [T]	3.87	$\pm$ 0.009
A_{11} [T/m]	-1.05	$\pm$ 0.14
A_{12} [T/m^2]	-9.1	$\pm$ 3.4
A_{13} [T/m^3]	-5.1	$\pm$ 100
A_{14} [T/m^4]	595.1	$\pm$ 6,000

7 CONCLUSIONS

The numerical computation of magnetic field multipoles and the effects of coil position tolerances for arbitrary geometry air-core magnets is greatly facilitated by the use of differential algebra software tools. We have developed a computer code to carry out this analysis for air core magnets composed of straight sections and arcs of rectangular cross section. The resulting code has been applied to the design of a superconducting dipole magnets for a compact storage ring.

ACKNOWLEDGEMENTS

The authors are grateful to Etienne Forest of the Center for Beam Physics at LBL for introducing us to the DA tools. We thank Sushil Sharma for help in preparing the input data files.

REFERENCES

1. J.B. Murphy, *X-Ray Lithography Sources: A Review*, in Proceedings of the 1989, IEEE Particle Accelerator Conference, p. 757.
2. J.D. Jackson, *Classical Electrodynamics*, 2nd ed. (Wiley, New York, 1975).
3. A. Schleich & A. Seggesseman, *"Calculation of the Electromagnetic Forces Acting on the Coils of An Electromagnet"*, in Proceedings of the 3rd International Conference on Magnet Technology **MT-3**, p. 198 (1971).
4. S.J. Sackett, EFFI - *"A Code for Calculating the Electromagnetic Field Force, and Inductance in Coil Systems of Arbitrary Geometry,"* Lawrence Livermore National Laboratory report UCRL-52402 (1978).
5. TOSCA, Vector Fields Ltd, Oxford, England.
6. K. Brown and R.V. Servranckx, *"First and Second Order Charged Particle Optics,"* Standard Linear Accelerator Center report, SLAC PUB 3381 (1984).
7. M. Berz, *"Differential Algebraic Descriptions of Beam Dynamics to Very High Orders,"* Part. Accel. **24**, (1989), pp. 109-124.
8. The LBL version of Berz's DA package was used to perform all the DA-related calculations.
9. M.F. Reusch, E. Forest and J.B. Murphy, *"Particle Tracking and Map Analysis for Compact Storage Rings,"* in Proceedings of the 1991 IEEE Particle Accelerator Conference, p. 1651.