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APPLICATION OF THE PHASE COMPRESSION – PHASE EXPANSION EFFECT FOR ISOCHRONOUS STORAGE RINGS

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In an isochronous cyclotron acceleration is accomplished through several acceleration gaps. A nonuniform voltage distribution along these gaps produces a time-varying magnetic field which affects the longitudinal bunch size. A radially increasing (decreasing) acceleration voltage compresses (expands) the bunch size. This effect can be utilized in an isochronous storage ring with $\lambda/2$ cavities, where particles undergo a cycle of acceleration, coasting at maximum energy, deceleration and coasting at minimum energy. This cycle could be repeated many times if H^- ions are injected into the ring and stripped to protons. Stored beams up to 100 A look feasible.

The phase expansion effect can also be utilized to improve the duty cycle of isochronous cyclotrons by a decreasing dee voltage towards the extraction radius or by insertion of decelerating *C*-electrodes like in the present improvement programs for several synchrocyclotrons.

1. INTRODUCTION

The phase compression–phase expansion effect was first mentioned by Mueller and Mahrt.[1] They derived a formula which is valid for small rf phases only. In this report a general formula will be given which includes the addition of 'flat-topping' harmonics and is valid for all phases. This extension leads to some interesting applications for isochronous storage rings.

2. THE PHASE COMPRESSION–PHASE EXPANSION EFFECT IN CYCLOTRONS

To get a simple picture of the phase compression effect we assume the acceleration gaps to be perpendicular to the orbits and of short azimuthal extent (small transit time effects). A closer analysis shows that oblique gap crossings do not affect the revolution time of a particle. This result was found in the 1930's, where the idea of a spiraling dee gap—to increase the energy of a classical cyclotron—was examined and soon discarded! The phase compression formula—derived for perpendicular gap crossings—is therefore also valid for oblique gap crossings.

Let us adopt a Cartesian coordinate system at the acceleration gap as shown in Figure 1 and let us assume the electric field of an acceleration gap to be constant over an effective gap width g and to be of the form:

$$\mathscr{E}_s(r,t) \equiv \mathscr{E}_s(r)\cos\omega_{\rm rf}\,t \equiv \mathscr{E}_s(r)\cos\phi \tag{1}$$

$\omega_{\rm rf} \equiv 2\pi\nu_{\rm rf} = h\omega_0$ is the frequency of the rf system

$\omega_0 \equiv \dfrac{2\pi}{\tau_0}$ is the isochronous revolution frequency of the particles

h (positive integer) is called the 'harmonic' of acceleration

ϕ is the relative rf phase and defined by

$$\frac{\mathrm{d}\phi}{\mathrm{d}n} = \omega_{\rm rf}\tau - 2\pi h = 2\pi h\frac{\tau-\tau_0}{\tau_0} \quad \begin{array}{l}(\text{leading phase: } \phi < 0\\ \text{lagging phase: } \phi > 0)\end{array} \tag{2}$$

$$\frac{\mathrm{d}E}{\mathrm{d}n} = eV_G(R)\cdot\cos\phi \quad \text{is the energy gain per turn} \tag{3}$$

$n =$ turn number
$E =$ kinetic energy of particle

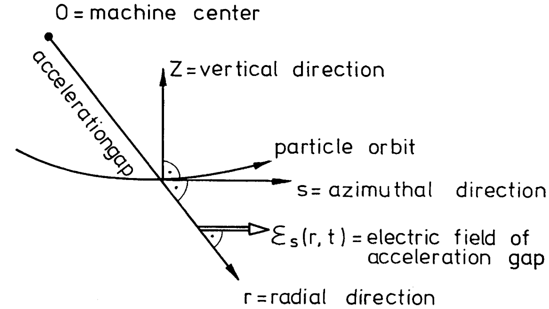

FIGURE 1 Coordinate system (r, s, z) adopted for the explanation of the phase compression–phase expansion effect. The assumptions are that the particle orbits are perpendicular to the acceleration gap and that the electric field is uniform across the gap.

$V_G(R)$ = peak voltage gain per turn
τ = revolution time of particle
R = average radius of particle orbit.

The magnetic field **b** produced by a nonuniform acceleration gap is given by Maxwell's equation:

$$\dot{\mathbf{b}} = -\operatorname{curl} \mathscr{E}$$

$$\dot{b}_z = -\frac{\partial \mathscr{E}_s(r)}{\partial r} \cos\phi$$

with the convention $' \equiv \frac{d}{dr}$ we get

$$b_z = -\frac{\mathscr{E}'_s}{\omega_{\mathrm{rf}}} \sin\phi. \tag{5}$$

Thus a transverse gradient of the electric field produces a time-varying magnetic field which is 90° out of phase with the electric field (in most cyclotrons this magnetic field is located far outside the useful volume for acceleration). This rf magnetic field points in the vertical direction as does the static cyclotron field. It gives a horizontal deflection α_B to those particles which do not arrive at the moment of peak voltage across the acceleration gap. Thus the orbits of particles with different phases ϕ become different with a consequent change in revolution time and phase. The deflection α_B across the acceleration gap g is given by

$$\alpha_B = \frac{b_z g}{(B\rho)} = -\frac{\mathscr{E}'_s g}{\omega_{\mathrm{rf}}(B\rho)} \sin\phi = -\frac{V'_{\mathrm{Dee}}}{\omega_{\mathrm{rf}}(B\rho)} \sin\phi \tag{6}$$

$V_{\mathrm{Dee}}(R)$ = peak voltage gain across the gap
$(B\rho)$ = magnetic rigidity of particle.

2.1. *Nonrelativistic Classical Cyclotron*

The effect of a nonuniform dee voltage on the beam is illustrated in Figures 2 and 3 for a homogeneous static magnetic field B_0. We assume two acceleration gaps at 0° and 180° with radically increasing peak voltage ($V'_{\mathrm{Dee}} > 0$).

For this special case the following results are easily obtained from geometrical considerations:

a) the fractional *change* in the *revolution time* is

$$\frac{\Delta\tau_\phi}{\tau(0)} \equiv \frac{\tau(\phi) - \tau(0)}{\tau(0)} = \frac{\alpha_B}{\pi} \tag{7}$$

which according to (2) leads to a contribution to the phase slip per turn of

$$\frac{d\phi}{dn} = -\frac{V'_G}{\omega_0(B\rho)} \sin\phi \tag{8}$$

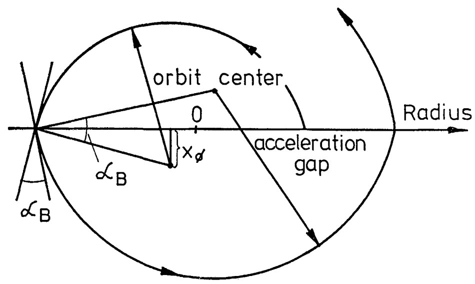

FIGURE 2 Orbit of accelerated particle in the case of a uniform external magnetic field in the z-direction and a 180° dee with radially increasing voltage. A particle with positive phase ($\phi > 0$) arrives late at the acceleration gap and gets a radially inward kick from the rf magnetic field. The ideal orbit center is displaced by a distance x_ϕ from the dee gap and jumps across the gap at each crossing.

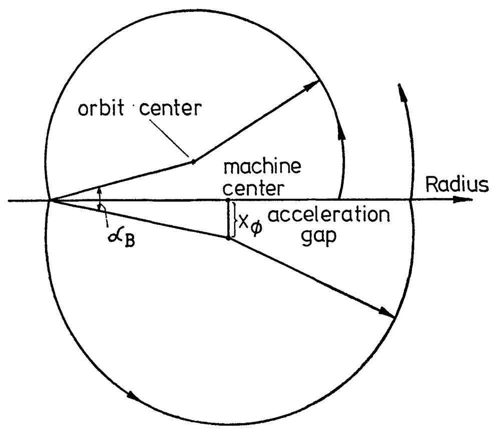

FIGURE 3 Same situation as in Figure 2, except that the particle arrives early ($\phi < 0$) at the acceleration gap and gets a radially outward kick which forces it on a longer path. Combining Figures 2 and 3 leads to a compression of the longitudinal bunch size.

b) the rf magnetic field displaces the orbit center perpendicular to the acceleration gap which leads to a *phase dependent center spread* x_ϕ given by

$$x_\phi = R\frac{\alpha B}{2} = -\frac{V'_G}{4\omega_{\mathrm{rf}} B_0} \sin\phi. \tag{9}$$

2.2 *Relativistic Isochronous Cyclotron with Single rf System*

We leave now the special case of Figures 2 and 3 and we will treat the *general case* of a *relativistic isochronous* cyclotron with a static magnetic field $B_z(r, \theta)$ and an arbitrary number of acceleration gaps. We assume that the change in voltage gain

$V_G(R)$ from turn to turn is small. This condition for *adiabatic* acceleration can be written as

$$E'_G \ll \frac{E}{R} \tag{10}$$

where we defined for convenience the peak energy gain per turn

$$E_G(R) \equiv eV_G(R). \tag{11}$$

The phase slip equation (2) can be split into two parts:

$$\frac{d\phi}{dn} \equiv \varphi_{ni}\,(\text{nonisochronism}) + \frac{d\phi}{dn}\,(\text{nonuniform energy gain}) \tag{12}$$

The first term φ_{ni} arises from deviations of the magnetic field and rf frequency from their isochronous values. We will mention it again in Eq. (26). For our purposes we need to treat only the second term which arises from the magnetic field of a nonuniform acceleration gap.

The rf field $b_z(r, \phi)$ changes the revolution time τ according to[2,3]

$$\frac{\Delta\tau(\phi)}{\tau(0)} = -\frac{1}{\gamma^2}\frac{\langle b_z\rangle}{\langle B_z\rangle} \tag{13}$$

where $\gamma = (1-\beta^2)^{-1/2}$ = relativistic factor, $\beta \equiv v/c$ $\langle\,\rangle$ means average over one revolution along the orbit.

From Eq. (5) we get:

$$\langle b_z\rangle = -\frac{\langle \mathscr{E}'_s\rangle}{\omega_{rf}}\sin\phi = -\frac{V'_G}{2\pi R\omega_{rf}}\sin\phi. \tag{14}$$

For positively charged particles and positive rotation of the beam around the cyclotron center $\langle B_z\rangle$ is negative. We therefore define

$$B_0(R) \equiv -\langle B_z(R)\rangle\ (\text{positive}). \tag{15}$$

Combining Eqs. (2), (13), (14) and (15) yields

$$\frac{d\phi}{dn}(\text{nonuniform energy gain}) = -\frac{V'_G}{\gamma^2 B_0 R\omega_0}\sin\phi. \tag{16}$$

In order to proceed further let us list some well-known relations for isochronous cyclotrons (see, e.g., Ref. 3)

$R_\infty = \dfrac{c}{\omega_0}$ = cyclotron unit of radius

$B_{cu} = \dfrac{m_0\omega_0}{e}$ = cyclotron unit of magnetic field

$R_\infty B_{cu} = \dfrac{m_0 c}{e} = \dfrac{E_0}{ec}$ (= 31.3 kg.m for protons)

$R = \beta R_\infty$

$B_0(R) = \gamma B_{cu}$

$B_0(R)R = \beta\gamma B_{cu} R_\infty = (B\rho)$ = magnetic rigidity

$$\frac{dE}{dR} = \frac{E_0}{R_\infty}\beta\gamma^3 = ecB_{cu}\beta\gamma^3. \tag{17}$$

With these relations Eq. (16) becomes

$$\frac{d\phi}{dn} = -\frac{dE_G}{dE}\sin\phi. \tag{18}$$

Combining Eqs. (3) and (18) we observe that the energy E and the rf phase ϕ are *canonically conjugate* variables with the Hamiltonian

$$H(E(n), \phi(n)) = E_G\sin\phi = \text{constant} \tag{19}$$

which leads to the 'equations of motion':

$$\frac{dE}{dn} = \frac{\partial H}{\partial\phi} = E_G\cos\phi \tag{20}$$

$$\frac{d\phi}{dn} = -\frac{\partial H}{\partial E} = -\frac{dE_G}{dE}\sin\phi. \tag{21}$$

Since the Hamiltonian H does not depend explicitly on the turn number $n\left(\dfrac{dH}{dn} \equiv \dfrac{\partial H}{\partial n} = 0\right)$, it is a constant of motion. This leads us to the important relation between peak energy gain and phase of a given particle during acceleration between two radii R_1 and R_2 of an isochronous cyclotron:

$$\boxed{E_G(R_1)\sin\phi(R_1) = E_G(R_2)\sin\phi(R_2)}. \tag{22}$$

This is a generalization of the result obtained by Mueller and Mahrt[1] for small phases ($\sin\phi \approx \phi$). We obtain further

$$\frac{dE}{dn}\tan\phi = \text{const} \tag{23}$$

whereas differentiating (20) again yields

$$\frac{d^2E}{dn^2} = E_G\frac{dE_G}{dE} = \frac{1}{2}\frac{d}{dE}(E_G^2),\ \text{independent of phase.} \tag{24}$$

The big effect of a nonuniform acceleration gap on the phase-width of the beam (as illustrated by Eq. (22)), will be seen in the second stage of the *Indiana cyclotron*[4] (operational in 1974). For protons the energy gain at injection (15 MeV) will be 4×75 keV $= 300$ keV, whereas at extraction (200 MeV) it will be 4×220 keV $= 880$ keV. For operation with a single rf frequency (no flat top) the relation between initial and final phase is given by (22):

$$\sin \phi_{\text{final}} = 0.35 \sin \phi_{\text{initial}}.$$

For example a bunch length of initially 20° will be compressed to about 7° during acceleration.

A slight phase compression will also occur in the SIN 590 MeV ring cyclotron which has four $\lambda/2$ rf cavities. Injection into the ring at 72 MeV occurs at 60 per cent, extraction at 85 per cent of the maximum energy gain per turn. An initial bunch length of about 20° will be compressed to 16°.

In many cases one is interested in an extracted beam with a high duty cycle for coincidence experiments. With a radially decreasing dee voltage one can indeed expand the phase width according to Eq. (22). At the same time one benefits from a smaller energy spread in the extracted beam. The disadvantage of this scheme is a lower extraction efficiency in conventional cyclotrons, where one relies on the turn separation for extraction with a septum. The improvement of the duty cycle with the phase expansion effect could however be most advantageous for H^- cyclotrons like TRIUMF, where the extraction by stripping is almost 100 per cent even with a low dee voltage.

The same phase expansion can also be achieved with decelerating C-electrodes at the extraction radius, as is foreseen for the improvement program for several synchrocyclotrons. In this case it is the time-dependent radial electric field at the inner edge of the C-electrodes which produces the required phase slip.

2.3 *Relativistic Isochronous Cyclotron with Two Radiofrequency Systems*

The addition of another radiofrequency system operating on the *m*th harmonic of the fundamental radiofrequency can alter the effective waveform of the acceleration voltage favorably for operation with high duty cycle or low energy spread (flat top). Second harmonic ($m = 2$) flat-topping will be available for the Indiana cyclotron,[4] while SIN[3] and TRIUMF[5] will use the third harmonic ($m = 3$).

The addition of this frequency will modify the energy gain per turn of a particle:

$$\frac{\mathrm{d}E}{\mathrm{d}n} = E_{G1}(R)\cos\phi + E_{Gm}(R)\cos m(\phi - \phi_m) \qquad (25)$$

where $E_{G1}(R)$ is the peak energy gain (positive in general) from the fundamental frequency, $E_{Gm}(R)$ the peak energy gain (negative for flat-topping) from the harmonic frequency.

Again the variables E, ϕ are canonically conjugate with the equations of motion derived from the *Hamiltonian for the general case*:

$$H = E_{G1}(R)\sin\phi + \frac{E_{Gm}(R)}{m}\sin m(\phi - \phi_m) - \int \varphi_{\text{ni}}(E)\,\mathrm{d}E \qquad (26)$$

$$\frac{\mathrm{d}E}{\mathrm{d}n} = \frac{\partial H}{\partial \phi} \quad \text{gives back Eq. (25)}$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}n} = -\frac{\partial H}{\partial E} = \varphi_{\text{ni}}(E) - \frac{\mathrm{d}E_{G1}}{\mathrm{d}E}\sin\phi - \frac{\mathrm{d}E_{Gm}}{m\,\mathrm{d}E}\sin m(\phi - \phi_m). \qquad (27)$$

φ_{ni} is the phase slip per turn in not quite isochronous cyclotrons as mentioned in (12). If we have a uniform acceleration gap and no flat-top harmonic Eq. (26) gives the well-known formula for the phase history of particles in a realistic cyclotron with imperfections:

$$\sin\phi(E_2) - \sin\phi(E_1) = \frac{1}{E_G}\int_{E_1}^{E_2} \varphi_{\text{ni}}(E)\,\mathrm{d}E. \qquad (28)$$

3. ISOCHRONOUS STORAGE RING

An interesting application of the phase compression–phase expansion effect is an isochronous storage ring with $\lambda/2$ rf resonators. To be a bit more specific let us demonstrate the basic principle with a four sector ring cyclotron illustrated in Figure 4. The drawing is not to scale, since we do not specify anything on energy and radiofrequency. The magnetic field of the sectors is shaped such that the revolution time for a closed orbit without acceleration is identical at all energies.

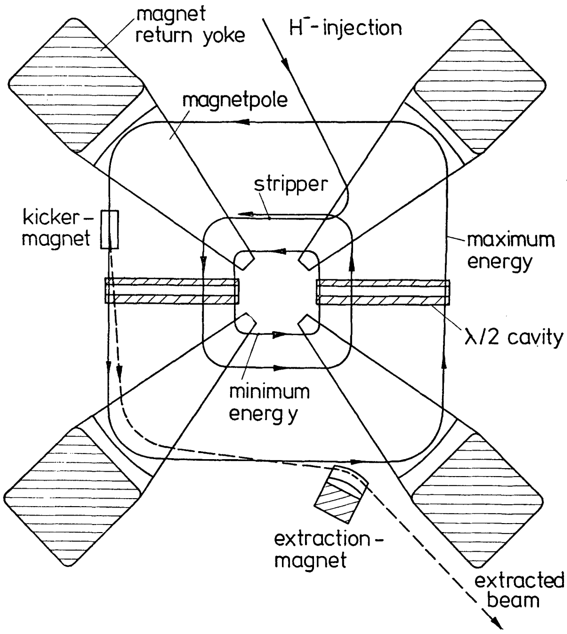

FIGURE 4 Schematic example of a four sector ring cyclotron used as a storage ring. H^- ions are injected into the cyclotron at a relatively low energy, stripped to protons and then accelerated up to a maximum energy, where they are decelerated again. The rf magnetic field of two $\lambda/2$ cavities drives the particles out of phase at the endpoints of the cavity. The protons oscillate thus between a minimum and maximum energy in the cyclotron. Extraction of the stored beam is not an easy problem and one possibility is indicated in the diagram with a pulsed kicker magnet.

For a really relativistic cyclotron there would be more than four sectors necessary and they would have to be spiralled to provide adequate vertical focusing. Between the magnet sectors are rectangular $\lambda/2$-cavities operating in the so-called H_{101}-mode (SIN-cavities[6]) as illustrated in Figure 5.

For a perfect rectangular box of length L and height H the resonant frequency for this mode is given by

$$\nu_{\mathrm{rf}} = \frac{c\sqrt{(L^2+H^2)}}{2LH}. \tag{29}$$

The minimum—or cutoff—frequency is obtained for $H = \infty$:

$$\nu_{\mathrm{rf}}(\min) = \frac{c}{2L} \tag{30}$$

The insertion of lips around the acceleration gap lowers this frequency without affecting the high Q-value substantially. These cavities provide a peak energy gain per turn of the form

$$E_G(R) = E_p \sin \pi \frac{(R-R_0)}{L} \tag{31}$$

With the definition

$$W \equiv \pi \frac{(R-R_0)}{L}, \qquad R = R(E) \tag{32}$$

the Hamiltonian (19) becomes

$$H = E_p \sin W \sin \phi = \text{const} \tag{33}$$

$$\frac{\mathrm{d}E}{\mathrm{d}n} = \frac{\partial H}{\partial \phi} = E_p \sin W \cos \phi \tag{34}$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}n} = -\frac{\partial H}{\partial E} = -\frac{\partial H}{\partial W}\frac{\mathrm{d}W}{\mathrm{d}E} = -\frac{\pi E_p}{L(\mathrm{d}E/\mathrm{d}R)} \cos W \sin \phi.$$

Using $\dfrac{\mathrm{d}E}{\mathrm{d}R}$ from (17) leads to

$$\frac{\mathrm{d}\phi}{\mathrm{d}n} = -\mu \cos W \sin \phi \tag{35}$$

$$\frac{\mathrm{d}W}{\mathrm{d}n} = \mu \sin W \cos \phi \tag{36}$$

$$\mu \equiv \frac{\pi}{\beta\gamma^3} \frac{R_\infty}{L} \frac{E_p}{E_0}. \tag{37}$$

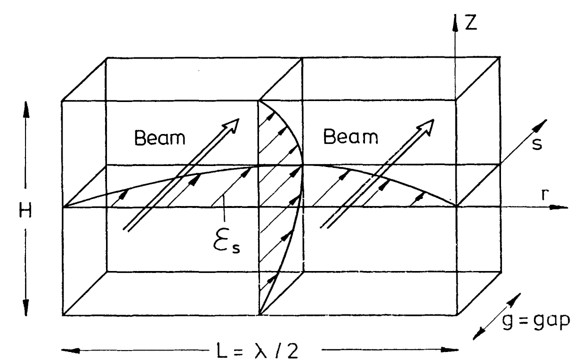

FIGURE 5 Geometry of a rectangular $\lambda/2$ cavity (SIN cavity operating in the H_{101}-mode). There is only an electric field component $\mathcal{E}_s$ with a sinusoidal distribution in the r- and z-direction. The rf magnetic field is confined in the (r, z)-plane and the beam passes through the cavity in the s-direction.

The curves $H = \text{constant}$ given by (33) are illustrated in Figure 6. The fixpoints are given by $\dfrac{d\phi}{dn} = \dfrac{dE}{dn} = 0$:

$$W = \pi/2 + j\pi,\ \phi = \pi/2 + l\pi \quad \text{stable fixpoint} \tag{38}$$

$$W = j\pi,\ \phi = l\pi \quad \text{unstable fixpoint} \tag{39}$$

$j, l = \text{integers}.$

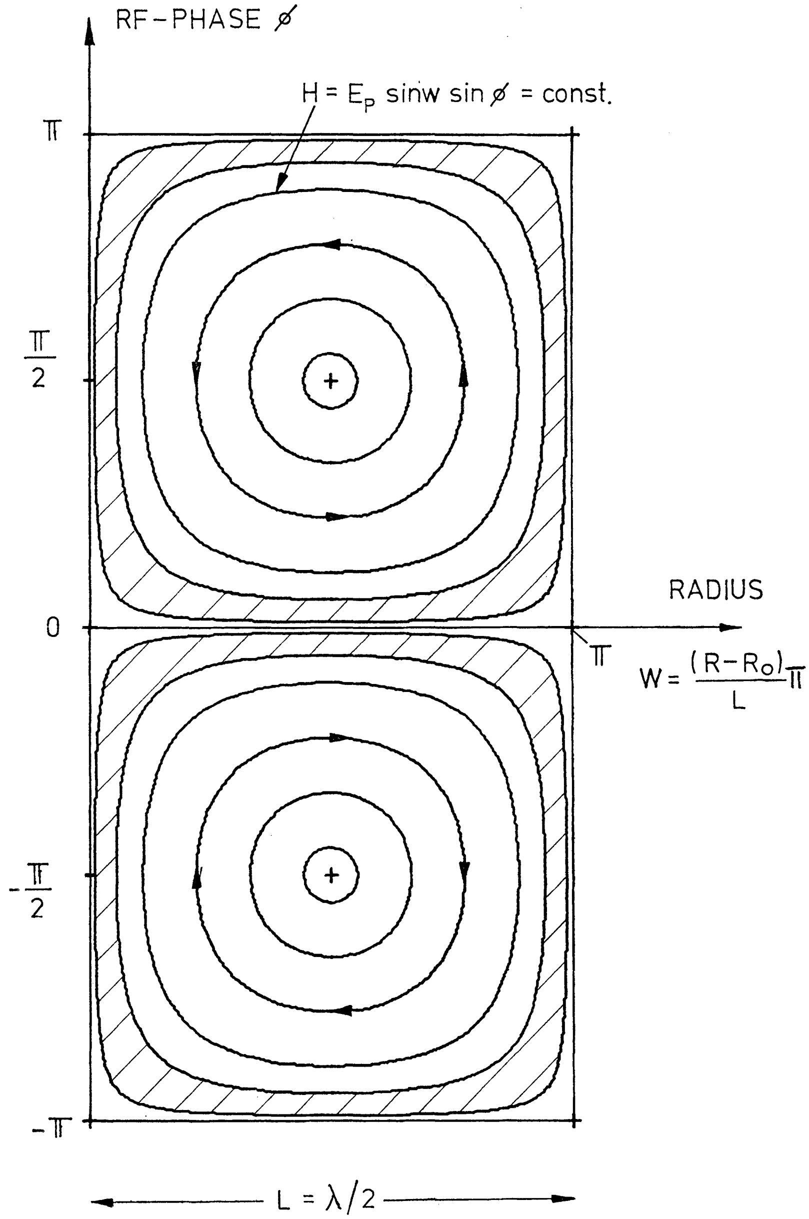

FIGURE 6 Particle trajectories in phase space (radius or W, rf phase ϕ) of an isochronous storage ring with cavities of a single frequency. The center of the squares represent stable fixpoints and the corners unstable fixpoints. The two hatched rings contain particles with phases between 2° and 10° respectively −2° and −10° in the center of the cavities at $W = \Pi/2$. The cavity walls are located at $W = 0$ and $W = \Pi$.

Around the stable fixpoint $W = \pi/2$, $\phi = \pi/2$, μ can be regarded as constant, which decouples Eqs. (34) and (35)

$$\frac{d^2E}{dn^2} \approx E_p\left(\frac{dW}{dE}\frac{dE}{dn}\cos W \sin\phi + \frac{d\phi}{dn}\sin W\cos\phi\right)$$

$$\approx \mu E_p \sin W \cos W$$

$$\frac{d^2E}{dn^2} \approx \frac{\mu}{2}E_p \sin 2W \tag{40}$$

and similarly

$$\frac{d^2\phi}{dn^2} \approx \frac{\mu^2}{2}\sin 2 \tag{41}$$

These two equations are similar to the equation of a rigid pendulum or to the longitudinal motion in a synchrotron (without acceleration).

Expanding the phase ϕ around $\frac{\pi}{2}$

$$\phi \equiv \frac{\pi}{2}+\psi \qquad \psi \ll 1 \tag{42}$$

leads to the harmonic oscillator equation:

$$\frac{d^2\psi}{dn^2}+\mu^2\psi = 0 \tag{43}$$

which shows that μ—given by (37)—is the oscillation frequency around the stable fixpoint. $2\pi/\mu$ is the number of revolutions necessary for one cycle.

Braun[7] at SIN did some numerical studies with a

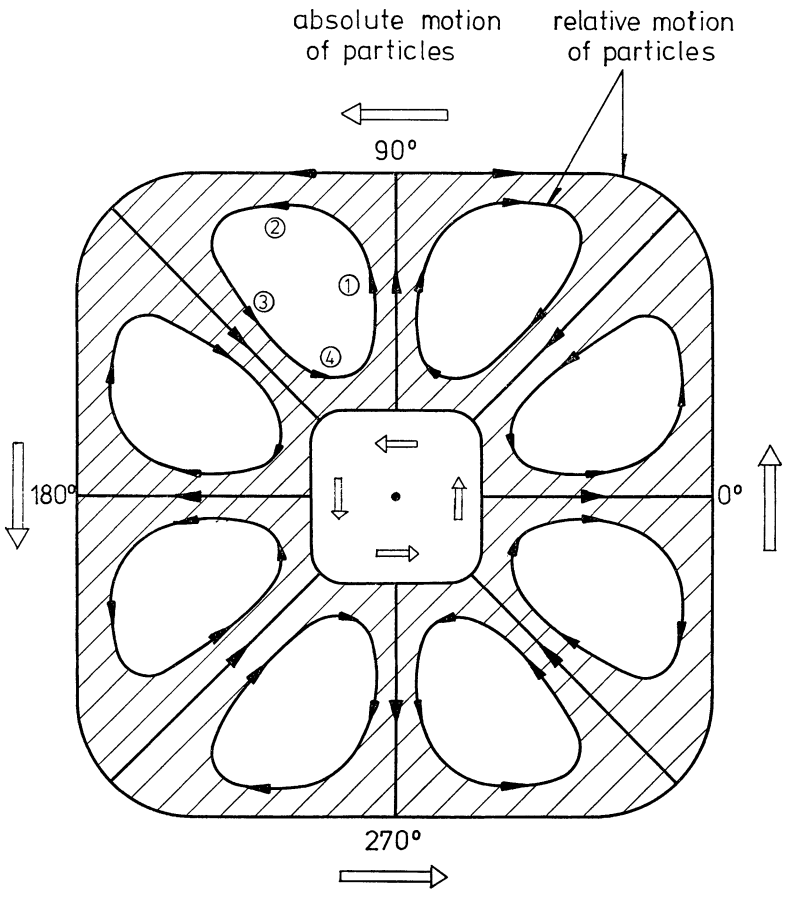

FIGURE 7 Snapshot of particle distribution in the four-sector cyclotron of Figure 4. For graphical simplicity the harmonic number of acceleration is chosen as 4. With uniform acceleration gaps we would thus have only the 4 bunches at the azimuths 0°, 90°, 180°, 270°. With $\lambda/2$ cavities all particles undergo a periodic cycle of acceleration ①, drifting at maximum energy ②, deceleration ③ and drifting at minimum energy ④. At intermediate energies the beam is strongly bunched while at the extreme energies we have practically a dc beam.

hypothetical short $\lambda/2$ cavity in the 590 MeV isochronous ring cyclotron. Formula (37) for the oscillation-frequency μ explains his results very well.

The (W, ϕ)-diagram of Figure 6 shows that even in a perfectly isochronous cyclotron the particles can never be accelerated into the cavity walls—if condition (10) is fulfilled. The rf magnetic field which is strongest at the walls is changing the revolution time of the particles till they get out of phase with the electric field and are decelerated again. This effect could thus be used for storing particles in an isochronous ring. The basic idea is illustrated in Figures 4 and 7.

H^- ions would be injected into the ring cyclotron at some intermediate energy and stripped to protons. These are accelerated to a maximum radius—which depends on the initial rf phase—decelerated to a minimum radius and accelerated again, performing thus a periodic cycle. To avoid particles with phase 0° interfering with the cavity walls at the extreme radii one can separate the cavity completely along the median plane into an upper and a lower part. This is possible since in the H_{101}-mode there are no surface currents inside the cavity across the median plane. One can also avoid particles getting close to the minimum and maximum radius by filling two separate phase-intervals on both sides of 0°. This could be done by injecting beam with negative phases for a while and then switching to positive phases. The stored particles fill finally two ring areas in (W, ϕ) phase space as indicated in Figure 6. For this example the particles occupy phases between $-2°$ and $-10°$ and between $+2°$ and $+10°$ in the middle of the cavity, whereas at the inner and outer radius the beam fills almost all phases.

Thus a bunched beam at intermediate energy is transformed into a practically dc beam at high and low energy. Since E, ϕ are canonically conjugate variables the flow of particles in (E, ϕ) phase space is governed by Liouville's theorem (constant phase-space density, incompressible fluid). Figure 6 and Eqs. (35), (36) show that the beam spends most of its time at the inner and especially at the outer radii. This feature is very desirable for *fast extraction* and for *keeping the stored beam away from the stripper*.

3.1. *Storage Ring with Addition of Flat-top Cavities*

The above-mentioned characteristic of an isochronous storage ring can be further improved with the addition of 3[rd] harmonic cavities. These flat-top cavities would have the same radial length L but three times the resonant frequency of the main cavities. They would operate in the so called H_{103}-mode with a voltage waveform given by

$$V_3(R, \phi) = V_{3p} \sin 3\pi \frac{(R-R_0)}{L} \cos 3\phi. \tag{44}$$

The combined peak energy gain per turn of the main and flat-top cavities is thus modified from (31) to

$$E_G(R) = E_p\left[\sin \pi \frac{(R-R_0)}{L} - \mathscr{E}_3 \sin 3\pi \frac{(R-R_0)}{L}\right] \tag{45}$$

and the new Hamiltonian $H(E(n), \phi(n))$ becomes

$$H = E_p\left[\sin W \sin \phi - \frac{\mathscr{E}_3}{3} \sin 3W \sin 3\phi\right] = \text{const.} \tag{46}$$

Curves $H = \text{const}$ for different initial conditions are plotted in Figure 8 with $\mathscr{E}_3 = 0.3$. The hatched rings contain phases from $-2°$ to $-10°$ and from 2° to 10° at $W = \pi/2$ in the middle of the cavity.

The equations of motion are given by:

$$\frac{dE}{dn} = \frac{\partial H}{\partial \phi} = E_p[\sin W \cos \phi - \mathscr{E}_3 \sin 3W \cos 3\phi] \tag{47}$$

or for W

$$\frac{dW}{dn} = \mu[\sin W \cos \phi - \mathscr{E}_3 \sin 3W \cos 3\phi] \tag{48}$$

$$\frac{d\phi}{dn} = -\frac{\partial H}{\partial E} = -\mu[\cos W \sin \phi - \mathscr{E}_3 \cos 3W \sin 3\phi]. \tag{49}$$

Note that μ is dependent on radius as given by (37) and that the connection between β and W is

$$\beta = (\beta_{\max} - \beta_{\min})\frac{W}{\pi} + \beta_{\min} \tag{50}$$

$$\beta_{\min} = \frac{R_0}{R_\infty} \text{ and } \beta_{\max} = \frac{R_0 + L}{R_\infty} \text{ determine}$$

the minimum and maximum energy of the storage ring.

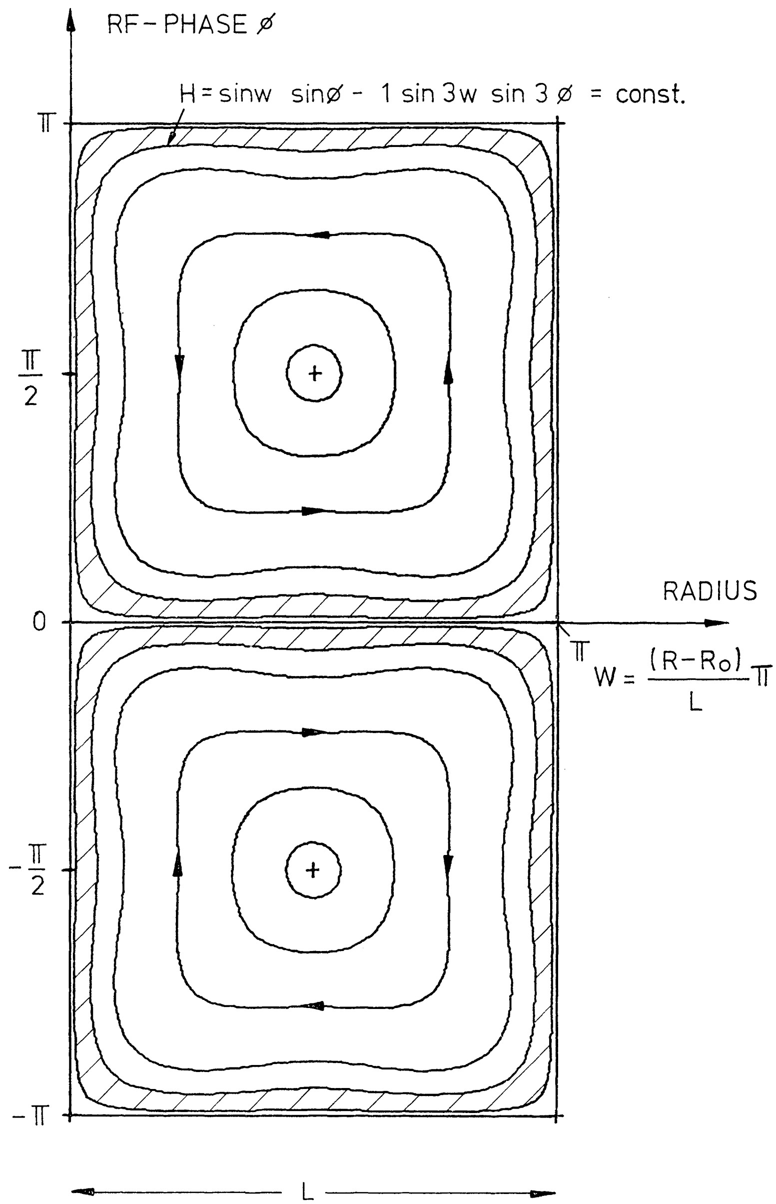

FIGURE 8 Same situation as in Figure 6 except that there are two different types of rf cavities with frequencies ν_{rf} and $3\nu_{rf}$ operating in the modes H_{101} and H_{103} respectively (flat-top operation).

3.2. *Numerical Example*

To get a better quantitative picture of the particle trajectories in (radius, ϕ)-phase space of a storage ring, let us look at a ring cyclotron operating between $\beta_{min} = 0.23$, $E_{min} = 26$ MeV and $\beta_{max} = 0.75$, $E_{max} = 500$ MeV. The orbital frequency and thus the size of the machine is still flexible, although one could have something in mind similar to the SIN 590 MeV cyclotron.

Figure 9 shows four representative particle trajectories—inside a hatched ring of Figure 8—obtained from numerical integration of Eqs. (48) and (49).

Figure 10 shows the radial particle density of a

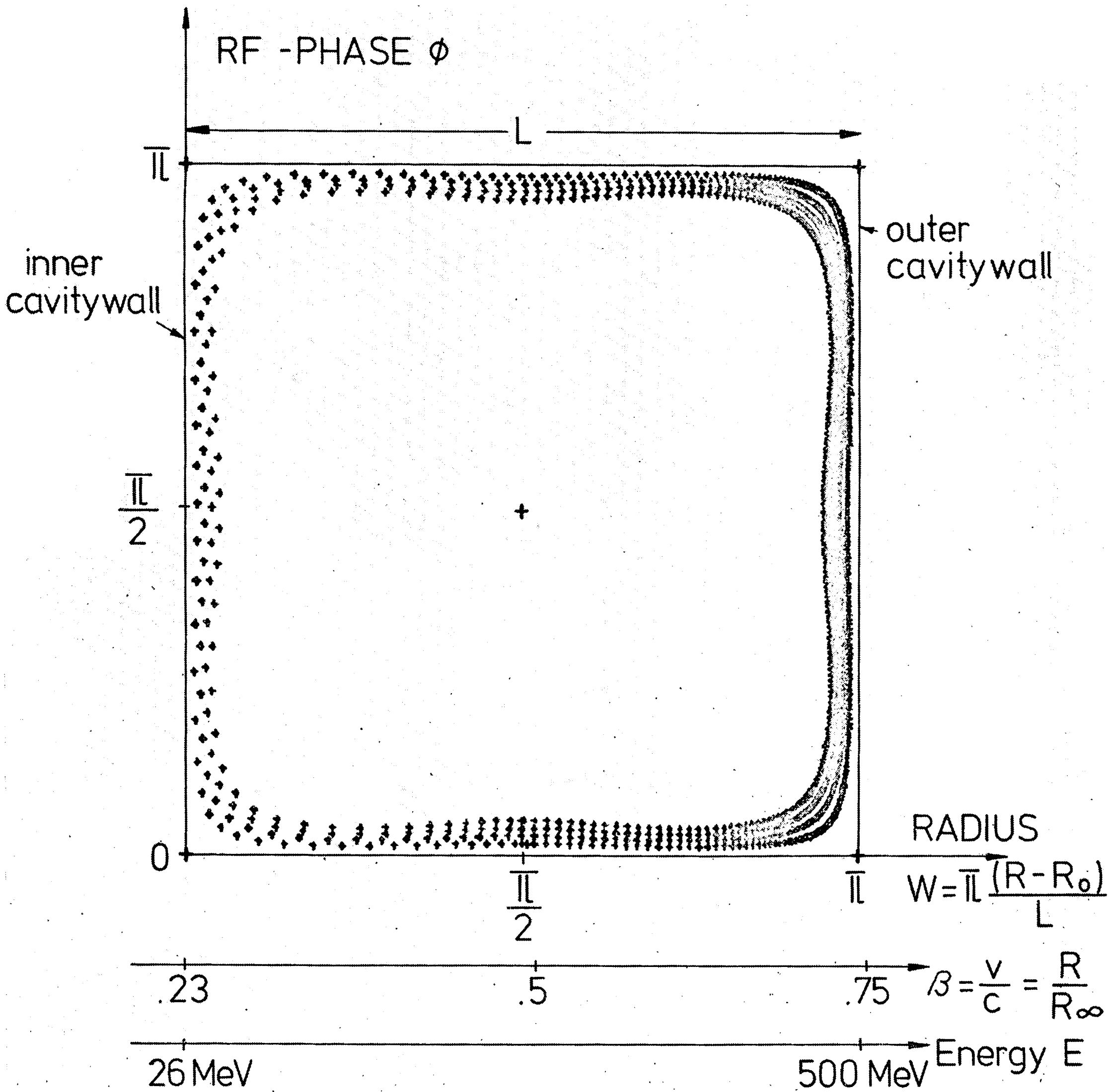

FIGURE 9 Four particle trajectories in (radius, ϕ) phase space inside hatched ring of Figure 8. The example of a storage ring operating between 26 and 500 MeV was chosen.

stored beam neglecting finite emittance effects. The ratio of the two peaks at the minimum and maximum energy is given approximately by the ratio of the corresponding μ-values:

$$\frac{\mu_{\text{max}}}{\mu_{\text{min}}}=\frac{\beta_{\text{max}}\,\gamma_{\text{max}}{}^3}{\beta_{\text{min}}\,\gamma_{\text{min}}{}^3}$$

which is about 9 for our example. About 50 per cent of the stored beam is contained in a small radial interval of $\Delta R = 0.04L$ between 420 and 470 MeV. This storage ring is actually an acceleration—and storage ring, since the H^- beam can be injected at an energy as low as 45 MeV.

The interesting question is of course: How much current can be stored in such a cyclotron?

Again, to get some feeling for the numbers involved, let us assume that the above-mentioned 500 MeV storage ring has a minimum radius of 1.5 m and a maximum radius of 5 m. The isochronous revolution time for a particle is then about 140 nsec. The rf cavities are then about 3.5 m long and could operate, e.g. on 50 MHz and 150 MHz.

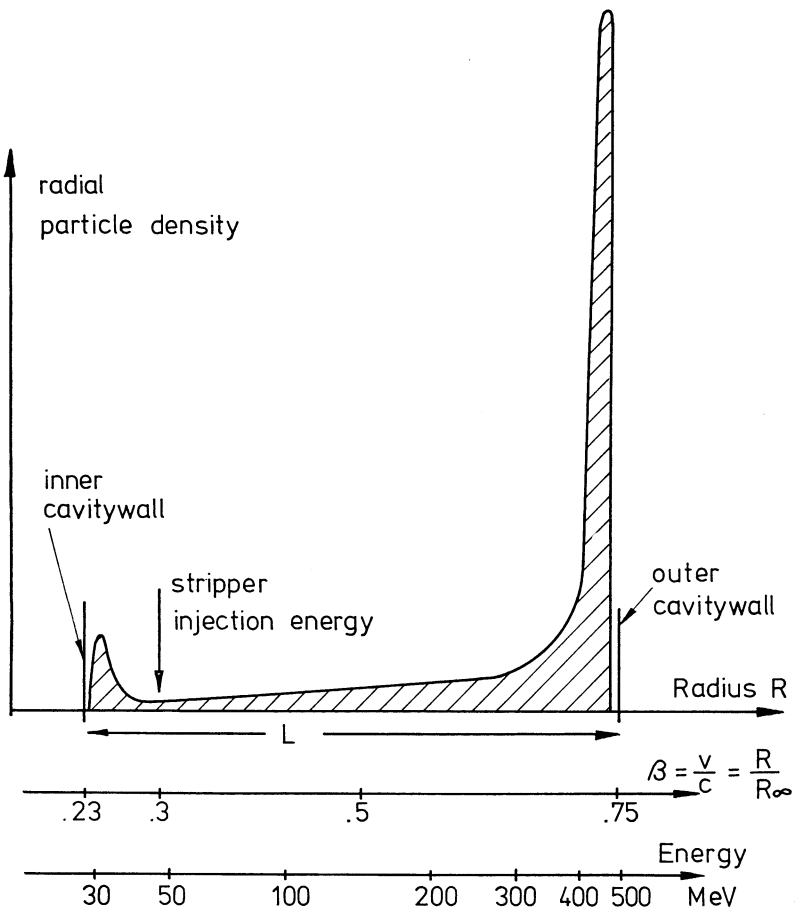

FIGURE 10 Storage ring with flat-top cavities ($\mathscr{E}_3 = 0.3$). Radial particle density for particles in hatched rings of Figure 8. The phase width of the injected beam is $2 \times 8°$ whereas at the extreme energies the beam is practically dc. About 50 per cent of the stored beam is contained in a radial interval of $\Delta R = 0.04\,L$ between 420 and 470 MeV.

Two main cavities with 500 kV and two flat-top cavities (150 MHz) with 150 kV peak voltage give about 5 mm turn separation at the injection energy of 45 MeV. This should be enough for the protons to avoid the stripper after one revolution. For the particle trajectories in Figure 9 the cycle period is then about 5000 revolutions or 0.7 msec. A continuously injected H^- beam of 100 μA average current would give in one storing cycle 500 mA of circulating beam, half of which would be inside a radial width of 14 cm between 420 and 470 MeV. But there is no need to stop injecting after one cycle. Let us assume that we use a carbon stripper of 100 μg/cm^2. The rms-Coulomb-scattering angle per traversal at 45 MeV is then 0.2 mrad. After 200 storing cycles or 140 msec the average scattering angle produced by the 400 stripper traversals amounts to 4 mrad, which should be tolerable. This means that from this point of view 100 A *of stored beam* are theoretically possible. Since ring cyclotrons have in general a strong vertical focusing (with ν_z around 1) the transversal space-charge limit is also in the 100 A region.

3.3. *Problems*

The great problem of this storage ring concept seems to be the *extraction.* Neutron physicists would be extremely interested if this large stored proton current could be extracted in a few microseconds. The stored current fills the circumference of the machine almost uniformly. Therefore even fast kicker magnets will produce beam spills, since the very large aperture of these magnets prevent very short risetimes. Another method could be a slow resonance extraction scheme. One could conceive even a vertical extraction, taking advantage of the free sectors in a ring cyclotron. At the anticipated high beam current one runs into activation problems. One may fill the storage ring therefore only once every few seconds to lower the activation level.

Since the particles coast a long time at maximum energy, the isochronism of the cyclotron has to be very good. The magnetic field has to be trimmed to an accuracy of typically 10^{-4} to 10^{-5}. This looks feasible for a fixed excitation of the magnets. Beam loading of the cavities should not be a big problem, since only the newly injected particles have to be accelerated to the average energy of the stored particles.

4. CONCLUSION

The phase compression–phase expansion effect has to be taken into account in cyclotrons with a radially varying energy gain per turn. A closer look at this effect reveals that it could be utilized in storing several amperes of protons using radial $\lambda/2$ resonators. A bunched H^- beam at injection energy is transformed into an almost dc beam at maximum energy. Since the author is familiar with the SIN 590 MeV isochronous ring cyclotron, some numerical calculations were done for a storage ring of similar dimensions. No obstacle can be seen yet why this storing concept could not work at higher energies too, using, e.g. the 800 MeV H^- beam of LAMPF for injection. At TRIUMF where the H^- ions are normally stripped to protons for extraction one could possibly use the $\nu_r = 3/2$ resonance to extract the H^- ions at about 400 MeV.[8] The protons could then be further accelerated in a ring cyclotron and stored at around 1500 MeV. More detailed calculations are necessary to determine if the problem of extraction from a storage ring can be solved.

The phase expansion effect can also be used to stretch the duty cycle of a cyclotron by lowering the dee voltage at extraction radius or by inserting C-electrodes at the maximum energy.

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