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PERIODIC TRANSMISSION-LINE MODE MEASUREMENTS

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Experimental measurements of group velocity and shunt impedance are presented for the $3\pi/4$ mode traveling-wave propagating at the velocity of light in a disk-loaded waveguide. Similar results are given for the same mode in a contoured cavity with drift-tube apertures, that is, with nose cones to control group velocity, intended to permit large apertures.

A discussion of measurement techniques is given, as well as some observations on higher-order mode suppression.

1. INTRODUCTION

It is well known that the locations of passbands in modular periodic structures are associated with the cutoff frequencies of the outer boundary, that is, with the waveguide before obstacle loading and, further, that Floquet's theorem establishes the propagation constants of the space harmonics in the structure. As a consequence, there are 'forbidden regions' on a dispersion diagram, for which at a specified frequency no real propagation constants exist.

The requirement of long pulse trains of high current for free-electron laser operation has motivated a search for a scheme to avoid an anticipated beam breakup,[1] particularly of the regenerative type. Because the mechanism of regenerative breakup presumably involves feedback and a forward wave 'slip' at nearly the velocity of light, the use of a traveling-wave acceleration mode for which the velocity of light line intersects the higher-order mode (HEM-11) passband in the forbidden zone (beyond the π-mode) suggests itself, since then there can be no higher-order mode propagation near the velocity of light (synchronism with the beam).

The above condition can be realized for the case of less than three obstacles per wavelength in the acceleration mode. Considerations of tuning the waveguide after fabrication indicate that the longitudinal mode choice ought to be a simple fraction; thus the $3\pi/4$-mode appears to be a practical choice.

It has been argued that such modes as this ought to be avoided for the reason that the Fourier spectrum of the rf pulse must then propagate in a highly dispersive structure.[2]

At some distance into the structure, the pulse is the sum of its frequency components, each component propagating with its own phase velocity and attenuation. Given an ideal pulse,

$$\begin{aligned} f(t) &= \cos\omega_0 t \qquad (-T < t < T) \\ &= 0 \qquad (T < t < -T) \end{aligned} \tag{1}$$

and its Fourier spectrum

$$F(\omega) = \int_{-\infty}^{+\infty} f(t)\bar{e}^{j\omega t}\, dt = \frac{\sin(\omega_0 - \omega)T}{\omega_0 - \omega} + \frac{\sin(\omega_0 + \omega)T}{\omega_0 + \omega}, \tag{2}$$

the pulse at some distance into the waveguide is the inverse transform with a modified kernel (the system transfer function)

$$f(z, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(\omega) e^{j(\omega t - \gamma z)}\, d\omega. \tag{3}$$

The real and imaginary components of the propagation constant ($\gamma = 2I + j\beta$) are not independent but are related by the Kramers-Kronig relation[3]

$$\beta_c = \frac{1}{\pi} \int_0^{\infty} \frac{d2I}{d\omega} \ln \left| \frac{\omega + \omega_c}{\omega - \omega_c} \right| d\omega, \tag{4}$$

where β_c is the propagation constant at frequency $\omega_c/2\pi$.

With a solution of Eq. (3), the energy gain of a particle transiting the waveguide is the line integral $V = \int f(z, t)\, dz$, from which it can be concluded that there will be little or no effect other than a slightly different energy gain than that theoretically anticipated from ideal wave propagation.

For the very long pulse lengths intended for free-electron lasers, these considerations are less important, since the spectral bandwidth is then much reduced.

In addition, because of wake-field effects,[4] what is desired is the largest beam aperture that can be tolerated considering the resulting decrement in shunt impedance. The obvious consequence of a wake field is, of course, a contribution to beam emittance; the effect can be so intense that each micropulse or bunch has a cometary appearance, sometimes referred to as a 'banana-shaped' bunch. The effect cannot be completely avoided in periodic structures of the sort presently used in microwave linacs, but may be ameliorated to some extent by means of large beam apertures. Large apertures, on the other hand, result in high group velocities and, therefore, long waveguides to obtain economical rf power usage. For the purpose of maintaining large beam apertures and acceptable group velocities, thick disks or mixed coupling (both electric and magnetic cell-to-cell coupling) may be used.

2. $3\pi/4$ CONTOURED CAVITIES

On the basis of contoured-cavity studies reported from Los Alamos National Laboratory[5] and unpublished data from the Accelerator Physics Branch, AECL, Chalk River, Ont., an investigation of the $3\pi/4$ traveling-wave mode with electric coupling in contoured cavities has been undertaken. For comparison, an experimental investigation was performed for a conventional disk-loaded waveguide; similar studies have been previously reported for 5, 4, 3, and 2 disks per wavelength but there does not appear to be any data for the $3\pi/4$-mode.[6]

The design of the basic cavity is shown in Fig. 1, where the cavity diameter (2b) was adjusted to provide propagation at the phase velocity of light for 2856 MHz, corresponding to the $3\pi/4$ mode for a specified aperture diameter (2a).

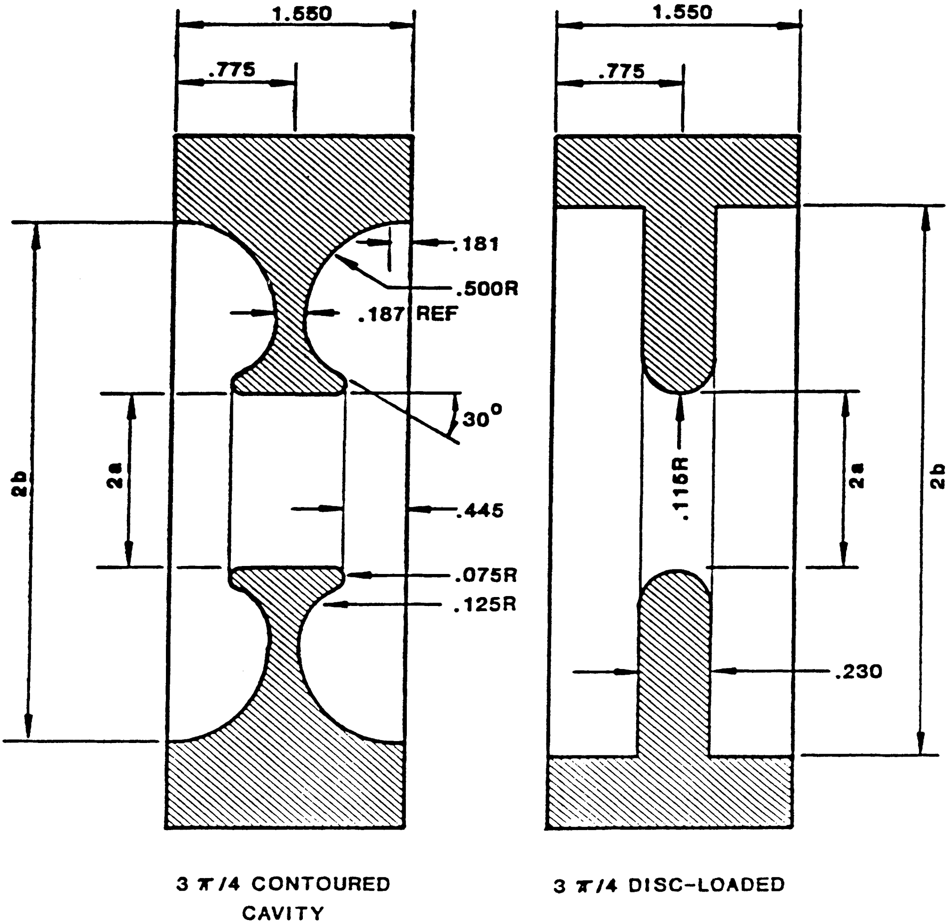

FIGURE 1 Drawings of cavity types studied. Dimensions are in inches.

A summary of the waveguide parameters is given in Table I and a display of the location of the next higher-order mode (HEM-11, lower branch) in Fig. 2. Referring to Table I, the group velocities can be represented reasonably well by the expression[7]

$$V_g/c = 0.44\left(\frac{a}{b}\right)^4. \tag{5}$$

Although a value for the loss factor (Q) is essential to interpretation of the data, a meaningful value cannot be obtained with the measurement techniques used. Microwave engineers would presumably estimate the order of magnitude from considerations of the TM-010 mode in a cylindrical cavity,

$$Q = \frac{\eta}{2R_s}\left(\frac{1}{P_{01}} + \frac{1}{\beta p}\right) \doteq \frac{\eta}{2R_s} \tag{6}$$

TABLE I

$2a$	$2b$	v_g/c	r/Q	$a_0^2/\sum a_n^2$
0.900	3.043	0.0034	48.3	0.618
1.000	3.059	0.0044	45.7	0.643
1.100	3.081	0.0066	43.8	0.667
1.200	3.108	0.0092	42.5	0.691
1.300	3.145	0.0130	39.5	0.716
1.400	3.185	0.0203	34.0	0.741
1.500	3.236	0.0257	31.3	0.763
1.600	3.297	0.0320	30.7	0.785
1.700	3.367	0.0451	29.0	0.806

$v_p = c$ at 2856 MHz
$2a$, aperture diameter (in)
$2b$, cavity diameter (in)
v_g/c, normalized group velocity
r/Q, shunt impedance (Ω/cm)
$a_0^2/\sum a_n^2$, fraction of power in synchronous space harmonic

where R_s is the surface resistivity (in ohms per square), and the approximation ignores the finite velocity of propagation.

The existence of a number of mesh computer programs,[8] on the other hand, permits a closer estimate of Q for an assumed surface resistivity. The Q of a particular structure for a specific longitudinal mode depends upon the coupling-aperture diameter, increasing slightly with increased coupling. For the $3\pi/4$ contoured cavity, a Q of 13,059 was obtained using URMEL, compared to 13,561 based upon Eq. (6). Scaling rules have been given for contoured cavity parameters operated resonantly,[9]

$$Q = 8.8 \times 10^8/\sqrt{f}$$
$$r = 2350\sqrt{f} \qquad \text{ohms/m},$$

where the gap to periodic length ratio is 0.6 and the beam channel diameter is small. For the data presented in Table I,

$$\frac{r}{Q} = \frac{0.03033f}{a} \qquad \text{ohms/cm},$$

where a is the aperture radius in centimeters, is a fairly accurate estimate before correction for space-harmonic content.

3. $3\pi/4$ DISK-LOADED CAVITIES

Precisely the same form of investigation has been completed for a conventional disk-loaded waveguide, that is, circular cylindrical waveguide periodically loaded with flat disks perforated with a fully-radiused aperture, as in Fig. 1. A summary of the waveguide parameters for this case is given in Table II.

A reasonably close fit to the group velocity data is given by

$$\frac{V_g}{c} = 1.54\left(\frac{a}{b}\right)^4,$$

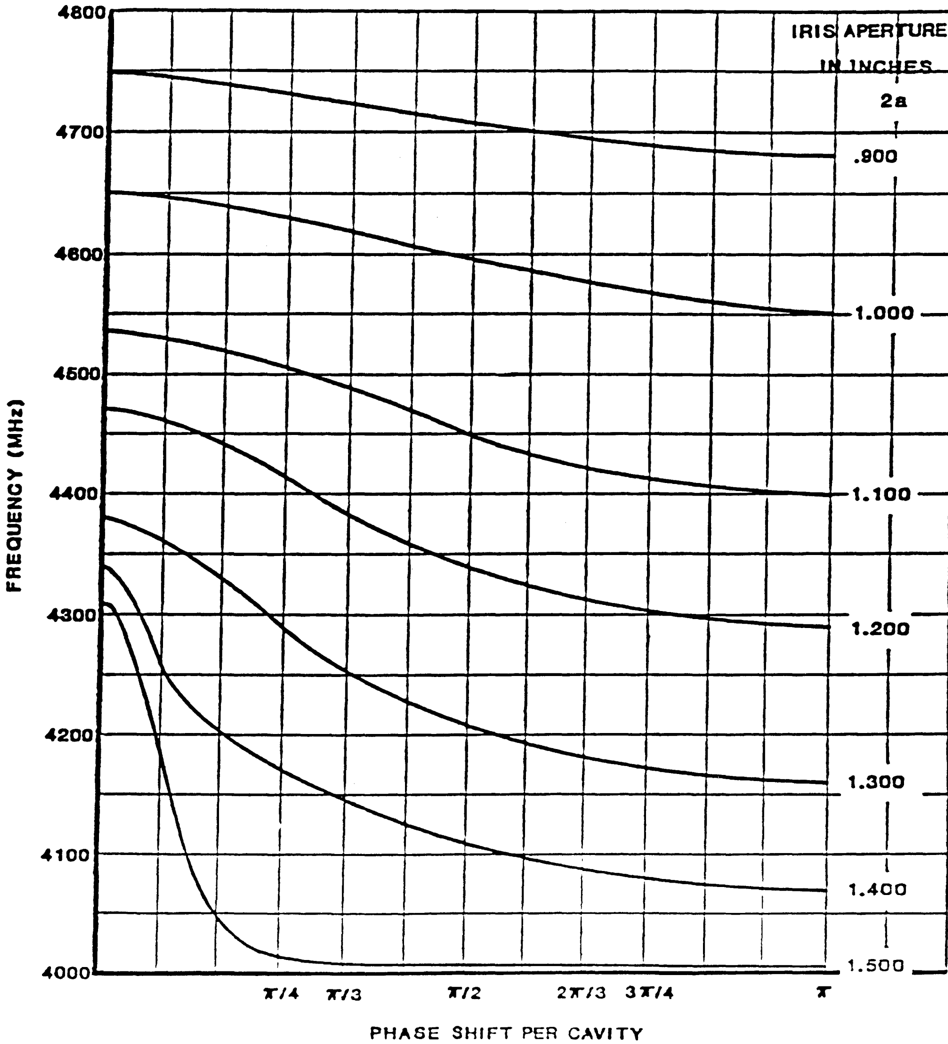

FIGURE 2 Location of HEM-11 mode for contoured cavities.

TABLE II

$2a$	$2b$	v_g/c	r/Q	$a_0^2/\sum a_n^2$
0.900	3.244	0.0093	38.4	0.63
1.000	3.263	0.0195	36.9	0.65
1.100	3.289	0.0257	36.5	0.68
1.200	3.319	0.0273	35.0	0.71
1.300	3.366	0.0336	33.4	0.74
1.400	3.404	0.0425	31.6	0.79

$v_p = c$ at 2856 MHz
$2a$, aperture diameter (in.)
$2b$, cavity diameter (in.)
v_g/c, normalized group velocity
r/Q, shunt impedance (Ω/cm)
$a_0^2/\sum a_n^2$, fraction of power in synchronous space harmonic

and to the r/Q at 2856 MHz by

$$r/Q = \frac{70}{a} \qquad \text{(ohms/cm)},$$

before correction for space-harmonic content.

Previously, theoreticians devoted a considerable effort to deriving such expressions, usually only correct to an order of magnitude because of simplifying assumptions (such as zero-thickness disks); but such expressions as those above, based upon experimental data, are useful for interpolation in waveguide design.

4. REMARKS ON MEASUREMENT TECHNIQUES

The fundamental observation that permits evaluating the properties of a traveling wave by means of measurements made on a resonant cavity is that the axial field of the traveling wave in the aperture may be written at the velocity of light

$$\begin{aligned} E_z &= \sum_{-\infty}^{\infty} a_n\, e^{j(\omega t - \beta_n z)} \\ \beta_n &= \beta_0 + 2\pi n/p \end{aligned} \tag{7}$$

If a cavity is formed by placing electric reflectors in transverse planes of symmetry, the propagating modes of the waveguide satisfy the boundary conditions of the cavity and the axial field in the cavity may be expressed as the sum of a forward wave and its reflection.

$$\begin{aligned} E_z &= \sum_{-\infty}^{\infty} a_n\, e^{j(\omega t - \beta_n z)} + \sum_{-\infty}^{\infty} a_n\, e^{j(\omega t + \beta_n z)} \\ &= \sum_{-\infty}^{\infty} 2\, a_n \cos \beta_n z\, e^{j\omega t} \end{aligned} \tag{8}$$

Perturbation techniques are among the most widely used in microwave engineering, particularly for field-strength measurements. Since the energy W stored in a resonant cavity at steady state is

$$W = \frac{PQ}{\omega}, \tag{9}$$

which is, alternatively, the definition of cavity Q, it follows that[10]

$$\frac{d\omega}{\omega} = -\frac{dW}{W}. \tag{10}$$

In microwave theory, the concept of shunt resistance was originally introduced to define the ratio of peak voltage squared across a cavity to the average energy loss per second, which was of course useful to estimate the required input power to maintain a specified gap voltage in a cavity,[11]

$$R = \frac{V^2}{2P}, \tag{11}$$

which, with Eq. (9), may be put in the form

$$\frac{R}{Q} = \frac{V^2}{2\omega W}. \tag{12}$$

However, in accelerator engineering the shunt impedance of a traveling wave is defined as the ratio of peak electric field intensity squared to the power dissipated per unit length,[12]

$$r = \frac{E^2}{dP/dz}. \tag{13}$$

By the definition of attenuation, $dP/dz = -2IP$ and of power flow, $P = wv_g$, $2I$ being the power attenuation constant, w the linear energy density and V_g the group (energy) velocity, it may be shown that

$$r/Q = \frac{E^2}{\omega w}. \tag{14}$$

In this it has also been assumed that the reader is familiar with the concept of the Q of a transmission line, defined as the ratio of stored energy per unit length to power loss per unit length in one radian,

$$Q = \frac{w}{2IP/\omega} = \frac{\omega}{2Iv_g} \tag{15}$$

Group velocity. Group velocity v_g is by definition the slope of the dispersion function, $v_g = d\omega/d\beta$; energy velocity v_e is defined by $P = wv_e$, where P is the power flux and w the linear energy density. In the lossless case, the two are equivalent. The conventional method of determination of group velocity is to measure the slope of the Brillouin diagram (normally plotted with frequency as a function of phase shift per period or mode); then

$$\frac{v_g}{c} = \frac{2\pi p}{c}\frac{\Delta f}{\Delta\beta\rho}. \tag{16}$$

It has been suggested that greater accuracy can be achieved analytically[13] since the dispersion diagram is periodic, by Fourier-analyzing the dispersion data and taking the derivative, that is, representing the dispersion diagram in the form[14]

$$f = \sum_{k=0}^{\infty} a_k \cos k\beta p \tag{17}$$

whereby

$$v_g = \frac{d\omega}{d\beta} = -2\pi p \sum_{k=1}^{\infty} ka_k \sin k\beta p. \tag{18}$$

But, while a periodic function can be accurately represented by means of a Fourier series in the least-squares error sense, it is well known that the slope of the function so composed may be considerably in error.

r/Q Measurements

The shunt impedance of a structure is needed by an accelerator designer to estimate the energy gain achievable for a specified power input. In principle, shunt impedance should be directly measurable. For example, if a cavity of shunt impedance R_0 and loss factor Q_0 is loaded with a resistance R (along the proposed trajectory), the observed loss factor Q will be lowered such that[15]

$$\frac{R}{Q} = \frac{R_0 + R}{Q_0}$$

or

$$R_0 = R\left(\frac{Q_0}{Q} - 1\right), \tag{19}$$

but a practical method has not been found as yet to insert the known resistance in such a way as to produce acceptable results, which would also involve a cavity construction as finally intended, rather than the present "cold-test" stack method.

Lacking such a technique, the customary procedure has been to determine the ratio r/Q, which is easily done by perturbation methods. Then a figure of merit Q derived from a separate measurement will provide an estimate of shunt impedance.

If a thin dielectric rod of cross-section A and dielectric constant ϵ is inserted along the axis of the cavity, traversing its entire length, the frequency perturbation is, from Eq. (10),

$$\frac{\Delta f}{f} = -\frac{(\epsilon - \epsilon_0)A \int E^2\, dz}{4W_c}. \tag{20}$$

The energy stored in the cavity can be expressed in terms of that in the traveling wave, $W_c = 2\,wL$, and the axial field in terms of the space-harmonic ensemble $\int E^2 \cdot dz = 2L \sum a_n^2$. Then Eq. (20) may be expressed in terms of the shunt impedance, Eq. (14), of the synchronous space harmonic by eliminating the stored energy between the two expressions,

$$\left(\frac{r}{Q}\right)_n = \frac{E_n^2}{\omega W} = \frac{240\lambda}{(K-1)A} \frac{\Delta f}{f} \frac{a_n^2}{\sum a_n^2}, \tag{21}$$

where K is the relative dielectric constant and $\epsilon_0 = 1/\eta c$, η being the impedance of free space.

The term $a_n^2/\sum a_n^2$, which is physically the fraction of power flux associated with the space harmonic of interest, must be determined by a separate measurement. This consists of drawing a bead (on a thread) along the same path as above, noting the frequency perturbation at several positions, which is proportional to the square of the electric field intensity at the bead location. A spatial Fourier analysis of the square root of the perturbation pattern will then provide coefficients proportional to the space harmonics (a_n).

It is also necessary in this method to know the dielectric constant of the rod. This can presumably be determined from the frequency perturbation in the case where the energy perturbation is calculable. For example, an axial dielectric rod of diameter $2a$ in a cylindrical cavity of diameter $2b$ supporting the TM-010 mode produces a time-average stored-energy perturbation

$$\Delta W = \frac{1}{4}(\epsilon - \epsilon_0)\pi a^2 E_0{}^2 L, \tag{22}$$

where L is the length of the cavity. The time-average stored energy in the cavity at resonance without the rod is[16]

$$W = \frac{\pi b^2}{2} L E_0{}^2 J_1{}^2\left(2\pi \frac{b}{\lambda}\right) \tag{23}$$

Thus, the frequency perturbation owing to the rod is

$$\frac{\Delta f}{f} = \left(\frac{a}{b}\right)^2 \frac{(K - 1)}{2J_1{}^2\left(\dfrac{2\pi b}{\lambda}\right)}, \tag{24}$$

from which the relative dielectric constant (K) can be found by means of two frequency measurements. Alternatively, the resonant frequency of the cavity with the dielectric rod may be calculated. With an axial dielectric core, the resonant frequency of the cavity is given by

$$\frac{J_0(kb)}{N_0(kb)} = \frac{J_0(\sqrt{\kappa}\, ka)J_1(ka) - \sqrt{\kappa}\, J_1(\sqrt{\kappa}\, ka)J_0(ka)}{J_0(\sqrt{\kappa}\, ka)N_1(ka) - \sqrt{\kappa}\, J_1(\sqrt{\kappa}\, ka)N_0(ka)}, \tag{25}$$

where $k = \omega/c$ and κ is the relative dielectric constant of the core. This solution is an elementary boundary-value problem, too tedious and lengthy to reproduce here. The two solutions, Eqs. (24) and (25), do not generally provide precisely the same estimate of relative dielectric constant because in the perturbation method it is essential that the cavity Q is not changed by insertion of the dielectric, but it is obvious that its presence changes the stored energy in the cavity and therefore its Q.

Another procedure is to insert the dielectric rod axially in a cavity of known R/Q, deducing therefrom the relative dielectric constant of the rod.[13] For a right-circular cylindrical cavity of length L and diameter $2b$ supporting the TM-010 mode, the theoretical R/Q is[16]

$$R/Q = \frac{\eta L}{\pi b p_{01} J_1^2(p_{01})}, \tag{26}$$

where $J_0(p_{01}) = 0$. By the same sort of argument as presented earlier, insertion of an axial dielectric rod of diameter $2a$ and relative dielectric constant E_0 will produce a perturbation

$$\frac{\Delta f}{f} = \frac{(k-1)\epsilon_0 \pi a^2 E_0^2 L}{4W}. \tag{27}$$

Eliminating the stored energy between Eqs. (12) and (27)

$$\frac{\Delta f}{f} = \frac{(k-1)\epsilon_0 \pi a^2 E_0^2 L}{4W}. \tag{27}$$

Then, from Eqs. (26) and (28), noting the resonance condition $2\pi b/\lambda = p_{01}$,

$$K = 1 + 2\left(\frac{b}{a}\right)^2 J_1^2(p_{01}) \frac{\Delta f}{f} \tag{29}$$

which is identical to Eq. (24), as expected.

For example, in an air-filled cylindrical cavity of 3.390 in. diameter and 3.960 in. length, the observed resonant frequency of the TM-010 mode was 2667.815 MHz compared to 2666.963 MHz theoretically (evacuated). Correcting for air dielectric ($K = 1.000544$, 20°C, 60 pcRH) the calculated resonant frequency would be 2667.688 MHz, the residue presumably owing to the excitation-probe apertures. With an 0.0625-in. diameter sapphire (corundum, Al_2O_3) rod inserted axially, the observed resonant frequency was 2650.403 MHz. By Eq. (24), the relative dielectric constant is 10.63 and by Eq. (25) 10.56.[17]

5. HIGHER-ORDER MODE SUPPRESSION

On the basis of studies using a computer code, it appears that the proposed contoured cavity structure will not exhibit beam breakup in S-band for 1A beam current within 100 μsec. Nevertheless, from examination of mode configurations, the disruption of surface currents representing the next higher-order mode (by means of slots) was experimentally investigated. Radial slots extending from the beam aperture showed complete suppression of the HEM-11 mode, at least to the level of detectability of the test equipment (20 dB in power).

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